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The Tjon-Wu model in extended kinetic theory: exact stationary solutions and trend to equilibrium (**)

A Bianca Manfredi con amicizia e stima

1 - Introduction

The present paper deals with the problem of equilibrium states in the kinetic theory of gases, and of their stability. In such a basic and classical investigation, the Boltzmann equation plays undoubtedly an essential role [4], [10]. On the other hand, because of its peculiar features, the Boltzmann equation has resisted solution for very long time.

The first significant nonequilibrium analytical solutions were discovered in 1976 in the isotropic and space homogeneous case [3], [7], and the first global existence proof is very recent [5]. Also quite recent is the appearance in the literature of the so called *extended kinetic theory*, in which gas particles are allowed to undergo other kinds of binary interactions, in addition to elastic scattering between themselves. In this nonconservative frame several physical phenomena of practical interest, like the presence of a background medium and of external sources, absorption collisions, chemical or nuclear reactions, can be described. The interested reader is referred to [8], where also previous work is reviewed.

Of course, solving the extended Boltzmann equation is a much harder task than solving the corresponding standard equation, which is known to be quite cumbersome to attack even from a numerical point of view [2], because of its

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five dimensional integral over an unbounded domain. The role of the Boltzmann models in the study of kinetic theory is widely recognized [6], since they have allowed significant analytical investigations, and have correctly predicted important effects.

In the frame described above, aim of the present article is the determination of exact solutions of the extended Tjon-Wu model, which is probably the simplest model retaining the most important features of the Boltzmann equation [9]. It is a maxwellian isotropic stochastic model in two dimensions [6], and the extended version examined here allows for the presence of a fixed background of field particles, and for the occurrence of elastic scattering and removal collisions. The idea is not new, since very recently Zanette [11] considered the linear version of the Tjon-Wu model to describe a gas of noninteracting test particles diffusing elastically in a host medium, and provided a stationary analytical solution.

It is shown here that his result can be generalized to the nonlinear conservative problem in which also t.p.-t.p. collisions take place, and that in the linear case also the time dependent nonconservative problem with removal and external source can be dealt with analytically.

2 - Extended Tjon-Wu equation

Let f denote the t.p. *distribution function*, depending on the energy and time variables x and t, both ranging from 0 to $+\infty$. The regular function f is assumed to have finite energy moments at least up to order one, in order to give physical meaning to the main macroscopic observables, i.e. number density

$$\rho(t) = \int_0^\infty f(x, t)\,\mathrm{d}x \tag{1}$$

and temperature (in energy units)

$$T(t) = \frac{1}{\rho(t)} \int_0^\infty x f(x, t)\,\mathrm{d}x\,. \tag{2}$$

The symbol C will denote the (constant) collision frequencies, and subscripts S and R will be used to label scattering and removal interactions, respectively, with $C = C_S + C_R$. All quantities relevant to f.p. are labeled by a star. Then the *Tjon-Wu model equation* for the extended problem described in Section **1** reads as [9], [8]

$$
(3) \qquad \begin{aligned} \frac{\partial f}{\partial t} &= -(C\rho(t) + C^*\rho^*) f(x, t) \\ &+ \int_x^\infty \frac{dy}{y} \int_0^y f(y - y', t)(C_S f(y', t) + C_S^* f^*(y'))\, dy' \end{aligned}
$$

with initial condition $f(x, 0) = f_0(x)$. The maxwellian distribution (equilibrium for the unextended case) corresponds to the exponential function

$$
(4) \qquad F(x) = \frac{\bar{\rho}}{\bar{T}} \exp\left(-\frac{x}{\bar{T}}\right).
$$

Direct integrations of equation (3) lead to the *continuity equation*

$$
(5) \qquad \frac{d\rho}{dt} + \rho^* C_R^* \rho + C_R \rho^2 = 0
$$

(a selfcontained Bernoulli equation for ρ), and to the *linear energy equation* for T

$$
(6) \qquad \frac{dT}{dt} + \frac{1}{2} \rho^* C_S^* (T - T^*) = 0
$$

which are not in conservation form, but express decay to zero of t.p. population (unless $\rho^* C_R^* = C_R = 0$) and relaxation of T to the background temperature at an exponential rate $\frac{1}{2} \rho^* C_S^*$ (unless $\rho^* C_S^* = 0$).

It is known [6] that a very convenient tool for the analysis of equation (3) is the Laplace transform, in spite of nonlinearity. Setting

$$
(7) \qquad f(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(z, t) e^{xz}\, dz \qquad \tilde{f}(z, t) = \int_0^\infty f(x, t) e^{-xz}\, dx
$$

with c in the half plane of absolute convergence, and denoting any Laplace transform by a superimposed tilde, one gets after some algebra the nonlinear equation with a single integral for $\tilde{f}$

$$
(8) \qquad \frac{\partial \tilde{f}}{\partial t} + (C\rho(t) + C^*\rho^*) \tilde{f}(z, t) = \frac{1}{z} \int_0^z (C_S \tilde{f}^2(z', t) + C_S^* \tilde{f}^*(z') \tilde{f}(z', t))\, dz'
$$

with $\tilde{f}(z, 0) = \tilde{f}_0(z)$. Now $\tilde{f}$ is at least once differentiable at $z = 0$ with

$$
(9) \qquad \rho(t) = \tilde{f}(0, t) \qquad T(t) = -\frac{1}{\rho(t)} \frac{\partial \tilde{f}}{\partial z}(0, t)
$$

and the maxwellian corresponds to the rational function

$$\tilde{F}(z) = \bar{\rho}(1 + \bar{T}z)^{-1} \tag{10}$$

with only a simple pole at $z = -\bar{T}^{-1}$.

In the sequel we will focus our attention on the solution to equation (8). Section 3 is devoted to the stationary conservative problem ($C_R = C_R^* = 0$) in order to study equilibrium under t.p.-f.p. and t.p.-t.p. elastic collisions. Zanette result is recovered as a particular case. Section 4 deals with relaxation to equilibrium for the same problem. Finally, in Section 5 the general time dependent solution to the linear problem ($C_S = C_R = 0$) is derived and briefly discussed.

3 - Steady state solutions

In stationary conditions, with $C_R = C_R^* = 0$, macroscopic equations yield at once $\rho = constant$, and $T = T^* = constant$, and, upon introducing

$$\varphi(z) = \frac{1}{\rho}\tilde{f}(z) \qquad \alpha = C_S\rho(C_S\rho + C_S^*\rho^*)^{-1} \tag{11}$$

with $0 \leq \alpha \leq 1$, equation (8) can be converted into a homogeneous Riccati equation with coefficients singular (simple pole) at $z = 0$, namely

$$\frac{d\varphi}{dz} + (1-\alpha)\frac{1-\varphi^*(z)}{z}\varphi(z) + \alpha\frac{1-\varphi(z)}{z}\varphi(z) = 0 \qquad \varphi(0) = 1 \tag{12}$$

where also $\varphi^*(0) = 1$. Of course the singularity disappears in the linear case $\alpha = 0$, in which the result of **[11]**

$$\varphi(z) = \exp\left(-\int_0^z \frac{1-\varphi^*(z')}{z'}\, dz'\right) \tag{13}$$

is immediately recovered.

For a general $\alpha > 0$, existence or uniqueness of solution could be compromised, and one should also bear in mind the additional physical requirement of existence of $\varphi'(0) = -T$, or, more precisely, $\varphi'(0) = \varphi^{*\prime}(0) = -T^*$ for $\alpha < 1$ (as it occurs in equation (13) for $\alpha = 0$). The further substitutions $\varphi = u^{-1}$ and $u = 1 + v$ lead to the singular linear equation

$$\frac{dv}{dz} - \left[(1-\alpha)\frac{1-\varphi^*(z)}{z} + \frac{\alpha}{z}\right]v(z) = (1-\alpha)\frac{1-\varphi^*(z)}{z} \qquad v(0) = 0\,. \tag{14}$$

In the unextended case $\alpha = 1$ (absence of background), the problem (14) has ∞^1 solutions $v(z) = Az$, A arbitrary constant, to be identified with the temperature, so that equilibrium distribution is, as well known, any maxwellian

$$\varphi(z) = (1 + \bar{T}z)^{-1}. \tag{15}$$

For $0 < \alpha < 1$, the additional physical requirement reads as

$v'(0) = -\varphi^{*\prime}(0) = T^*$, and the ansatz

$$v(z) = (1-\alpha)\int_0^z \left(\frac{z}{z'}\right)^\alpha \frac{1-\varphi^*(z')}{z'} \exp\left((1-\alpha)\int_{z'}^z \frac{1-\varphi^*(z'')}{z''}\, dz''\right) dz' + w(z) \tag{16}$$

yields for w

$$\frac{dw}{dz} - \left((1-\alpha)\frac{1-\varphi^*(z)}{z} + \frac{\alpha}{z}\right) w(z) = 0 \qquad w(0) = 0. \tag{17}$$

which again admits ∞^1 solutions

$$w(z) = Az^\alpha \exp\left((1-\alpha)\int_0^z \frac{1-\varphi^*(z')}{z'}\, dz'\right). \tag{18}$$

But, if the physical condition about temperature (reading as $w'(0) = 0$) has to be satisfied, the integration constant is uniquely determined as $A = 0$, and we are left with the unique admissible equilibrium distribution

$$\varphi(z) = \left[1 + (1-\alpha)\int_0^1 \frac{1-\varphi^*(zy)}{y}\, y^{-\alpha} \exp\left((1-\alpha)\int_y^1 \frac{1-\varphi^*(zy')}{y'}\, dy'\right) dy\right]^{-1} \tag{19}$$

holding for any $0 \leqslant \alpha < 1$ (it collapses to (13) for $\alpha = 0$). The equilibrium distribution function is then determined analytically in terms of the f.p. distribution.

If in particular the latter is a maxwellian, one gets after some algebra

$$\varphi(z) = \left[1 + (1-\alpha)\int_0^z \frac{T^*}{1+T^*z'}\left(\frac{z}{z'}\right)^\alpha \left(\frac{1+T^*z}{1+T^*z'}\right)^{1-\alpha} dz'\right]^{-1} = (1+T^*z)^{-1}, \tag{20}$$

namely again a maxwellian at the same temperature as field particles, independent of α. However, the expression of φ becomes easily very complicated even for very simple f.p. distributions. For instance, for a superposition of maxwellians at temperature T_1 and T_2, i.e.

$$\varphi^*(z) = \beta(1 + T_1 z)^{-1} + (1-\beta)(1 + T_2 z)^{-1} \qquad 0 < \beta < 1 \tag{21}$$

with $T = T^* = \beta T_1 + (1-\beta) T_2$, it is obtained, after some lengthy manipulations,

$$
(22)\qquad \begin{aligned}
&\varphi(z) = \{1 + (1-\alpha) z (1 + T_1 z)^{(1-\alpha)\beta} (1 + T_2 z)^{(1-\alpha)(1-\beta)} A\}^{-1} \\
&A = I_\alpha(z, T_1, T_2, \beta) + I_\alpha(z, T_2, T_1, 1-\beta) \\
&I_\alpha(z, T_1, T_2, \beta) = \beta T_1 ((T_1 - T_2) z)^{\alpha - 1} B_\xi (1-\alpha, \beta + \alpha - \alpha\beta)
\end{aligned}
$$

where B denotes *incomplete beta function* [1], and $\xi = (T_1 - T_2) z (T_1 z + 1)^{-1}$.

It is easily seen that if $\varphi^*(z)$ is analytic at $z = 0$, the same occurs for $\varphi(z)$, since the Cauchy-Riemann conditions are satisfied. This implies, in particular, the existence of all energy moments of the equilibrium distribution function.

4 - Trend to equilibrium

In the same conservative situation of Section 3, macroscopic quantities, in time dependent conditions, may still be determined independently of f, and read as

$$
(23)\qquad \rho = \text{constant} \qquad T = T^* + (T_0 - T^*)\ \exp\left(-\frac{1}{2}\, C_S^* \rho^* t\right).
$$

Setting $\tilde{f}(z, t) = \rho[\varphi(z) + \psi(z, t)]$, with φ given by (19), equation (8) yields for ψ

$$
(24)\qquad \frac{\partial \psi}{\partial t} + \psi(z, t) = \frac{1}{z} \int_0^z [\alpha \psi(z', t) + 2\alpha \varphi(z') + (1-\alpha)\varphi^*(z')]\, \psi(z', t)\, \mathrm{d}z'
$$

in terms of the dimensionless time variable $(C_S \rho + C_S^* \rho^*) t$, labeled again by t.

We can prove now that ψ tends to zero for $t \to \infty$, at least under the hypothesis that $f(x, t)$ admits moments of any order, namely that $\tilde{f}(z, t)$ is analytic at $z = 0$. If $\psi_0(z)$ denotes the initial condition to be associated to equation (24), we may write

$$
(25)_a \qquad \psi(z, t) = \sum_{n=0}^{\infty} a_n(t) z^n \qquad \psi_0(z) = \sum_{n=0}^{\infty} b_n z^n
$$

with $a_0 = b_0 = 0$, and analogously

$$(25)_b \qquad 2\alpha\varphi(z) + (1-\alpha)\,\varphi^*(z) = G(z) = \sum_{n=0}^{\infty} c_n z^n$$

with $c_0 = 1 + \alpha$.

We have to determine the unknown coefficients a_n, subject to initial conditions $a_n(0) = b_n$. Plugging (25) into (24) leads to the hierarchy of first order linear ODE's

$$(26) \qquad \frac{da_n}{dt} + \frac{n-\alpha}{n+1}\,a_n(t) = \frac{1}{n+1}\sum_{k=1}^{n-1} a_k(t)[c_{n-k} + \alpha a_{n-k}(t)] \qquad n = 1, 2, \dots$$

which shares with other sets in extended kinetic theory the nice feature of solvability in cascade [8], starting from

$$(27) \qquad \frac{da_1}{dt} + \frac{1-\alpha}{2}\,a_1 = 0 \qquad \frac{da_2}{dt} + \frac{2-\alpha}{3}\,a_2 = \frac{1}{3}\,a_1(c_1 + \alpha a_1).$$

Since $a_1 = b_1 \exp[-\frac{1}{2}(1-\alpha)t]$, the inhomogeneous term in the second equation is the sum of two exponentials, and $\exp[-\frac{1}{2}(1-\alpha)t]$ is the most persistent in time. The equation being linear, and $(n-\alpha)(n-1)^{-1}$ being monotonically increasing with n, a_2 is in turn a sum of exponential terms (or possibly exponentials times a pover of t), among which the dominant one for $t \to \infty$ is again $\exp[-\frac{1}{2}(1-\alpha)t]$. This fact can be proved by induction at any step. Assuming that all $a_k(t)$ are finite sums of exponentials or exponentials times powers, and are $O\{\exp[-\frac{1}{2}(1-\alpha)t]\}$ for $t \to \infty$, up to $k = n-1$, the n-th equation provides the same structure and the same asymptotic behaviour for $a_n(t)$. Thus exponential asymptotic stability follows for the equilibrium (19) with $\alpha < 1$: the perturbation ψ vanishes for $t \to \infty$ at a rate $\exp[-\frac{1}{2}(1-\alpha)t]$, whatever the initial condition.

In the unextended case $\alpha = 1$, temperature is also conserved, and relaxation to the Maxwellian at the initial density ρ_0 and temperature T_0 has to be expected. Proceeding in the same way as before, now with $a_1 = b_1 = 0$ and $c_n = 2(-1)^n T_0^n$, one readily realizes that again all expansion coefficients a_n, $n \geqslant 2$, vanish for $t \to \infty$ at an exponential rate $\exp(-\frac{t}{3})$.

An analytical solution is indeed possible for the linearized equation which is obtained from equation (24) by dropping the quadratic term under the integral

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Sommario

Si generalizza l'equazione di Tjon-Wu spazialmente omogenea, tenendo conto anche della presenza di un mezzo ospite, al caso in cui effetti non conservativi come l'assorbimento hanno rilevanza fisica. Si determina la distribuzione di equilibrio, nel caso conservativo, come soluzione di un problema singolare, in cui l'unicità segue dall'imposizione delle naturali condizioni fisiche, che sarebbero sovrabbondanti in un problema regolare. Viene analizzata, poi, anche la tendenza all'equilibrio. Infine si esamina il caso lineare non conservativo e non stazionario, che viene risolto analiticamente in termini di funzioni di Bessel.

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