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$\mathcal{K}$-analyticity of spaces of compact and nuclear operators (**)

1 - Introduction

Following [1] we shall say that a topological space is $\mathcal{K}$-analytic if it is the continuous image of a $F_{\sigma\delta}$ set of a compact space. We refer to [1] or [3] for the stability properties of $\mathcal{K}$-analytic sets. A Banach space E is said to be WKA (for weakly $\mathcal{K}$-analytic) if it is $\mathcal{K}$-analytic for the weak topology. It is known that E is $\mathcal{K}$-analytic for the norm topology if and only if E is separable.

It is well known that $l_1(I)$ for I uncountable and l_∞ are not WKA. Let us give a sketch for the proof of this facts: Let $\omega \notin I$ and I_ω the set $I \cup \{\omega\}$ endowed with the discrete topology; let $\mathcal{T}$ be the coarser topology on $l_1(I)$ which makes continuous the coordinates and the map $(a_i) \to \Sigma_i a_i$; then I_ω is homeomorphic to a closed subset of $l_1(I)$ for the $\mathcal{T}$-topology, hence $l_1(I)$ cannot be weakly Lindelof (hence it is not WKA). Now, we denote by D the set of finite sequences of odd lenght of rational numbers and by χ_A the characteristic function of $A \subset \mathbb{R}$; for $r = (r_1, \ldots, r_n) \in D$ we write $f_r = 2 \cdot \chi_{A_r} - 1$, where $A_r =]-\infty, r_1] \cup [r_2, r_3] \cup \ldots \cup [r_{n-1}, r_n]$. The map g: $l_1(\mathbb{R}) \to l_\infty(D)$ defined for $a = (a_t)_{t \in \mathbb{R}}$ by

$$g(a)(r) = \sum_{t \in \mathbb{R}} f_r(t) \cdot a_t \qquad r \in D$$

is a linear isometry onto a subspace of $l_\infty(D)$; this shows that l_∞ is not weakly Lindelof.

By E, F we shall denote Banach spaces. $L(E, F)$ (resp., $K(E, F)$, $A(E, F)$)

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shall mean the space of continuous (resp., compact, approximable) operators from E to F. For a subspace $S(E, F)$ we shall write $S^W(E, F)$ (resp., $S^s(E, F)$) the space $S(E, F)$ endowed with the weak operator topology (resp., the strong operator topology), i.e., the topology on $S(E, F)$ induced by the product topology of F^E when F is endowed with its weak topology (resp., the norm topology). When F is a dual Banach space we can consider on $S(E, F)$ the weak* operator topology defined in an obvious way and we shall denote it by $S^{W^*}(E, F)$; it is easy to see that the unit ball of $L(E, F)$ is compact for this topology.

It is known from [4]$_2$ that $L^W(E, F)$ (resp., $L^s(E, F)$) is $\mathcal{K}$-analytic if E is separable and F is WKA (resp., if E and F are separable) and that $A^W(E, F)$ (resp., $A^s(E, F)$) is $\mathcal{K}$-analytic if F is WKA (resp., separable).

We shall denote, for $1 \leqslant p \leqslant \infty$,

$$l_p[E] = \{(x_n) \in E^{\mathrm{N}} / (x^*(x_n)) \in l_p, \ \forall x^* \in E^*\}.$$

The expression

$$\|(x_n)\| = \sup\{(\Sigma_n |x^*(x_n)|^p)^{1/p} / \|x^*\| \leqslant 1\}$$

$$(\text{resp. } \|(x_n^*)\| = \sup\{(\Sigma_n |x_n^*(x)|^p)^{1/p} / \|x\| \leqslant 1\})$$

define a norm on $l_p[E]$ (resp., $l_p[E^*]$).

2 - The case of nuclear operators

The following definition is taken from [2].

Def. Let $0 < r \leqslant \infty$ and $1 \leqslant p, \ q \leqslant \infty$ such that $1 + r^{-1} \geqslant p^{-1} + q^{-1}$. An operator $T \in L(E, F)$ is said *(r, p, q)-nuclear* if we can write $T = \Sigma_n a_n x_n^* \otimes y_n$ for some $(a_n) \in l_r$, $(x_n^*) \in l_{q^*}[E^*]$ and $(y_n) \in L_{p^*}[F]$, q^* and p^* being the conjugated exponents of q and p. In the case $r = \infty$, one supposes that $(a_n) \in c_0$. We shall denote by $N_{(r,p,q)}(E, F)$ the space of these operators.

We have the following result about the $\mathcal{K}$-analyticity of the space of (r, p, q)-operators.

Theorem. *Let r, p, q as above. Then $N_{(r,p,q,)}(E, F)$ is $\mathcal{K}$-analytic for the weak operator topology if F is* WKA.

Proof. We shall denote by B the unit ball of $l_{q^*}[E^*]$ endowed with the topology of identification with the weak* operator topology of $L(E, l_{q^*})$ by means of the one-to-one and onto map

$$(x_n^*) \in l_{q^*}[E^*] \to S_{(x_n^*)} \in L(E, l_{q^*})$$

where $S_{(x_n^*)}(x) = (x_n^*(x))$ for all $x \in E$. Thus, a net $((x_n^{*j})_n)_j$ in $l_{q^*}[E^*]$ converges to (x_n^*) for this topology if $\Sigma_n b_n x_n^{*j}(x)$ converges to $\Sigma_n b_n x_n^*(x)$ for every $x \in E$ and every $(b_n) \in l_q$ (or every $(b_n) \in c_0$ if $q^* = 1$). B is compact.

We shall write C for the unit ball of $l_{p^*}[F]$ endowed with the topology of identification with the weak operator topology of the unit ball of $L(l_p, F)$ (or $L(c_0, F)$ if $p^* = 1$) by means of the one-to-one and onto map

$$(y_n) \in l_{p^*}[F] \to T_{(y_n)}$$

where $T_{(y_n)}((a_n)) = \Sigma_n a_n y_n \in F$ for $(a_n) \in l_p$ (or c_0 if $p^* = 1$). C is $\mathcal{K}$-analytic since it is closed in $L^W(l_p, F)$ and F is WKA.

Last, we shall write A for the space l_r (or c_0 if $r = \infty$) endowed with the norm topology if $1 \leqslant r \leqslant \infty$; if $0 < r < 1$ we consider A endowed with the topology induced by the $\|\cdot\|_1$-topology of l_1. A is $\mathcal{K}$-analytic in every case.

The map

$$((a_n),\ (x_n^*),\ (y_n)) \in A \times B \times C \to \Sigma_n a_n x_n^* \otimes y_n \in L^w(E,\ F)$$

is continuous. This finishes the proof as the image of this map is $N_{(r,p,q,)}(E,\ F)$.

Remarks. (1) Since F for the weak topology is canonically homeomorphic to a closed subspace of $N^W_{(r,p,q)}(E,\ F)$ the $\mathcal{K}$-analyticity of this space implies that F is WKA.

(2) The method of the precedent theorem shows that $N^s_{(r,p,q)}(E,\ F)$ is $\mathcal{K}$-analytic if and only if F is separable.

(3) We cannot expect, in this case, positive results for other finer topologies on $N_{(r,p,q)}(E,\ F)$. In fact, the space $N(E,\ F)$ of nuclear operators from E to F is not WKA if E^* and F are WKA: it is well known that $l_2(I)\hat{\otimes}_\pi l_2(I)$ contains isometrically $l_1(I)$ and, therefore, it cannot be weakly Lindelof if I is uncountable.

3 - The case of compact operators

A compact space K is said a *Talagrand compact* if the space $C(K)$ of real continuous functions on K is WKA. For a Banach space E to be $\mathcal{K}$-analytic is necessary and sufficient that the unit ball of its dual is a Talagrand compact for the weak* topology ($[4]_1$, Th. 3.6).

The following results shows that the space $K(E, F)$ of compact operator is $\mathcal{K}$-analytic when $A(E, F)$ is (for the topologies that we consider).

Theorem. *If F is* WKA *(resp., separable) then $K^W(E, F)$ (resp., $K^s(E, F)$) is $\mathcal{K}$-analytic.*

Proof. We write V for the unit ball of F^* endowed with the weak* topology and j for the canonical embedding of F into $C(V)$. The map $G: T \in K(E, F) \to j_0 T \in K(E, C(V))$ is a linear embedding. The topology induced on the image C of G by the weak operator topology on $K(E, C(V))$ is finer than the topology of identification with that of $K^W(E, F)$. Since $K(E, C(V)) = A(E, C(V))$ and since $C(V)$ is WKA if F is WKA, it will be enough to show that C is closed in $K^W(E, C(V))$. But this is a consequence of the fact that we can identify F with the subspace of $C(V)$ which consists in the affine functions in V and null at zero. The proof for the strong operator topology is analogous.

The method of the precedent theorem also shows the following results. Nevertheless, let us give an alternative proof.

Theorem. *$K(E, F)$ is* WKA *if E^* and F are* WKA.

Proof. The product of two Talagrand compacts being also a Talagrand compact, the product $U \times V$ is, under the hypothese, a Talagrand compact where U and V denote the unit balls of E^{**} and F^*, resp., endowed with the weak* topology. The proof finishes as the map

$$T \in K(E, F) \to f_T \in C(U \times V)$$

is a linear isometry where $f_T(u, v) = \langle u, T^* v \rangle$, $u \in U$ and $v \in V$, T^* being the adjoint map of T.

Remark. It is obvious that E^* and F are WKA if $K(E, F)$ is WKA. Hence, we can replace «if» by «iff» in the previous theorem.

References

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Resumen

En este trabajo se estudia la $\mathcal{K}$-analiticidad de los espacios de operadores compactos and (r, p, q)-nucleares entre dos espacios de Banach E and F para la topología de la convergencia puntual débil sobre $L(E, F)$.

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