Riv. Mat. Univ. Parma (4) **17** (1991), 29-39

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Einstein metrics on some special manifolds (**)

1 - Introduction

A well known problem in Riemannian geometry is to find Einstein metrics on a particular manifold. Many Authors studied this subject from different points of view (see [1] and the extensive bibliography enclosed therein).

The main purpose of this paper is to give the explicit construction of Einstein metrics by means of the deformation of a given Riemannian metric along a unit Killing vector field. More precisely, in 2, if (M, g) is a Riemannian manifold admitting a globally defined unit Killing vector field ξ, the connection and the curvature forms of the deformed metric

$$g_u = g + u\eta \otimes \eta \qquad u \in \mathrm{R} - \{0\} \qquad u > -1$$

(η: 1-form dual to ξ) are related to the corresponding forms of g. This is done applying the Cartan's theory of moving frame since the calculations can be performed in a compact way.

In 3 looking for conditions such that g_u is an Einstein metric, the following is proved

Main Theorem. *Let (M, g) be a non Einstein Riemannian manifold of dimension $m+1$ and let ξ be a unit Killing vector field with dual 1-form η. The Riemannian metric $g_u = g + u\eta \otimes \eta$ $(u > -1)$ is Einstein if and only if the follo-*

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(**) Work supported by the Italian M.U.R.S.T. and G.N.S.A.G.A. (C.N.R.).
M.R. classification: 53C25; 53C15. - Ricevuto: 4-VII-1990.

wing condition holds

$$\operatorname{Ric}(g) = \sum_{\alpha;\beta;\gamma} \{(1+u)\|\nabla_{e_\alpha} e_{m+1}\|^2 \delta_{\alpha\beta}$$

$$-2u\omega^\alpha{}_{m+1}(e_\gamma)\,\omega^\gamma{}_{m+1}(e_\beta)\}\,\omega^\alpha \otimes \omega^\beta + \sum_\alpha \|\nabla_{e_\alpha} e_{m+1}\|^2\, \omega^{m+1} \otimes \omega^{m+1}$$

where

$$u = \frac{\operatorname{Scal}(g) - (m+1)\sum_\alpha \|\nabla_{e_\alpha} e_{m+1}\|^2}{(m+2)\sum_\alpha \|\nabla_{e_\alpha} e_{m+1}\|^2},$$

$(e_1, \ldots, e_{m+1} = \xi)$ *and* $(\omega^1, \ldots, \omega^{m+1} = \eta)$ *are respectively a local orthonormal frame for* g *and the dual coframe.*

Of course, the existence of a globally defined unit Killing vector field imposes some topological restrictions on the manifold M; for example, its Euler-Poincaré characteristic must be zero. Nevertheless, as it is shown in **4**, the Main Theorem can be applied to a broad class of manifolds, precisely the so called K-contact spaces (see [2] for the examples).

A typical and important instance of this situation is given by the tangent sphere bundle $T_1 S^m$ of the standard m-dimensional sphere S^m, where we recover the Einstein metric found by Kobayashi [3], who considered $T_1 S^m$ as an S^1-bundle over the Grassmann manifold $SO(m+1)/SO(m-1) \times SO(2)$ (see also [4] for an alternative description).

The Authors wish to thanks Prof. D. Perrone and Prof. F. Tricerri for useful discussions during the preparation of this paper.

2 - Deformation of a metric tensor along a vector field

Let (M, g) be a Riemannian manifold which admits a globally defined unit vector field ξ. Let η be the corresponding dual 1-form given by

(2.1) $$\eta(X) = g(X, \xi)$$

for any vector field X on M. The covariant tensor field

(2.2) $$g_u = g + u\eta \otimes \eta$$

is still a Riemannian metric on M for any real constant $u > -1$. We may think of

g_u as the metric obtained from g after a deformation in the direction of the vector field ξ and, to avoid the trivial case, we shall always assume $u \neq 0$.

By means of the Cartan's structural equations, it is relatively easy to find the relations between the Riemannian connections of g and g_u. Let $\{e_1, \ldots, e_{m+1} = \xi\}$ be a local orthonormal frame for the metric g and let $\{\omega^1, \ldots, \omega^{m+1} = \eta\}$ be the associated coframe. If $t^2 = 1 + u$,

$$\begin{aligned} &\{e'_1 = e_1, \ \ldots, \ e'_m = e_m, \ e'_{m+1} = \frac{1}{t}\xi\} \\ &\{\omega'^1 = \omega^1, \ \ldots, \ \omega'^m = \omega^m, \ \omega'^{m+1} = t\eta\} \end{aligned} \tag{2.3}$$

are respectively an orthonormal frame and the relative coframe for the metric g_u.

We adopt the following convention about the indices

$$\alpha, \beta, \gamma = 1, 2, \ldots, m \qquad A, B, C = 1, 2, \ldots, m+1.$$

The Riemannian metrics g and g_u are (locally) given by

$$g = \sum_A \omega^A \otimes \omega^A \qquad g_u = \sum_\alpha \omega^\alpha \otimes \omega^\alpha + (1+u)\,\omega^{m+1} \otimes \omega^{m+1}. \tag{2.4}$$

Moreover, the connection 1-forms of ∇, Riemannian connection of g, are defined by

$$\omega^A{}_B(X) = g(\nabla_X e_B, \ e_A) = \omega^A(\nabla_X e_B) \tag{2.5}$$

for any vector field X om M, and they satisfy the Cartan's structural equations

$$d\omega^A = -\sum_B \omega^A{}_B \wedge \omega^B \qquad \omega^A{}_B + \omega^B{}_A = 0. \tag{2.6}$$

Similar formulas hold for the connection 1-forms of ∇', Riemannian connection of g_u. We have the following

Theorem 2.1. *The relations between the connection 1-forms of ∇ and ∇' are given by*

$$\begin{aligned} \omega'^\alpha{}_\beta &= \omega^\alpha{}_\beta + \frac{t^2-1}{2}\{\omega^\alpha{}_{m+1}(e_\beta) - \omega^\beta{}_{m+1}(e_\alpha)\}\,\omega^{m+1} \\ \omega'^\alpha{}_{m+1} &= t\omega^\alpha{}_{m+1} - \frac{t^2-1}{2t}\sum_\beta\{\omega^\alpha{}_{m+1}(e_\beta) + \omega^\beta{}_{m+1}(e_\alpha)\}\,\omega^\beta \end{aligned} \tag{2.7}$$

where $t^2 = 1 + u$.

Proof. The differentials $d\omega'^A{}_B$ can be computed directly from (2.3) and (2.6). Comparing such expressions with the structural equations of ∇', the following conditions must hold

$$(2.8)\qquad \begin{aligned} &\sum_\beta(\omega'^\alpha{}_\beta-\omega^\alpha{}_\beta)\,\omega'^\beta+(\omega'^\alpha{}_{m+1}-\frac{1}{t}\,\omega^\alpha{}_{m+1})\,\omega'^{m+1}=0\\ &\sum_\alpha(\omega'^{m+1}{}_\alpha-t\omega^{m+1}{}_\alpha)\,\omega'^\alpha=0\,. \end{aligned}$$

Because of Cartan's Lemma, we have

$$(2.9)_1\qquad \omega'^\alpha{}_\beta-\omega^\alpha{}_\beta=\sum_\gamma F^\alpha{}_{\beta\gamma}\,\omega'^\gamma+F^\alpha{}_{\beta m+1}\,\omega'^{m+1}$$

$$(2.9)_2\qquad \omega'^\alpha{}_{m+1}-\frac{1}{t}\,\omega^\alpha{}_{m+1}=\sum_\gamma F^\alpha{}_{m+1\gamma}\,\omega'^\gamma+F^\alpha{}_{m+1m+1}\,\omega'^{m+1}$$

$$(2.9)_3\qquad \omega'^{m+1}{}_\alpha-t\omega^{m+1}{}_\alpha=\sum_\beta F^{m+1}{}_{\alpha\beta}\,\omega'^\beta$$

for some functions $F^A{}_{BC}$ such that

$$F^\alpha{}_{BC}=F^\alpha{}_{CB}\qquad F^{m+1}{}_{\alpha\beta}=F^{m+1}{}_{\beta\alpha}\qquad F^\alpha{}_{\beta C}=-F^\beta{}_{\alpha C}\,.$$

It follows immediately that $F^\alpha{}_{\beta\gamma}=0$ and from (2.9) we get

$$(2.10)\qquad \frac{t^2-1}{t}\,\omega^\alpha{}_{m+1}=\sum_\beta(F^\alpha{}_{m+1\beta}+F^{m+1}{}_{\alpha\beta})\,\omega^\beta+tF^\alpha{}_{m+1m+1}\,\omega^{m+1}\,.$$

Applying the 1-forms (2.10) to the vector field e_{m+1}, we find

$$F^\alpha{}_{m+1m+1}=\frac{t^2-1}{t^2}\,\omega^\alpha{}_{m+1}(e_{m+1})\,.$$

In a similar way, we have

$$\begin{aligned} F^\alpha{}_{\beta m+1}&=\frac{t^2-1}{2t}\,\{\omega^\alpha{}_{m+1}(e_\beta)-\omega^\beta{}_{m+1}(e_\alpha)\}\\ F^{m+1}{}_{\alpha\beta}&=\frac{t^2-1}{2t}\,\{\omega^\alpha{}_{m+1}(e_\beta)+\omega^\beta{}_{m+1}(e_\alpha)\}\,. \end{aligned}$$

Inserting such expressions into the equations (2.9), we finally obtain the theorem.

Now let us assume that ξ be a Killing vector field, that is its local one-parameter group of transformations consists of local isometries. It is well known that such condition is equivalent to $L_\xi g = 0$, where L_ξ denotes the Lie derivative with respect to ξ. Then, ξ is a Killing vector field if and only if

$$g(\nabla_X \xi,\ Y) + g(X,\ \nabla_Y \xi) = 0 \tag{2.11}$$

for all vector fields X and Y on M. This is equivalent to

$$\omega^A{}_{m+1}(e_B) + \omega^B{}_{m+1}(e_A) = 0 \tag{2.12}$$

and, in this case, the relations (1.7) can be simplified into

$$\omega'^\alpha{}_\beta = \omega^\alpha{}_\beta + (t^2-1)\,\omega^\alpha{}_{m+1}(e_\beta)\,\omega^{m+1} \qquad \omega'^a{}_{m+1} = t\omega^\alpha{}_{m+1}\,. \tag{2.13}$$

The curvature tensor field R and the curvature 2-forms $\Omega^A{}_B$ of ∇ are respectively defined as follows

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X,\ \nabla_Y] \qquad 2\Omega^A{}_B(X,\ Y) = -\,\omega^A(R_{XY}e_B) \tag{2.14}$$

for all vector fields X and Y on M. Such 2-forms are skew-symmetric with respect to the indices A and B and satisfy the structural equations

$$\Omega^A{}_B = d\omega^A{}_B + \sum_C \omega^A{}_C \wedge \omega^C{}_B\,. \tag{2.15}$$

Similar equations also hold for the curvature 2-forms $\Omega'^A{}_B$ of the connection ∇'. We have the

Theorem 2.2. *The relations between the curvature 2-forms of ∇ and ∇' are given by*

$$\begin{aligned} \Omega'^\alpha{}_\beta &= \Omega^\alpha{}_\beta + (t^2-1)\{d[\omega^\alpha(\nabla_{e_\beta} e_{m+1})] \\ &\quad + \sum_\gamma[\omega^\gamma(\nabla_{e_\beta} e_{m+1})\,\omega^\alpha{}_\gamma - \omega^\alpha(\nabla_{e_\gamma} e_{m+1})\,\omega^\gamma{}_\beta]\} \wedge \omega^{m+1} \\ &\quad + (t^2-1)[\omega^\alpha(\nabla_{e_\beta} e_{m+1}) \sum_\gamma \omega^\gamma{}_{m+1} \wedge \omega^\gamma + \omega^\gamma{}_{m+1} \wedge \omega^{m+1}{}_\beta] \\ \Omega'^\alpha{}_{m+1} &= t\Omega^\alpha{}_{m+1} + t(t^2-1) \sum_\beta \omega^\alpha(\nabla_{e_\gamma} e_{m+1})\,\omega^{m+1} \wedge \omega^\beta{}_{m+1} \end{aligned} \tag{2.16}$$

where $t^2 = 1 + u$.

Proof. It is enough to compute the structural equations (2.15) using the relations (2.13).

The Ricci tensor and the scalar curvature of the metric g are respectively defined by

$$\mathrm{Ric}(g) = 2 \sum_{A,B,C} \Omega^A{}_B(\mathrm{e}_C,\ \mathrm{e}_B)\,\omega^A \otimes \omega^C \tag{2.17}$$

$$\mathrm{Scal}(g) = \sum_A \mathrm{Ric}(g)(\mathrm{e}_A,\ \mathrm{e}_A)\,. \tag{2.18}$$

Similar expressions hold for the Ricci tensor $\mathrm{Ric}(g_u)$ and for the scalar curvature $\mathrm{Scal}(g_u)$ of the metric g_u. In order to find the relations between the Ricci curvature of the two metrics, we first state the following

Lemma 2.3. *Let $(M,\ g)$ be a Riemannian manifold which admits a unit Killing vector field ξ. Then*

$$\mathrm{Ric}(g)(\xi,\ \xi) = \sum_\alpha \|\nabla_{\mathrm{e}_\alpha}\xi\|^2 \tag{2.19}$$

where $\{\mathrm{e}_1,\ \ldots,\ \mathrm{e}_{m+1} = \xi\}$ is a local orthonormal frame for the metric g.

Proof. From (2.15), (2.17) and the fact that ξ is a unit Killing vector field, we get

$$\mathrm{Ric}(g)(\mathrm{e}_{m+1},\ \mathrm{e}_{m+1}) = 2\sum_\alpha \Omega^\alpha{}_{m+1}(\mathrm{e}_\alpha,\ \mathrm{e}_{m+1})$$

$$= 2\sum_\alpha[\mathrm{d}\omega^\alpha{}_{m+1} + \sum_\beta \omega^\alpha{}_\beta \wedge \omega^\beta{}_{m+1}](\mathrm{e}_\alpha,\ \mathrm{e}_{m+1})$$

$$= -\sum_\alpha \omega^\alpha{}_{m+1}([\mathrm{e}_\alpha,\ \mathrm{e}_{m+1}]) - \sum_{\alpha,\beta} \omega^\alpha{}_\beta(\mathrm{e}_{m+1})\,\omega^\beta{}_{m+1}(\mathrm{e}_\alpha)$$

$$= -\sum_\alpha \omega^\alpha{}_{m+1}([\mathrm{e}_\alpha,\ \mathrm{e}_{m+1}]) - \sum_{\alpha,\beta} g(\nabla_{\mathrm{e}_{m+1}}\mathrm{e}_\alpha,\ \mathrm{e}_\beta)\,\omega^\alpha{}_{m+1}(\mathrm{e}_\beta)$$

$$= -\sum_\alpha \omega^\alpha{}_{m+1}([\mathrm{e}_\alpha,\ \mathrm{e}_{m+1}]) - \sum_{\alpha,\beta}\{g(\nabla_{\mathrm{e}_\alpha}\mathrm{e}_{m+1},\ \mathrm{e}_\beta)$$

$$+ g([\mathrm{e}_{m+1},\ \mathrm{e}_\alpha],\ \mathrm{e}_\beta)\}\,\omega^\alpha{}_{m+1}(\mathrm{e}_\beta) = \sum_{\alpha,\beta}\{\omega^\alpha{}_{m+1}(\mathrm{e}_\beta)\}^2$$

$$= \sum_\alpha g(\nabla_{\mathrm{e}_\alpha}\mathrm{e}_{m+1},\ \nabla_{\mathrm{e}_\alpha}\mathrm{e}_{m+1})\,.$$

We are now able to prove the

Theorem 2.4. *In the hypothesis of the previous Lemma, we have*

$$\mathrm{Ric}(g_u) = \mathrm{Ric}(g) + 2u \sum_{\alpha,\beta,\gamma} \omega^\alpha{}_{m+1}(\mathrm{e}_\gamma)\,\omega^\gamma{}_{m+1}(\mathrm{e}_\beta)\,\omega^\alpha \otimes \omega^\beta \tag{2.20}$$

$$+u \sum_\alpha \mathrm{Ric}(g)(\mathrm{e}_\alpha,\ \mathrm{e}_{m+1})(\omega^\alpha \otimes \omega^{m+1} + \omega^{m+1} \otimes \omega^\alpha)$$

$$+u(u+2) \sum_\alpha \|\nabla_{\mathrm{e}_\alpha} \mathrm{e}_{m+1}\|^2\, \omega^{m+1} \otimes \omega^{m+1}$$

$$\mathrm{Scal}(g_u) = \mathrm{Scal}(g) - u \sum_\alpha \|\nabla_{\mathrm{e}_\alpha} \mathrm{e}_{m+1}\|^2. \tag{2.21}$$

Proof. The formula (2.20) follows from the definition of $\mathrm{Ric}(g_u)$, (2.16) and the previous Lemma. The (2.21) follows immediately from the definition of $\mathrm{Scal}(g_u)$ and (2.20).

3 - Einstein metrics

The main purpose of this paragraph is to find the necessary and sufficient conditions such that the deformed metric g_u becomes an Einstein metric, i.e.

$$\mathrm{Ric}(g_u) = E_u g_u \qquad E_u = \frac{\mathrm{Scal}(g_u)}{m+1}. \tag{3.1}$$

To avoid trivial cases, we assume that the given metric g is non Einstein. Then we have the following

Theorem 3.1. *Let $(M,\ g)$ be a non Einstein Riemannian manifold of dimension $m+1$ and let ξ be a unit Killing vector field with dual 1-form η. The Riemannian metric $g_u = g + u\eta \otimes \eta$ $(u > -1)$ is Einstein if and only if the following condition holds*

$$\mathrm{Ric}(g) = \sum_{\alpha,\beta,\gamma} \{(1+u)\|\nabla_{\mathrm{e}_\alpha} \mathrm{e}_{m+1}\|^2 \delta_{\alpha\beta} \tag{3.2}$$

$$-2u\omega^\alpha{}_{m+1}(\mathrm{e}_\gamma)\,\omega^\gamma{}_{m+1}(\mathrm{e}_\beta)\}\omega^\alpha \otimes \omega^\beta + \sum_\alpha \|\nabla_{\mathrm{e}_\alpha} \mathrm{e}_{m+1}\|^2\, \omega^{m+1} \otimes \omega^{m+1}$$

where

$$\text{(3.3)} \qquad u = \frac{\operatorname{Scal}(g) - (m+1)\sum_\alpha \|\nabla_{e_\alpha} e_{m+1}\|^2}{(m+2)\sum_\alpha \|\nabla_{e_\alpha} e_{m+1}\|^2}$$

$\{e_1, \ldots, e_{m+1} = \xi\}$ *and* $\{\omega^1, \ldots, \omega^{m+1} = \eta\}$ *are respectively a local orthonormal frame for g and the dual coframe.*

Proof. Because of (2.20) and (3.1), the metric g_u is Einstein if and only if

$$\text{(3.4)} \qquad \operatorname{Ric}(g_u) = u\{2 \sum_{\alpha,\beta,\gamma} \omega^{m+1}{}_\alpha(e_\gamma)\,\omega^\gamma{}_{m+1}(e_\beta)\,\omega^\alpha \otimes \omega^\beta$$

$$- \sum_\alpha \operatorname{Ric}(g)(e_\alpha,\ e_{m+1})(\omega^\alpha \otimes \omega^{m+1} + \omega^{m+1} \otimes \omega^\alpha)$$

$$- (u+2)\sum_\alpha \|\nabla_{e_\alpha} e_{m+1}\|^2 \omega^{m+1} \otimes \omega^{m+1}\} + E_u g_u\,.$$

Then we compute E_u by means of (3.1) and (2.21) and we get

$$\text{(3.5)} \qquad E_u = (u+1)\sum_\alpha \|\nabla_{e_\alpha} e_{m+1}\|^2\,.$$

Inserting such expression into (2.4) and recalling that $u \neq 0$, we obtain the (3.2). On the other hand, a direct computation of Scal(g) from (3.4) and (3.5) gives (3.3).

Remarks. (i) It follows from (3.3) that a necessary condition for finding such Einstein metrics is that

$$\operatorname{Scal}(g) > -\sum_\alpha \|\nabla_{e_\alpha} e_{m+1}\|^2\,.$$

(ii) If we drop the request that g_u be positive definite, the same construction permits to find Lorentian metrics which may be of some interest for physical applications. Of course, in the present situation, the above remark does not apply.

4 - Applications

The theorem of the previous paragraph can be applied to some special classes of Riemannian manifolds. Here we consider, in particular, the case of K-contact manifolds. We first recall some definitions (see [2] for more details and examples).

A contact metric manifold (M, g, ϕ, η, ξ) is a Riemannian manifold (M, g) of odd dimension $2n+1$ endowed with three tensor fields ϕ, η, ξ of type (1, 1), (0, 1), (1, 0) respectively, which satisfy the following conditions

$$\phi^2 = -I + \eta \otimes \xi \qquad \eta(\xi) = 1 \tag{4.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \qquad g(X, \phi Y) = d\eta(X, Y) \tag{4.2}$$

for all vector fields X, Y on M (I is the Kronecker tensor of type (1.1)). Note that the above relations imply that

$$\eta(X) = g(X, \xi) \qquad \|\xi\| = 1 \tag{4.3}$$

for each vector field X on M.

A contact metric manifold M is said *K-contact* if, in addition, ξ is a Killing vector field, which is equivalent to

$$\nabla_X \xi = -\phi X \tag{4.4}$$

for each vector field X on M. On the other hand, a contact metric manifold M is called *Sasakian* if

$$(\nabla_X \phi)(X) = g(X, Y)\xi - \eta(Y)X \tag{4.5}$$

for all vector fields X, Y on M. It follows immediately that every K-contact manifold is Sasakian.

Let us consider a K-contact manifold M. It is known that there exist local orthonormal frames $\{e_1, \ldots, e_{2n+1} = \xi\}$ adapted to the contact structure, i.e.

$$\phi(e_i) = e_{i+n} \qquad i = 1, 2, \ldots, n.$$

If $\{\omega^1, \ldots, \omega^{2n+1} = \eta\}$ is the corresponding coframe, we have

$$\omega^{n+1} = -\omega^i \circ \phi \qquad i = 1, 2, \ldots, n \qquad \omega^{2n+1} = \eta.$$

The condition (3.4) is equivalent to

$$\omega^\alpha{}_{2n+1} = -\omega^\alpha \circ \phi \qquad \alpha = 1, 2, \ldots, n$$

which imply

$$\|\nabla_{e_\alpha} e_{2n+1}\| = 1$$

and, because of Lemma 2.3,

$$\mathrm{Ric}\,(g)(e_{2n+1},\ e_{2n+1}) = 2n\,.$$

Hence Theorem 3.1 can be reformulated as follows

Theorem 4.1. *Let $(M,\ g,\ \phi,\ \eta,\ \xi)$ be a K-contact Riemannian manifold of dimension $2n+1$. The Riemannian metric $g_u = g + u\eta \otimes \eta$ $(u > -1)$ is Einstein if and only if*

$$\mathrm{Ric}\,(g) = 2[(n+1)\,u + n] \sum_{\alpha=1}^{2n} \omega^\alpha \otimes \omega^\alpha + 2n\omega^{2n+1} \otimes \omega^{2n+1}$$

where

$$u = \frac{\mathrm{Scal}\,(g) - 2n(2n+1)}{4n(n+1)}\,.$$

Remark. Since two homothetic metrics have the same Ricci tensor, we find here a result due to Tanno (see [5]), who calls η-Einstein the manifolds which satisfy the hypotheses of the previous theorem.

A concrete application of Theorem 4.1 is given by $T_1 S^m$, the unit tangent bundle of the m-dimensional sphere S^m, endowed with the standard Sasakian structure $(g,\ \phi,\ \eta,\ \xi)$. Here $g = (1/4)\,g_S$, where g_S denotes the metric induced on $T_1 S^m$ by the Sasaki metric of TS^m. It can be shown that

$$\mathrm{Ric}\,(g) = 2(2m-3) \sum_{\alpha=1}^{2m-2} \omega^\alpha \otimes \omega^\alpha + 2(m-1)\,\eta \otimes \eta$$

$$\mathrm{Scal}\,(g) = 2(m-1)(4m-5)\,.$$

It follows from the above theorem that the metric $g_u = g + u\eta \otimes \eta$ is Einstein when $u = (m-2)/m$.

The result is in accordance with Theorem 7.1 of [4] and this metric coincides with the one found, in a completely different way, by Kobayashi (see [3]).

References

[1] A. L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin-Heidelberg, 1987.

[2] D. E. Blair, *Contact manifolds in Riemannian geometry*, Springer-Verlag, Berlin-New York, Lect. Notes in Math. **509** (1976).

[3] S. Kobayashi, *Topology of positively pinched Kähler manifolds*, Tôhoku Math. J. **15** (1963), 121-139.

[4] E. Musso and F. Tricerri, *Riemannian metrics on tangent bundles*, Ann. Mat. Pura Appl. **150** (1988), 1-19.

[5] S. Tanno, *Geodesic flows on C_L-manifolds and Einstein metrics on $S^3 \times S^2$*, in «Minimal submanifolds and geodesics», Kagai Publications, Tokyo (1978), 283-292.

Sunto

Sono costruite metriche di Einstein su varietà Riemanniane che ammettono un campo di Killing globale e unitario. Tali metriche sono ottenute deformando la metrica originaria in direzione di tale campo. Sono, inoltre, considerate alcune applicazioni alle varietà K-contatto.

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