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On common contractivity of non-linear type (**)

Alla memoria di ANTONIO MAMBRIANI

1 - Introduction

In the past few years many generalizations have appeared in literature concerning the famous Contraction Theorem of Banach, relating to metric spaces, especially for a single function. Among these a few (in our opinion among the most interesting) generalize the classic hypothesis $d(f(x_1), f(x_2)) \leqslant \alpha d(x_1, x_2)$ $(0 \leqslant \alpha < 1)$ into a contractive type hypothesis in which:

(1) though maintaining as its second member the linear function structure of type $t \mapsto \alpha t : \mathfrak{R}^+ \to \mathfrak{R}^+$ $(0 \leqslant \alpha < 1,\ \mathfrak{R}^+ \equiv 0 \vdash\!\dashv + \infty)$, the number of distances within this increases until $d(x_1, x_2)$ is involved either with all four possible distances of x_i from $f(x_h)$ $(i,\ h = 1,\ 2)$ (see for example [3] and [7]), or with all the infinite distances of the type $d(f^{h-1}(x_1), f^{i-1}(x_2))$ $(h,\ i = 1, 2, 3, \ldots)$ (¹) (see for example [5]₂);

(2) its second member is changed into a non-linear function of the type $t \mapsto \sigma(t) : \mathfrak{R}^+ \to \mathfrak{R}^+$, where σ is non decreasing, continuous on the right and such that $\sigma(t) < t$ if $t > 0$, involving the above mentioned distances in the argument of σ (see for example [4], [6], [9]).

We have been generalizing various results for several years now, both considering as an ambient space a H-space a special case of which is the ordinary metric space, and examining two functions rather than one (see for example [2], [8], [1]).

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(¹) By f^n here we mean $f \circ f \circ \ldots \circ f$ n times with the convention $f^0(y) = y$.

The definition of this space is as follows

E is a set and $d: E \times E \to \mathfrak{R}^+$ satisfies the following properties:

(a) $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$ for every $x_1, x_2 \in E$;

(b) $d(x_1, x_2) = d(x_2, x_1)$ for every $x_1, x_2 \in E$;

(c) there exist a subset A of $\mathfrak{R}^+$ containing an interval $0 \,\lceil\!\rceil\, a$ $(a > 0)$, a real constant $\tau \geq 1$ and a function $\varphi: A \to \mathfrak{R}^+$ which is infinitesimal at zero, such that for every $x_1, x_2, x_3 \in E$, $d(x_1, x_2) \in A \Rightarrow d(x_1, x_3) \leq \varphi[d(x_1, x_2)] + \tau d(x_2, x_3)$ (*generalized triangular property*, g.t.p.).

On such a H-space, which is a Hausdorff space, it is possible to introduce topological and completeness notions thereby treating them in the same manner as ordinary metric spaces. The mapping d however is uniformly continuous if and only if $\tau = 1$; and, in general, it is not even continuous. It is obvious that metric spaces are particular H-spaces with $A \equiv \mathfrak{R}^+$, $\tau = 1$, and φ the identity function.

In these spaces we have considered the hypothesis of common generalized contractivity (above all for two functions) where the left hand side of the inequality is given the lowest number of the six distances $d(f_r(x_s), f_h(x_k))$ $(r = 1, 2;\ s = 3 - r, 2;\ h = 1, r \wedge s;\ k = 1, r \wedge (3 - h))$ compatible with the highest number of the nine distances $d(x_1, x_2)$, $d(x_i, f_j(x_h))$ $(i, j, h = 1, 2)$ in the right hand side. Our study has at times been impeded and at times helped by the placing of certain coefficients $\frac{1}{\tau}$; indeed it will be shown in **5.8**, that the number and placing of these very coefficients may or may not contribute to the result hoped for.

Here in particular we consider the hypothesis of common generalized contractivity of a non linear type with at the first member always and only the two distances $d(f_1(x_1), f_1(x_2))$, $d(f_1(x_1), f_2(x_2))$ (2) and at second member (in the argument of σ) the infinite distances

$$d(x_1, x_2),\ d(f_1^{h-1}(x_1), f_j(f_1^{i-1}(x_2))) \qquad j = 1, 2;\ h = 1, 2, 3, \ldots$$

finding fixed point theorems which have, among others, special cases in H-spaces some of the most significant of which theorems have been already described in previous works; and in the case of metric spaces for one function only, the theorems quoted above.

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(2) On the appropriateness of this choice see, for instance, [8].

2 - Preliminary considerations and lemmas

Let $f_1, f_2 : E \to E$ be two applications, $\sigma : \mathfrak{R}^+ \to \mathfrak{R}^+$ a non-decreasing function for which we have $\sigma(t) < t$ if $t > 0$ [3], and two sequences of points for $E : u_0(x) = x$, $u_n(x) = f_1(f_1^{n-1}(x))$; $f_2(f_1^{n-1}(x))$ $(n = 1, 2, 3, \ldots)$; i.e.

(1) $u_0(x) = x \qquad u_n(x) = f_1(u_{n-1}(x)) \qquad f_2(u_{n-1}(x)) \qquad (n = 1, 2, 3, \ldots)$.

For two given points (not necessarily distinct) $x_1, x_2 \in E$, let us consider the following l.u.b.'s (*assuming them to be finite*)

(2) $\delta_1(x_1, x_2) = \sup\{d(x_1, f_j(f_1^{i-1}(x_2))): j = 1, 2;\ i = 1, 2, 3, \ldots\}$

(3) $\gamma_1(x_1, x_2) = \sup\{d(f_1^h(x_1), f_j(f_1^{i-1}(x_2))): j = 1, 2;\ h, i = 1, 2, 3, \ldots\}$

and let

(4) $$\rho_1(x_1, x_2) = \delta_1(x_1, x_2) \vee \gamma_1(x_1, x_2)$$
$$= \sup\{d(f_1^{h-1}(x_1), f_j(f_1^{i-1}(x_2))): j = 1, 2;\ h, i = 1, 2, 3, \ldots\};$$

let $\delta(x_1, x_2)$, $\gamma(x_1, x_2)$, $\rho(x_1, x_2)$ be given respectively by

$(5)_1$ $\delta(x_1, x_2) = d(x_1, x_2) \vee \delta_1(x_1, x_2) \vee \delta_1(x_2, x_2) \vee \delta_1(x_2, x_1) \vee \delta_1(x_1, x_1)$

$(5)_2$ $\gamma(x_1, x_2) = \gamma_1(x_1, x_2) \vee \gamma_1(x_2, x_2) \vee \gamma_1(x_2, x_1) \vee \gamma_1(x_1, x_1)$

$(5)_3$ $\rho(x_1, x_2) = \delta(x_1, x_2) \vee \gamma(x_1, x_2)$

or by

$(5)'_1$ $\delta(x_1, x_2) = d(x_1, x_2) \vee \delta_1(x_1, x_2) \vee \dfrac{1}{\tau}\delta_1(x_2, x_2) \vee \delta_1(x_2, x_1) \vee \delta_1(x_1, x_1)$

$(5)'_2$ $\gamma(x_1, x_2) = \gamma_1(x_1, x_2) \vee \dfrac{1}{\tau}\gamma_1(x_2, x_2) \vee \gamma_1(x_2, x_1) \vee \gamma_1(x_1, x_1)$

$(5)'_3$ $\rho(x_1, x_2) = \delta(x_1, x_2) \vee \gamma(x_1, x_2)$

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[3] It is obvious that $\sigma(0) = 0$.

or by

$$(5)''_1 \quad \delta(x_1,\ x_2)=d(x_1,\ x_2)\vee\delta_1(x_1,\ x_2)\vee\frac{1}{\tau}\delta_1(x_2,\ x_2)\vee\delta_1(x_2,\ x_1)\vee\frac{1}{\tau}\delta_1(x_1,\ x_1)$$

$$(5)''_2 \quad \gamma(x_1,\ x_2)=\gamma_1(x_1,\ x_2)\vee\frac{1}{\tau}\gamma_1(x_2,\ x_2)\vee\gamma_1(x_2,\ x_1)\vee\frac{1}{\tau}\gamma_1(x_1,\ x_1)$$

$$(5)''_3 \quad \rho(x_1,\ x_2)=\delta(x_1,\ x_2)\vee\gamma(x_1,\ x_2)$$

or by

$$(5)'''_1 \quad \delta(x_1,\ x_2)=d(x_1,\ x_2)\vee\frac{1}{\tau}\delta_1(x_1,\ x_2)\vee\frac{1}{\tau}\delta_1(x_2,\ x_2)\vee\frac{1}{\tau}\delta_1(x_2,\ x_1)\vee\delta_1(x_1,\ x_1)$$

$$(5)'''_2 \quad \gamma(x_1,\ x_2)=\frac{1}{\tau}\gamma_1(x_1,\ x_2)\vee\frac{1}{\tau}\gamma_1(x_2,\ x_2)\vee\frac{1}{\tau}\gamma_1(x_2,\ x_1)\vee\gamma_1(x_1,\ x_1)$$

$$(5)'''_3 \quad \rho(x_1,\ x_2)=\delta(x_1,\ x_2)\vee\gamma(x_1,\ x_2).$$

Let us now consider the following two hypotheses for common generalized contractivity

$$(6) \quad \begin{aligned} &d(f_1(x_1),\ f_r(x_2))\leq\sigma(\rho(x_1,\ x_2)) \qquad r=1,\ 2 \qquad x_1,\ x_2\in E \\ &d(f_1(x_1),\ f_r(x_2))\leq\sigma(\rho(x_2,\ x_1)) \qquad r=1,\ 2 \qquad x_1,\ x_2\in E \end{aligned}$$

$$(7) \quad \begin{aligned} &d(f_1(x_1),\ f_r(x_2))\leq\sigma(\delta(x_1,\ x_2)) \qquad r=1,\ 2 \qquad x_1,\ x_2\in E \\ &d(f_1(x_1),\ f_r(x_2))\leq\sigma(\delta(x_2,\ x_1)) \qquad r=1,\ 2 \qquad x_1,\ x_2\in E \end{aligned}$$

and observe that

(1) δ, γ, ρ of (5) and (5)″ are symmetric for x_1 and x_2. Therefore with these ρ and δ, (6) and (7) are reduced, both, to only one condition; δ, γ, ρ of (5)′ and (5)‴ are not symmetric in x_1 and x_2.

(2) δ, γ, ρ given by (5) are greater than or equal to those given by (5)′, which in turn are greater than or equal to those given by (5)″ and (5)‴. Therefore (6) and (7) with ρ and δ given by (5)″ or (5)‴ obviously yield (6) and (7) with ρ and δ given by (5)′, which in turn yield (6) and (7) with ρ and δ given by (5).

The following lemma hold.

Lemma 1. *If* (6) [(7)] *holds and* ρ *and* δ *are of the form* (5), *or* (5)′, *or* (5)″, *then with these respective forms we have* $\rho(x_1, x_2) = \delta(x_1, x_2)$ *and* (6) *coincides with* (7).

Lemma 2. *If* ρ *has the form* (5)‴ *and the first in* (6) *holds, then*

$$(8)\qquad \begin{gathered} \gamma_1(x_1, x_2) \leqslant \sigma(\tau\rho(x_1, x_2)) \\ \gamma_1(x_1, x_1) \leqslant \sigma(\rho(x_1, x_2)) \qquad \gamma_1(x_2, x_2) \leqslant \sigma(\tau\rho(x_1, x_2)). \end{gathered}$$

Proof of Lemma 1. Suppose (6) is true. Taking ρ and δ in the three forms (5), or (5)′, or (5)″, the following is true $\delta(x_1, x_2) \leqslant \rho(x_1, x_2)$. To prove the reciprocal inequality note that

$$(9)\qquad \rho(u_{n+k-2}(x_1), u_{m+s-2}(x_2)) \leqslant \rho(u_{n-2}(x_1), u_{m-2}(x_2))$$

$$k = 0, 1, 2, \ldots \quad s = 1, 2, 3, \ldots \quad m, n = 2, 3, 4, \ldots \quad x_1, x_2 \in E$$

$u_0(x)$, $u_n(x)$ being the first of (1) (it is sufficient to compare the expressions of the two sides). Then let us identify three cases corresponding to the three forms called ρ and δ, considering first the two cases of symmetry.

I - The case of ρ and δ (symmetric) given by (5). By (6)(4), and (9), and the non decreasing of σ, we have that

$$(10)\qquad \begin{gathered} \rho(u_{h-1}(x_1), u_{i-1}(x_2)) \\ \leqslant \sigma(\rho(u_{h-2}(x_1), u_{i-2}(x_2))) \vee \sigma(\rho(u_{i-2}(x_2), u_{i-2}(x_2))) \vee \sigma(\rho(u_{i-2}(x_2), u_{h-2}(x_1))) \\ \vee\, \sigma(\rho(u_{h-2}(x_1), u_{h-2}(x_1))) \qquad h, i = 2, 3, 4, \ldots \end{gathered}$$

which, by the symmetry of ρ, the fact of being $\rho(u_{i-2}(x_2), u_{i-2}(x_2)) \leqslant \rho(u_{h-2}(x_1), u_{i-2}(x_2))$ and $\rho(u_{h-2}(x_1), u_{h-2}(x_1)) \leqslant \rho(u_{h-2}(x_1), u_{i-2}(x_2))$ and

(4) Which here, as we have already said, is reduced to a single hypothesis.

also because σ is non decreasing, becomes

(11) $$\rho(u_{h-1}(x_1),\ u_{i-1}(x_2)) \leqslant \sigma(\rho(u_{h-2}(x_1),\ u_{i-2}(x_2))) \qquad h,\ i=2,\ 3,\ 4,\ \ldots$$

from which also

(12) $$\rho(u_{h-1}(x_1),\ u_{i-1}(x_2)) \leqslant \rho(u_{h-2}(x_1),\ u_{i-2}(x_2)) \qquad h,\ i=2,\ 3,\ 4,\ \ldots .$$

From which in turn follows that

$$\rho(u_{h-1}(x_1),\ u_{i-1}(x_2)) \leqslant \rho(u_0(x_1),\ u_{i-h}(x_2)) \qquad \text{if } 2 \leqslant h \leqslant i$$

$$\rho(u_{h-1}(x_1),\ u_{i-1}(x_2)) \leqslant \rho(u_{h-i}(x_1),\ u_0(x_2)) \qquad \text{if } 2 \leqslant i < h$$

and clearly

(13) $$\rho(u_{h-1}(x_1),\ u_{i-1}(x_2)) \leqslant \rho(u_{0\vee(h-i)}(x_1),\ u_{0\vee(i-h)}(x_2)) \qquad \text{with } h,\ i \geqslant 1 .$$

But since $\rho(u_{0\vee(h-i)}(x_1),\ u_{0\vee(i-h)}(x_2)) \leqslant \rho(x_1,\ x_2)$ we obtain

$$\rho(u_{h-1}(x_1),\ u_{i-1}(x_2)) \leqslant \rho(x_1,\ x_2) \qquad h,\ i \geqslant 1 .$$

Therefore by the first of (1) and (6)

$$\max\{d(u_h(x_1),\ f_r(u_{i-1}(x_2))):\ r=1,\ 2\}$$
$$\leqslant \sigma(\rho(u_{h-1}(x_1),\ u_{i-1}(x_2))) \leqslant \sigma(\rho(x_1,\ x_2)) \qquad h,\ i \geqslant 1$$

and from (3)

(14) $$\gamma_1(x_1,\ x_2) \leqslant \sigma(\rho(x_1,\ x_2)) .$$

If we now consider $\rho(x_1,\ x_2)$ with its expression (5), i.e.

$$\rho(x_1,\ x_2) = \delta(x_1,\ x_2) \vee \gamma_1(x_1,\ x_2) \vee \gamma_1(x_2,\ x_2) \vee \gamma_1(x_2,\ x_1) \vee \gamma_1(x_1,\ x_1)$$

for (14) (taking into account that $\rho(x,\ x) = \rho_1(x,\ x)$ and $\rho(x,\ y) = \rho(y,\ x)$) we have

(15) $$\rho(x_1,\ x_2) \leqslant \delta(x_1,\ x_2) \vee \sigma(\rho(x_1,\ x_2)) \vee \sigma(\rho_1(x_2,\ x_2)) \vee \sigma(\rho_1(x_1,\ x_1))$$

from which, since $\rho_1(x, x) \leqslant \rho(x, y)$, because σ is non decreasing,

$$\rho(x_1, x_2) \leqslant \delta(x_1, x_2) \vee \sigma(\rho(x_1, x_2)); \tag{16}$$

since this cannot be $\rho(x_1, x_2) \leqslant \sigma(\rho(x_1, x_2))$ (5), (16) gives us $\rho(x_1, x_2) \leqslant \delta(x_1, x_2)$.

II: Case of ρ and δ (symmetric) given by (5)″. Using the same reasoning as in the previous case (6), first (12) is obtained, but with

$$\begin{aligned}(10)'' \qquad & \rho(u_{h-1}(x_1), u_{i-1}(x_2)) \\ & \leqslant \sigma(\rho(u_{h-2}(x_1), u_{i-2}(x_2)) \vee \frac{1}{\tau}\sigma(\rho(u_{i-2}(x_2), u_{i-2}(x_2))) \vee \sigma(\rho(u_{i-2}(x_2), u_{h-2}(x_1))) \\ & \vee \frac{1}{\tau}\sigma(\rho(u_{h-2}(x_1), u_{h-2}(x_1))) \qquad h, i = 2, 3, 4, \dots\end{aligned}$$

and

$$\begin{aligned}(11)'' \qquad & \rho(u_{h-1}(x_1), u_{i-1}(x_2)) \\ & \leqslant \sigma(\rho(u_{h-2}(x_1), u_{i-2}(x_2))) \vee \frac{1}{\tau}\sigma(\tau\rho(u_{h-2}(x_1), u_{i-2}(x_2))) \qquad h, i = 2, 3, 4, \dots\end{aligned}$$

instead of (10) and (11) (being here simply $\rho(u_{i-2}(x_2), u_{i-2}(x_2)) \leqslant \tau\rho(u_{h-2}(x_1), u_{i-2}(x_2))$ and $\rho(u_{h-2}(x_1), u_{h-2}(x_1)) \leqslant \tau\rho(u_{h-2}(x_1), u_{i-2}(x_2))$). Then (14) is obtained and, bearing in mind expression (5)″ of $\rho(x_1, x_2)$, we arrive at

$$(15)'' \qquad \rho(x_1, x_2) \leqslant \delta(x_1, x_2) \vee \sigma(\rho(x_1, x_2)) \vee \frac{1}{\tau}\sigma(\rho_1(x_2, x_2)) \vee \frac{1}{\tau}\sigma(\rho_1(x_1, x_1))$$

from which, since $\rho_1(x, x) \leqslant \tau\rho(x, y)$, and σ is non decreasing,

$$(16)'' \qquad \rho(x_1, x_2) \leqslant \delta(x_1, x_2) \vee \sigma(\rho(x_1, x_2)) \vee \frac{1}{\tau}\sigma(\tau\rho(x_1, x_2))$$

and then again $\rho(x_1, x_2) \leqslant \delta(x_1, x_2)$.

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(5) Unless it is $\rho(x_1, x_2) = 0$.

(6) Here too (6) is reduced to a single hypothesis.

III. Case of ρ and δ (non-symmetric) given by (5)′. Using a reasoning that is practically identical to the above (7), we obtain first

$$(10)' \qquad \rho(u_{h-1}(x_1),\ u_{i-1}(x_2))$$

$$\leqslant \sigma(\rho(u_{h-2}(x_1),\ u_{i-2}(x_2))) \vee \frac{1}{\tau}\sigma(\rho(u_{i-2}(x_2),\ u_{i-2}(x_2))) \vee \sigma(\rho(u_{h-2}(x_1),\ u_{h-2}(x_1)))$$
$$h,\ i=2,\ 3,\ 4,\ \dots;$$

then (11)″, (12) and (13) (in this case of course with ρ given by (5)′). But since $\rho(u_{0\vee(h-i)}(x_1),\ u_{0\vee(i-h)}(x_2)) \leqslant \rho(x_1,\ x_2)$ and $\rho(u_{0\vee(i-h)}(x_2),\ u_{0\vee(h-i)}(x_1)) \leqslant \rho(x_2,\ x_1)$ (h, $i=1,\ 2,\ 3,\ \dots$), we obtain

$$\rho(u_{h-1}(x_1),\ u_{i-1}(x_2)) \leqslant \rho(x_1,\ x_2) \qquad \text{and}$$

$$\rho(u_{i-1}(x_2),\ u_{h-1}(x_1)) \leqslant \rho(x_2,\ x_1) \qquad h,\ i=1,\ 2,\ 3,\ \dots;$$

therefore, from the first of (1), from (6) (8) and σ non decreasing,

$$\max\{d(u_h(x_1),\ f_r(u_{i-1}(x_2))):\ r=1,\ 2\}$$
$$\leqslant \sigma(\rho(u_{h-1}(x_1),\ u_{i-1}(x_2)) \wedge \sigma(\rho(u_{i-1}(x_2),\ u_{h-1}(x_1)))$$
$$\leqslant \sigma(\rho(x_1,\ x_2)) \wedge \sigma(\rho(x_2,\ x_1)) \qquad h,\ i \geqslant 1;$$

whence, from (3), we obtain immediately

$$(14)' \qquad \gamma_1(x_1,\ x_2) \leqslant \sigma(\rho(x_1,\ x_2)) \wedge \sigma(\rho(x_2,\ x_1)).$$

Bearing in mind the expression of $\rho(x_1,\ x_2)$ given by (5)′ we obtain

$$(15)' \qquad \rho(x_1,\ x_2) \leqslant \delta(x_1,\ x_2) \vee \sigma(\rho(x_1,\ x_2)) \vee \frac{1}{\tau}\sigma(\rho_1(x_2,\ x_2)) \vee \sigma(\rho_1(x_1,\ x_1))$$

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(7) Bearing in mind that here ρ is not symmetric and therefore (6) is *not* reduced to a single hypothesis.

(8) See note (7).

from which, since $\rho_1(x, x) \leqslant \rho(x, y) \wedge \tau\rho(y, x)$, and σ is non decreasing, we have (16)″ again, and therefore also $\rho(x_1, x_2) \leqslant \delta(x_1, x_2)$.

Thus the equality $\rho(x_1, x_2) = \delta(x_1, x_2)$ with ρ and δ in the forms (5) or (5)′ or (5)″ is obtained; therefore, since (6) holds, (7) becomes identical to it.

Suppose now (7) holds. Since σ is non decreasing and $\delta(x_1, x_2) \leqslant \rho(x_1, x_2)$, (6) is also true and therefore $\rho(x_1, x_2) = \delta(x_1, x_2)$ holds; therefore since (7) is true (6) becomes identical to it.

Lemma 1 is thus proved (9).

Proof of Lemma 2. Taking ρ in its (5)‴ form and considering again the first of sequences (1) of points of E, if we suppose for sake of brevity

$$\bar{\delta}(x_1, x_2) = \frac{1}{\tau}\delta_1(x_1, x_2) \vee \frac{1}{\tau}\delta_1(x_2, x_2) \vee \frac{1}{\tau}\delta_1(x_2, x_1) \vee \delta_1(x_1, x_1)$$

we obtain immediately

$$\begin{aligned} &\rho(u_{h-1}(x_1), u_{i-1}(x_2)) \\ &= d(u_{h-1}(x_1), u_{i-1}(x_2)) \vee \bar{\delta}(u_{h-1}(x_1), u_{i-1}(x_2)) \vee \gamma(u_{h-1}(x_1), u_{i-1}(x_2)) \\ &\qquad h, i = 2, 3, 4, \ldots, \quad x_1, x_2 \in E; \end{aligned}$$

but from the first of (6) and from a simple comparison we obtain, respectively, $d(u_{h-1}(x_1), u_{i-1}(x_2)) \leqslant \sigma(\rho(u_{h-2}(x_1), u_{i-2}(x_2))$ and $\bar{\delta}(u_{h-1}(x_1), u_{i-1}(x_2)) \vee \gamma(u_{h-1}(x_1), u_{i-1}(x_2)) \leqslant \delta(u_{h-2}(x_1), u_{i-2}(x_2)) \vee \gamma(u_{h-2}(x_1), u_{i-2}(x_2)) = \rho(u_{h-2}(x_1), u_{i-2}(x_2))$, whence

$$\begin{aligned} (10)''' \quad &\rho(u_{h-1}(x_1), u_{i-1}(x_2)) \\ &\leqslant \sigma(\rho(u_{h-2}(x_1), u_{i-2}(x_2))) \vee \rho(u_{h-2}(x_1), u_{i-2}(x_2)) \qquad h, i = 2, 3, 4, \ldots . \end{aligned}$$

Therefore (12) as in Lemma 1 [with ρ given by (5)‴ of course]. If we procede as in the above lemma we obtain again (13) and, from this, other three inequalities (with $h, i \geqslant 1$)

$$\rho(u_{h-1}(x_1), u_{i-1}(x_2)) \leqslant \tau\rho(x_1, x_2)$$

(9) From this Lemma for $f_1 = f_2$ and within the metric spaces we can obtain (6) of $[5]_1$.

$$\rho(u_{h-1}(x_1),\ u_{i-1}(x_1)) \leqslant \rho(x_1,\ x_1) \leqslant \rho(x_1,\ x_2)$$

$$\rho(u_{h-1}(x_2),\ u_{i-1}(x_2)) \leqslant \rho(x_2,\ x_2) \leqslant \tau\rho(x_1,\ x_2)$$

which, from the first of (1) and the first of (6) and thereafter from (3), lead to (8).

3 - A converging lemma

Suppose now the space E is complete and the function σ is also continuous from the right: the following holds

Lemma 3. *Suppose* (6) *holds with* ρ *in any of its four forms; in this case however* x *is taken in* E *the sequences* (1) *converge to a single point of* E.

Proof. Limiting ourselves to the case of ρ given by (5) (see 1st and 2nd part of 2) (10) and repeating in part the considerations made at the beginning of proof of Lemma 1, we obtain (11). From (11) written for $h-1=i-1=n$ we find immediately that if the sequence $\rho(u_n(x_1),\ u_n(x_2))$ $(n=0, 1, 2, \dots)$ has all its terms different from zero, then it is decreasing; therefore from the continuity from the right of σ we have

$$\lim_{n\to+\infty} \rho(u_n(x_1),\ u_n(x_2)) = 0 \tag{17}$$

which, given $y = x_1 = x_2$, produces, in particular (11)

$$\lim_{n,\ m\to+\infty} d(u_n(y),\ u_m(y)) = 0\,.$$

Therefore, from the hypothesis of completeness, we have $l_0 \in E$ such that $\lim_{n\to+\infty} u_n(y) = l_0$.

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(10) In the case of non symmetric ρ produced by (5)′ or (5)‴ it is sufficient to consider either the first or the second of (6) to be true.

(11) Remember that $\rho(u_n(x_1),\ u_n(x_1)) = \sup\{d(f_1^{h-1}(u_n(x_1)),\ f_j(f_1^{i-1}(u_n(x_1)))\colon j=1,\ 2;\ h,\ i \geqslant 1\}$.

Fix x in E with $x \neq y$. For the g.t.p. of (c) ([12]), the first of (1) and (6) we obtain

$$d(l_0, f_r(u_n(x))) \leqslant \varphi[d(l_0, u_{n+1}(y))] + \tau d(f_1(u_n(y)), f_r(u_n(x)))$$

$$\leqslant \varphi[d(l_0, u_{n+1}(y))] + \tau\rho(u_n(y), u_n(x)) \qquad r = 1, 2;$$

thence the sequences $f_r(u_n(x))$ ($r = 1, 2$; $n = 0, 1, 2, \ldots$) converge to the same $l_0 \in E$ whatever x happens to be: which is what we wanted to prove ([13]).

Remark. If (7) were true with δ in any of its four forms, then

$$\lim_{n \to +\infty} \delta(u_n(x_1), u_n(x_2)) = 0 \tag{18}$$

would also be true, and clearly the sequences (1) converge to the same point of E.

4 - Theorems of common fixed point

Since E is still complete, the l.u.b.'s (2) and (3) are finite and σ is also continuous from the right, we obtain the following theorems.

Theorem 1. *Suppose* (6) *holds with* ρ *in the form* (5)'. *Then the mappings* f_1 *and* f_2 *have a single common fixed point, which is also the only fixed point of* f_1.

Proof. For Lemma 1 we can refer indifferently to (6) or (7) and consider ρ or δ. Since l_0 is the common limit of the sequences (1) (see Lemma 3), and (see (5)')

$$\delta(u_n(l_0), l_0) = d(u_n(l_0), l_0) \vee \delta_1(u_n(l_0), l_0) \vee \frac{1}{\tau}\delta_1(l_0, l_0) \tag{19}$$

$$\vee\, \delta_1(l_0, u_n(l_0)) \vee \delta_1(u_n(l_0), u_n(l_0)) \qquad n = 0, 1, 2, \ldots$$

([12]) Bear in mind that from a certain n onwards this is $d(l_0, u_{n+1}(y)) \leqslant a$, i.e. $d(l_0, u_{n+1}(y)) \in A$, a and A given in (c) of **1**.

([13]) Note that if for a certain index $i \in \mathfrak{N} \cup \{0\}$, $\rho(u_i(x_1), u_i(x_2)) = 0$ was true, then $\rho(u_n(x_1), u_n(x_2))$ from the i-th onwards would be zero: therefore (17) would be obvious. Furthermore if it was already $\rho(u_0(x_1), u_0(x_2)) = 0$ we would also have the two sequences (1) constant (with all terms equal to $x_1 = x_2$): thus the lemma would be trivial.

we show that

(20) $$\lim_{n\to+\infty} \delta(u_n(l_0),\ l_0) = 0 .$$

To this end we observe that:

(1) we have, for (2), (3) and (14)′ [14], $\delta_1(u_n(l_0),\ l_0) \leqslant \gamma_1(u_{n-1}(l_0),\ l_0) \leqslant \sigma(\rho(u_{n-1}(l_0),\ l_0)) \wedge \sigma(\rho(l_0,\ u_{n-1}(l_0))) \leqslant \sigma(\rho(u_{n-1}(l_0),\ l_0)) = \sigma(\delta(u_{n-1}(l_0),\ l_0))$;

(2) from a certain index n onwards we have for the g.t.p. of (c) and for (1) above $\frac{1}{\tau}\delta_1(l_0,\ l_0) \leqslant \frac{1}{\tau} \sup\{\varphi[d(l_0,\ u_n(l_0)] + \tau d(u_n(l_0),\ f_j(u_{i-1}(l_0)) \colon j = 1,\ 2;\ i = 1,\ 2,\ 3,\ \ldots\} = \frac{1}{\tau}\varphi[d(l_0,\ u_n(l_0))] + \delta_1(u_n(l_0),\ l_0) \leqslant \frac{1}{\tau}\varphi[d(l_0,\ u_n(l_0)] + \sigma(\delta(u_{n-1}(l_0),\ l_0))$;

(3) clearly we have $\delta_1(u_n(l_0),\ u_n(l_0)) \leqslant \delta(u_n(l_0),\ u_n(l_0))$;

(4) we have for (2), from a certain index n onwards, for the g.t.p. of (c) and for (3) above $\delta_1(l_0,\ u_n(l_0)) \leqslant \varphi[d(l_0,\ u_n(l_0))] + \tau \sup\{d(u_n(l_0), f_j(u_{n+i-1}(l_0))) \colon j = 1, 2;\ i = 1,\ 2,\ 3,\ \ldots\} = \varphi[d(l_0,\ u_n(l_0))] + \tau\delta_1(u_n(l_0),\ u_n(l_0)) \leqslant \varphi[d(l_0,\ u_n(l_0))] + \tau\delta(u_n(l_0),\ u_n(l_0))$.

Therefore from (19) we have

$$\begin{aligned} &\lim_{n\to+\infty}{}'' \delta(u_n(l_0),\ l_0) \\ &\leqslant \lim_{n\to+\infty}{}'' \{d(u_n(l_0),\ l_0) \vee [\frac{1}{\tau}\varphi[d(l_0,\ u_n(l_0))] + \sigma(\delta(u_{n-1}(l_0),\ l_0))] \\ &\qquad \vee [\varphi[d(l_0,\ u_n(l_0))] + \tau\delta(u_n(l_0),\ u_n(l_0))]\} \end{aligned}$$

from which (see also (18)) since σ is continuous from the right, $\lim''_{n\to+\infty} \delta(u_n(l_0),\ l_0) \leqslant \sigma(\lim''_{n\to+\infty} \delta(u_n(l_0),\ l_0))$. This implies $\lim''_{n\to+\infty} \delta(u_n(l_0),\ l_0) = 0$, i.e. (20).

But from the expression (2) and (19) of δ_1 and δ we find immediately that

$$d(l_0,\ f_r(l_0)) \leqslant \delta_1(l_0,\ l_0) \leqslant \tau\delta(u_n(l_0),\ l_0) \qquad r = 1,\ 2$$

therefore for (20) l_0 is a fixed point both of f_1 and f_2.

If f_1 and f_2 had two distinct common fixed points l_0 and z both

$$d(l_0,\ z) = d(f_1(l_0),\ f_r(z)) \leqslant \sigma(\delta(l_0,\ z)) \qquad r = 1,\ 2$$

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[14] Place $x_1 = u_{n-1}(l_0)$, $x_2 = l_0$ in (14)′.

and (see (5)′ and (2) knowing that also $l_0 = f_1^{i-1}(l_0)$, $z = f_1^{i-1}(z)$, $i \in \mathfrak{N}$ is true)

$$\delta(l_0, z) = d(l_0, z)$$

would be true; from which $d(l_0, z) \leq \sigma(d(l_0, z))$, which is not possible. Therefore the common fixed point is unique. This point is also the only fixed point of f_1; in fact if y were another fixed point of f_1 we would have (from (7))

$$d(y, f_2(y)) = d(f_1(y), f_2(y)) \leq \sigma(\delta(y, y)).$$

But $\delta(y, y) = \sup\{d(y, f_j(f_1^{i-1}(y))): j = 1, 2; i = 1, 2, 3, \ldots\} = \max\{d(y, f_r(y)): r = 1, 2\} = d(y, f_2(y))$ and therefore we would have $d(y, f_2(y)) \leq \sigma(d(y, f_2(y)))$, i.e. $y = f_2(y)$ which is not possible.

The theorem is thus proved.

Theorem 2. *Suppose* (6) *holds with* ρ *in the form* (5)″. *Then the mappings* f_1 *and* f_2 *have a single common fixed point, which is also the only fixed point of* f_1.

Theorem 3. *Suppose* (6) *holds with* ρ *in the form* (5)‴. *Then the mappings* f_1 *and* f_2 *have a single common fixed point, which is also the only fixed point of* f_1.

Both theorems follow from Theorem 1, bearing in mind (2) of 2.

Theorem 4. *Suppose the first of* (6) *holds with* ρ *in the form* (5)‴. *Then the mappings* f_1 *and* f_2 *have a single common fixed point, which is also the only fixed point of* f_1.

Proof. Since l_0 is still the common limit of sequences (1) (see Lemma 3), and (see (5)‴) ([15])

$$\rho(u_n(l_0), l_0) = d(u_n(l_0), l_0) \vee \frac{1}{\tau}\delta_1(u_n(l_0), l_0) \vee \frac{1}{\tau}\gamma_1(u_n(l_0), l_0) \vee \frac{1}{\tau}\delta_1(l_0, l_0) \tag{21}$$

$$\vee \frac{1}{\tau}\gamma_1(l_0, l_0) \vee \frac{1}{\tau}\delta_1(l_0, u_n(l_0)) \vee \frac{1}{\tau}\gamma_1(l_0, u_n(l_0)) \vee \delta_1(u_n(l_0), u_n(l_0)) \vee \gamma_1(u_n(l_0), u_n(l_0))$$

$$n = 0, 1, 2, \ldots$$

([15]) Since Lemma 1 is not true with ρ and δ in the form (5)‴ we cannot procede at first as we did in Theorem 1 where ρ or δ were used indifferently and either (6) or (7) taken.

we show that

(22) $$\lim_{n\to+\infty} \rho(u_n(l_0),\ l_0)=0\,.$$

To this end we observe that

(1) by (2) and (3) and by the first of (8) of Lemma 2 ([16]), we obtain both $\frac{1}{\tau}\delta_1(u_n(l_0),\ l_0)\leqslant\frac{1}{\tau}\gamma_1(u_{n-1}(l_0),\ l_0)\leqslant\frac{1}{\tau}\sigma(\tau\rho(u_{n-1}(l_0),\ l_0))$ and $\frac{1}{\tau}\gamma_1(u_n(l_0),\ l_0)\leqslant\frac{1}{\tau}\sigma(\tau\rho(u_n(l_0),\ l_0))$;

(2) from a certain index n onwards, by (2), from the g.t.p. of (c) and from the first of (6) we have $\frac{1}{\tau}\delta_1(l_0,\ l_0)\leqslant\frac{1}{\tau}\sup\{\varphi(d(l_0,\ u_n(l_0)))+\tau d(u_n(l_0),\ f_j(f_1^{i-1}(l_0))):\ j=1,\ 2;\ i=1,\ 2,\ 3,\ \dots\}\leqslant\frac{1}{\tau}\varphi(d(l_0,\ u_n(l_0)))+\sup\{\sigma(\rho(u_{n-1}(l_0),\ u_{i-1}(l_0))):\ i=1,\ 2,\ 3,\ \dots\}$, i.e. (since $\rho(u_{n-1}(l_0),\ u_{i-1}(l_0))\leqslant\rho(u_{n-1}(l_0),\ l_0)\vee\sup\{d(u_{n-1}(l_0),\ u_{s-1}(l_0)):\ s=1,\ 2,\ 3,\ \dots\}$), and σ is non decreasing, $\frac{1}{\tau}\delta_1(l_0,\ l_0)\leqslant\frac{1}{\tau}\varphi[d(l_0,\ u_n(l_0))]+\sigma(\rho(u_{n-1}(l_0),\ l_0)\vee\sup\{d(u_{n-1}(l_0),\ u_{s-1}(l_0)):\ s=1,\ 2,\ 3,\ \dots\})$;

(3) by the third of (8) in Lemma 2 we have immediately $\frac{1}{\tau}\gamma_1(l_0,\ l_0)\leqslant\frac{1}{\tau}\sigma(\tau\rho(u_n(l_0),\ l_0))$;

(4) clearly, from (5)''', $\delta_1(u_n(l_0),\ u_n(l_0))\leqslant\rho(u_n(l_0),\ u_n(l_0))$ and $\gamma_1(u_n(l_0),\ u_n(l_0))\leqslant\rho(u_n(l_0),\ u_n(l_0))$;

(5) by (2), from a certain index n onwards, from the g.t.p. of (c) and from the first inequality of (4) above, we have $\frac{1}{\tau}\delta_1(l_0,\ u_n(l_0))\leqslant\frac{1}{\tau}\varphi[d(l_0,\ u_n(l_0))]+\sup\{d(u_n(l_0),\ f_j(f_1^{i-1}(u_n(l_0)))):\ j=1,\ 2;\ i=1,\ 2,\ 3,\ \dots\}=\frac{1}{\tau}\varphi[d(l_0,\ u_n(l_0))]+\delta_1(u_n(l_0),\ u_n(l_0))\leqslant\frac{1}{\tau}\varphi(d(l_0,\ u_n(l_0)))+\rho(u_n(l_0),\ u_n(l_0))$;

(6) by (3) of 2 and by (3) above we have $\frac{1}{\tau}\gamma_1(l_0,\ u_n(l_0))\leqslant\frac{1}{\tau}\gamma_1(l_0,\ l_0)\leqslant\frac{1}{\tau}\sigma(\tau\rho(u_n(l_0),\ l_0))$.

([16]) Where we suppose $x_1=u_{n-1}(l_0)$, $x_2=l_0$.

Therefore from (21)

$$\lim_{n\to+\infty}{}'' \rho(u_n(l_0),\ l_0) \leqslant \lim_{n\to+\infty}{}'' [d(u_n(l_0),\ l_0) \vee \frac{1}{\tau}\sigma(\tau\rho(u_{n-1}(l_0),\ l_0))$$

$$\vee \frac{1}{\tau}\sigma(\tau\rho(u_n(l_0),\ l_0)) \vee (\frac{1}{\tau}\varphi(d(l_0,\ u_n(l_0))) + \sigma(\rho(u_{n-1}(l_0),\ l_0)$$

$$\vee \sup\{d(u_{n-1}(l_0),\ u_{s-1}(l_0)):\ s = 1,\ 2,\ 3,\ \ldots\}))$$

$$\vee (\frac{1}{\tau}\varphi(d(l_0,\ u_n(l_0))) + \rho(u_n(l_0),\ u_n(l_0))) \vee \rho(u_n(l_0),\ u_n(l_0))]$$

whence, since σ is continuous from the right and φ is infinitesimal at zero, and the sequence $u_n(l_0)$ $(n = 0,\ 1,\ 2,\ \ldots)$ is a Cauchy-sequence,

$$\lim_{n\to+\infty}{}'' \rho(u_n(l_0),\ l_0) \leqslant \frac{1}{\tau}\sigma(\tau \lim_{n\to+\infty}{}'' \rho(u_n(l_0),\ l_0)) \vee \sigma(\lim_{n\to+\infty}{}'' \rho(u_n(l_0),\ l_0))$$

which implies $\lim_{n\to+\infty}{}'' \rho(u_n(l_0),\ l_0) = 0$, i.e. (22).

But from the expressions (2) and (21) of δ_1 and ρ it is clear that

$$d(l_0,\ f_r(l_0)) \leqslant \delta_1(l_0,\ l_0) \leqslant \tau\rho(u_n(l_0),\ l_0) \qquad r = 1,\ 2\,.$$

Therefore by (22) l_0 is the fixed point both of f_1 and f_2. Proceeding then exactly as in Theorem 1 ([17]) we conclude the proof of Theorem 4.

5 - Remarks and other theorems

5.1 - As we know (see (2), 2), the first of (6) with ρ in the form (5)‴ yields the first of (6) with ρ in the form (5)′; however Theorem 4 can not be deduced from Theorem 1 where both the first and second of (6) are requested.

5.2 - Note that Theorem 3 also results directly from Theorem 4.

([17]) Bearing in mind of course that ρ should be kept (in the form (5)‴), since it is not possible to substitute it by δ (see also note ([15])).

5.3 - The four theorems above are true even if ρ is substituted by δ in the respective common generalized contractivity hypotheses: in the first two theorems, because of Lemma 1; in the other two because of the trivial inequality $\delta(x_1, x_2) \leqslant \rho(x_1, x_2)$ and the non decreasing of σ. In other words these four theorems can be reformulated according to the hypotheses of common generalized contractivity (7) ([18]).

5.4. Theorem 5 ([19]). *Suppose*

$$d(f_1(x_1), f_r(x_2)) \tag{23}$$

$$\leqslant \sigma(\max\{d(x_1, x_2), \frac{1}{\tau} d(x_i, f_j(x_i)), d(x_i, f_j(x_{3-i})) : i, j = 1, 2\}) \qquad r = 1, 2; \qquad x_1, x_2 \in E$$

holds. Then the mappings f_1 and f_2 have a single fixed point in common, which is also the single fixed point of both.

Proof. Since, clearly, the argument of σ in (23) is less than or equal to δ given from (5)″, (23) implies (7) with such a δ ([20]); therefore, from Theorem 2, by 5.3, we find immediately that f_1 and f_2 have a single fixed point in common, which is also the only fixed point of f_1.

Let y be a fixed point of f_2; for this point the following is true

$$d(y, f_1(y)) = d(f_1(y), f_2(y))$$

$$\leqslant \sigma(\max\{d(y, f_j(y)) : j = 1, 2\}) = \sigma(d(y, f_1(y))).$$

Therefore f_2 also has a single fixed point which is the common fixed point with f_1.

5.5 - Two other theorems resembling Theorem 5 can be obtained with a few easy calculations from Theorem 1 and Theorem 3.

([18]) In these four theorems, furthermore, the only common fixed point in f_1 and f_2 would also be the only fixed point of f_2 if we had also $\delta_1(x_1, x_1) = d(x_1, f_1(x_1))$.

([19]) Assuming of course the conditions set out at the beginning of 4.

([20]) Note the symmetry compared with x_1 and x_2 of the argument of σ in (23) similar to that of δ in (5)″; and remember that (7) is reduced to a single hypothesis (see (1), 2).

5.6 - If the preceding five theorems are rewritten in metric spaces (where the first four are of course reduced to one only), we see immediately that the first four generalize, for two functions, the following theorems: 1 of $[5]_2$ (21), 2 of [6], and 1 of [9]; whilst all five generalize, still for two functions, the assertions of Theorem 1 of [4] (note that in metric spaces the hypothesis $\rho_1(x_1, x_1) < +\infty$, which appears in [4], [6] and [9], is equivalent to the hypothesis $\rho_1(x_1, x_2) < +\infty$, which has been put forward here by us and which appears in $[5]_2$ (22) also).

5.7 - In theorems of $[2]_2$, $[2]_3$, [8], and [1], in addition to the hyphoteses of common generalized contractivity, the limitation $\delta_1(x_1, x_1) < +\infty$ (23) exist as does, sometimes, the need that it be $d(x_1, f_1(x_1)) \in A$ or $d(x_1, f_1(x_2)) \in A$ (24). If we now observe that in the H-spaces $\rho_1(x_1, x_1) < +\infty$ together with $d(x_1, x_2) \in A$ and (23) implies $\rho_1(x_1, x_2) < +\infty$ for each $x_1, x_2 \in E$ (25) and therefore $\delta_1(x_1, x_1) < +\infty$ and $\delta_1(x_1, x_2) < +\infty$ (26), from Theorem 5 we obtain immediately theorems similar to those of $[2]_2$ and $[2]_3$. More precisely we obtain

Theorem 6. *Let $f_1, f_2 \colon E \to E$ be such that for all $x_1, x_2 \in E$ we have*

$$d(f_1(x_1), f_r(x_2)) \tag{24}$$

$$\leqslant \alpha \max\{d(x_1, x_2), \frac{1}{\tau} d(x_i, f_j(x_i)), d(x_i, f_j(x_{3-i})) \colon i, j = 1, 2\} \qquad r = 1, 2 \qquad 0 \leqslant \alpha < 1 .$$

If for every $x_1, x_2 \in E$, $\rho_1(x_1, x_1) < +\infty$ and $d(x_1, x_2) \in A$ hold, then f_1 and f_2 have only one fixed point in common which is also the only fixed point of each.

And so, for a single function, we have

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(21) Relative to what the Author calls the «generalized Banach contraction».

(22) The four limitations $\rho_1(x_1, x_1) < +\infty$, $\rho_1(x_1, x_2) < +\infty$, $\delta_1(x_1, x_1) < +\infty$, $\delta_1(x_1, x_2) < +\infty$ are actually equivalent in metric spaces.

(23) See observations made at $[2]_4$ in relation to theorems of $[2]_2$ and $[2]_3$.

(24) Note that in metric spaces whatever $x, y \in E$ are, $d(x, y) \in A \equiv \mathfrak{R}^+$ is always true.

(25) This would occur also if in (23) the coefficients $\frac{1}{\tau}$ did not exist.

(26) Note that limitation $\delta_1(x_1, x_1) < +\infty$ together with $d(x_1, x_2) \in A$ implies $\delta_1(x_1, x_2) < +\infty$ without the aid of (23).

Theorem 7. *Let us assume a mapping* $f\colon E \to E$. *If*

$$d(f(x_1),\ f(x_2)) \tag{25}$$

$$\leq \alpha \max\{d(x_1,\ x_2),\ \frac{1}{\tau} d(x_1,\ f(x_1)),\ \frac{1}{\tau} d(x_2,\ f(x_2)),\ d(x_1,\ f(x_2)),\ d(x_2,\ f(x_1))\}$$

$$0 \leq \alpha < 1,\ \rho_1(x_1,\ x_1) < +\infty \ \textit{and}\ d(x_1,\ x_2) \in A \ \textit{for every}\ x_1,\ x_2 \in E$$

then there is precisely one fixed point for f.

5.8 - The explicit statements of Theorem 6 and Theorem 7 (even if they are specific cases of Theorem 5) allow us to observe that:

(1) they are similar (in the sense specified in 5.7) respectively to Theorem 6 of $[2]_3$ and Theorem 1 of $[2]_2$ (27), coinciding with the latter in metric spaces where the condition $\delta_1(x_1,\ x_1) < +\infty$ (and therefore $\rho_1(x_1,\ x_1) < +\infty$: see note (22)) descends directly (28) from the hypotheses of generalized contractivity of type (24) or (25), and where Theorem 7 coincides with Theorem 1(a) and (b) of [3] or with the Theorem in [7].

(2) the position of the coefficients $\frac{1}{\tau}$ [four in (24) and two in (25)] makes the right hand sides of (24) and (25) symmetric in x_1 and x_2 as they are already in metric spaces.

Therefore we can consider Theorems 6 and 7 as the *real natural extension* of the similar metrics quoted in (1) above, in that, if we were to think of going over the same paths that led us to the generalization of Theorems 1(a) and (b) of [3] or the theorem of [7] we would start from Theorem 7, first opening to Theorem 6 and Theorem 5 and then to Theorem 2; passing easily thereafter from hypotheses of linear contractivity to non-linear hypotheses with second members having either an infinite or a finite number of distances.

In our previous studies however, starting at Theorem 1 of $[2]_2$ we arrived at Theorem 6 of $[2]_3$ and at the results of [1] (29) without succeeding in our intention of opening further to an infinite number of distances the argument of σ in the common generalized contractivity hypothesis.

(27) See note (23).
(28) With a similar procedure as that of $[2]_4$.
(29) These too generalize Theorem 1 of [4].

We believe that this different and incomplete result could depend on the «less fortunate» position of the coefficient $\frac{1}{\tau}$ which does not render the right hand side of the hypothesis symmetric in x_1 and in x_2 already in Theorem 1 of $[2]_2$ (and so successively in the others).

All these remarks (which are obviously only possible a posteriori) lead us to believe not only that «the best» position for the coefficients $\frac{1}{\tau}$ is that identified here but also that their number can not be less than two in theorems of type 7, and less than four in theorems of type 6 and 5 (unless the left hand side of the hypothesis of generalized contractivity is not considered with other distances which are not included yet. But this is a field still to be explored).

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Sommario

Vengono dati, in spazi metrici generalizzati, detti H-spazi, teoremi di punto unito comune per due applicazioni f_1 *e* f_2 *con ipotesi di contrattività comune di tipo non lineare. Tali teoremi generalizzano, fra gli altri, teoremi di Daneš, Hegedüs, Kasahara, Tasković, relativi ad una sola applicazione in spazi metrici.*

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