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An existence theorem for a class of nonlinear Volterra integral equations (**)

1 - Introduction

Numerous papers have been appeared on the theory of Volterra integral equations [4]. Brauer [3] obtained a non-linear variation of constant formula for Volterra equations. Kartsatos [6]discussed the existence of bounded solutions and asymptotic relationship for nonlinear Volterra integral equations. Okrasinski [7] studied the nonnegative solution of nonlinear Volterra integral equation. Recently Banas [1] proved an existence theorem for nonlinear Volterra integral equation with deviating argument without assuming the Lipschitz condition. In this paper we shall prove an existence theorem for more general class of nonlinear Volterra integral equation with deviating argument. The result generalises the previous result.

2 - Basic assumptions

Let us introduce the following notations. Put

$$I = [0,\ \infty) \qquad J = (0,\ \infty) \qquad \Delta = \{(t,\ s): 0 \leqslant s \leqslant t < \infty\}.$$

Let $p(t)$ be a given continuous function defined on the interval I with values in J. We will denote by $C_p = C(I,\ p(t);\ R^n)$ the space of all continuous functions

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from I into R^n such that

$$\sup[|x(t)|\, p(t): t \geq 0] < \infty .$$

It has been shown [5] that C_p form the real Banach space with respect to the norm

$$\|x\| = \sup[|x(t)|\, p(t): t \geq 0] .$$

Consider the following form of nonlinear Volterra integral equation with deviating argument

$$x(t) = g(t, \ x(t), \ Tx(t)) + \int_0^t K(t, s, x(\alpha(s)) \, \mathrm{d}s \tag{1}$$

where x, g and K are n-vectors. Assume the following conditions:

(i) K: $\Delta \times R^n \to R^n$ is continuous and there exists continuous functions m: $\Delta \to I$, $a: I \to J$, $b: I \to J$ such that

$$|K(t, \ s, \ x)| \leq m(t, \ s) + a(t) \ b(s)|x| \qquad \forall (t, \ s) \in \Delta \qquad x \in R^n .$$

Let us put $L(t) = \int_0^t a(s)\, b(s) \, \mathrm{d}s$ and take an arbitrary number $M > 0$ and consider the space C_p with

$$p(t) = [a(t) \exp(ML(t) + t)]^{-1} .$$

(ii) g: $I \times R^n \times R^n \to R^n$ is continuous and there exists a constant A and a continuous functions $c: I \to J$ such that

$$|g(t, \ x(t), \ Tx(t))| \leq Ac(t)[|x| + |Tx|] \qquad R = \int_0^\infty c(t) \, \mathrm{d}t .$$

(iii) T is a continuous operator which maps R^n into R^n such that

$$|Tx(t)| \leq M \int_0^t b(s)|x(s)| \, \mathrm{d}s .$$

(iv) There exists a constant $B \geq 0$ such that for any $t \in I$ the following inequality holds

$$\int_0^t m(t, s)\, \mathrm{d}s \leq B a(t) \exp(ML(t)).$$

(v) $\alpha: I \to I$ is a continuous function satisfying the condition

$$L(\alpha(t)) - L(t) \leq N$$

where N is a positive constant.

(vi) $$\frac{a(\alpha(t))}{a(t)} \leq M(1 - 2AR - B)\exp(-MN) \qquad 2AR + B < 1.$$

In what follows we will employ the following criterion of compactness of sets in the space C_p.

Lemma [2]. *Let E be a bounded set in the space C_p. If all the functions belonging to E are equicontinuous on each interval* $[0, \eta]$ *and*

$$\lim_{\eta \to \infty} \sup[|x(t)|p(t): t \geq \eta] = 0 \textit{ uniformly with respect to } E$$

then E is relatively compact in C_p.

If $x \in C_p$, we will denote by $\omega(x, h)$, its modulus of continuity on the interval $[0, \eta]$ as $\omega(x, h) = \sup[|x(t) - x(s)| : |t - s| \leq \text{ for } t, s \in [0, \eta]]$.

3 - Existence theorem

Theorem. *Under the assumptions* (i) *to* (vi) *the equation* (1) *has at least one solution x in the space C_p such that*

$$|x(t)| \leq a(t) \exp(ML(t)) \quad \textit{for any} \quad t \geq 0.$$

Proof. Consider the following transformation defined on the space C_p by

$$(Fx)(t) = g(t, x(t), Tx(t)) + \int_0^t K(t, s, x(\alpha(s)))\, \mathrm{d}s.$$

Observe that our assumption imply that $(Fx)(t)$ is continuous in I and define

$$G = \{x \in C_p : |x(t)| \leqslant a(t) \exp(ML(t))\}.$$

Obviously G is nonempty, bounded, closed and convex in the space C_p. Now we show that F maps the set G into itself. Take $x \in G$. Then from our assumptions we have

$$|(Fx)(t)| \leqslant |g(t, \ x(t), \ Tx(t))| + \int_0^t |K(t, \ s, \ x(\alpha(s)))| \, \mathrm{d}s$$

$$\leqslant Ac(t)(|x(t)| + |Tx(t)|) + \int_0^t [m(t, \ s) + a(t)\, b(s) |x(\alpha(s))|] \, \mathrm{d}s$$

$$\leqslant Ac(t)(|x(t)| + M \int_0^t b(s)|x(s)| \, \mathrm{d}s) + \int_0^t [m(t, \ s) + a(t)\, b(s)|x(\alpha(s))|] \, \mathrm{d}s$$

$$\leqslant Ac(t)[a(t) \exp(ML(t)) + M \int_0^t b(s)\, a(s) \exp(ML(s)) \, \mathrm{d}s] + Ba(t) \exp(ML(t))$$

$$+ a(t) \int_0^t b(s)\, a(\alpha(s)) \exp(ML(\alpha(s))) \, \mathrm{d}s$$

$$\leqslant (2AR + B)\, a(t) \exp(ML(t))$$

$$+ \frac{1}{M} a(t) \int_0^t Mb(s) \exp(ML(s))\, a(\alpha(s)) \exp(M(L(\alpha(s)) - L(s))) \, \mathrm{d}s$$

$$\leqslant (2AR + B)\, a(t) \exp(ML(t))$$

$$+ a(t)(1 - 2AR - B) \int_0^t Ma(s)\, b(s) \exp(ML(s)) \exp(-MN) \exp(MN) \, \mathrm{d}s$$

$$\leqslant (2AR + B)\, a(t) \exp(ML(t)) + (1 - 2AR - B)\, a(t) \exp(ML(t)) = a(t) \exp(ML(t)).$$

Which shows that $FG \subset G$.

Next we show that F is continuous on the set G. For this let us fix $h > 0$ and $x, \ y \in G$ such that $\|x - y\| + \|Tx - Ty\| \leqslant h$. Further take an arbitrary fixed $\eta > 0$. Then using the fact that the functions $g(t, x(t), Tx(t))$ and $K(t, s, x)$ are respectively, uniformly continuous on

$$[0, \ \eta] \times [-\gamma(\eta), \ \gamma(\eta)] \times [-\gamma(\eta), \ \gamma(\eta)]$$

and

$$[0, \ \eta] \times [0, \ \eta] \times [-\gamma(\alpha(\eta)), \ \gamma(\alpha(\eta))]$$

where $\gamma(\eta) = \max[a(s) \exp(ML(s)) : s \in [0, \ \eta]]$, we obtain for $t \in [0, \ \eta]$

(2)
$$\begin{aligned} &|(Fx)(t) - (Fy)(t)| \\ &\leqslant |g(t, \ x(t), \ Tx(t)) - g(t, \ y(t), \ Ty(t))| \\ &\quad + \int_0^t |K(t, \ s, \ x(\alpha(s))) - K(t, \ s, \ y(\alpha(s)))| \, \mathrm{d}s \\ &\leqslant \beta_1(h) + \beta_2(h) \end{aligned}$$

where $\beta_i(h)$ are some continuous functions such that $\lim\limits_{h \to \infty} \beta_i(h) = 0$. Let us take $t \geqslant \eta$. Then

$$\begin{aligned} &|(Fx)(t) - (Fy)(t)|[a(t) \exp(ML(t) + t)]^{-1} \\ &\leqslant [|(Fx)(t)| + |(Fy)(t)|][a(t) \exp(ML(t)]^{-1} e^{-t} \leqslant 2e^{-t}. \end{aligned}$$

Hence for sufficiently large η we have

$$|(Fx)(t) - (Fy)(t)| \, p(t) \leqslant h \qquad \text{for} \quad t \geqslant \eta\delta.$$

Thus inview of (2) and (3) we deduce that F is continuous on the set G.

Now we show that FG is relatively compact. In the set G, note that

$$|(Fx)(t)| \, p(t) \leqslant e^{-t}$$

which implies that

(4)
$$\lim_{\eta \to \infty} \sup[|(Fx)(t)| \, p(t) : t \geqslant \eta] = 0$$

uniformly with respect to $x \in G$.

Furthermore, let us fix $h > 0$, $\eta > 0$, $t, \ s \in [0, \ \eta]$ such that $|t - s| \leqslant h$. Then for $x \in G$, we get

$$|(Fx)(t)-(Fx)(s)| \leqslant |g(t,\ x(t),\ Tx(t))-g(s,\ x(s),\ Tx(s))|$$

$$+\left|\int_0^t K(t,\ u,\ x(\alpha(u)))\,\mathrm{d}u-\int_0^s K(s,\ u,\ x(\alpha(u)))\,\mathrm{d}u\right|$$

$$\leqslant \omega(g,\ h)+\left|\int_0^t K(t,\ u,\ x(\alpha(u)))\,\mathrm{d}u-\int_0^s K(t,\ u,\ x(\alpha(u)))\,\mathrm{d}u\right|$$

$$+\left|\int_0^s K(t,\ u,\ x(\alpha(u)))\,\mathrm{d}u-\int_0^s K(s,\ u,\ x(\alpha(u)))\,\mathrm{d}u\right|$$

$$\leqslant \omega(g,\ h)+\int_s^t |K(t,\ u,\ x(\alpha(u)))|\,\mathrm{d}u+\int_0^s |K(t,\ u,\ x(\alpha(u)))-K(s,\ u,\ x(\alpha(u)))|\,\mathrm{d}u$$

$$\leqslant \omega(g,\ h)+h\max\left[m(t,\ u)+a(u)\,b(u)\,p(\alpha(u)):0\leqslant u\leqslant t\leqslant \eta\right]+\eta\omega(K,\ h)\to 0$$

as $h\to 0$ since $\lim_{h\to 0}\omega(g,\ h)=\lim_{h\to 0}\omega(K,\ h)=0$.

Hence we deduce that all the functions belonging the set FG are equicontinuous on each interval $[0,\ \eta]$ and by (4), using the Lemma we infer that FG is relatively compact. Thus the Schauder fixed point theorem guarantees that F has a fixed point $x \in G$ such that $(Fx)(t)=x(t)$.

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Abstract

We prove an existence theorem for a special class of nonlinear Volterra integral equations with deviating argument without assuming the Lipschitz condition.

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