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Some remarks on the hypoellipticity of the $\bar{\partial}$-problem and the local extendibility of CR-functions (**)

Let Ω be a relatively compact pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with boundary $b\Omega$ smooth of class C^h $(1 \leq h \leq \infty)$.

It is known that the $\bar{\partial}$-problem for $L^2_{(p,q)}$-forms on Ω is said to be hypoelliptic in case, for every $\bar{\partial}$-closed $L^2_{(p,q)}$-form α on Ω $(1 \leq q \leq n-1)$, the equation $\bar{\partial}\beta = \alpha$ can be solved by an $L^2_{(p,q-1)}$-form β on Ω such that $\operatorname{sing\,supp}(\beta) = \operatorname{sing\,supp}(\alpha)$. We recall that the singular support of a differential form on Ω with square integrable coefficients is the complement in $\bar{\Omega}$ of the set of all points $z \in \bar{\Omega}$ such that, for a sufficiently small open neighbourhood U of z in $\mathbb{C}^n$, every coefficient of the form is in $C^h(U \cap \bar{\Omega})$.

Catlin $[2]_2$ and Diederich-Pflug [3] proved independently that a necessary condition for the $\bar{\partial}$-problem to be hypoelliptic for $L^2_{(p,q)}$-forms on Ω is that no complex-analytic variety of dimension $\geq q$ is contained in bΩ.

Hence a necessary condition for hypoellipticity common to all q is that no complex-analytic variety of codimension one is contained in bΩ.

On the other hand, according to a theorem of Trépreau [9], when $h \geq 2$ this latter condition is also necessary and sufficient for another property of Ω, namely the local extension property of CR-functions at every point of bΩ. Let us recall that a CR-function on a relatively open subset Σ of bΩ is a complex-valued function f on Σ that satisfies $\bar{\partial}_b f = 0$. Here we consider continuous CR-functions, for which $\bar{\partial}_b$ is understood in the weak sense. One says that the local extension property of CR-functions is valid at a point $z^0 \in$ bΩ in case, for every open

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neighbourhood U of z^0 in $\mathbb{C}^n$, there is an open neighbourhood $U' \subset U$ of z^0 so that every CR-function f on $b\Omega \cap U$ has a unique extension $F \in C^0(\bar{\Omega} \cap U') \cap \mathcal{O}(\Omega \cap U')$.

Hence it follows from the results in the quoted papers that, when $h \geq 2$, the hypoellipticity for some q of the $\bar{\partial}$-problem for $L^2_{(p,q)}$-forms on Ω implies the validity of the local extension property of CR-functions at every point of $b\Omega$. On the other hand, since Trépreau's theorem is a rather hard one, it seems reasonable to ask whether one could do without resorting to that theorem and achieve directly and by simpler tools some properties of the above kind. In this note we just consider the case $q=1$, however we allow $h=1$ too, whereas Trépreau's theorem works for $h \geq 2$.

In the first place we prove

Proposition 1. *If the $\bar{\partial}$-problem is hypoelliptic for $L^2_{(p,1)}$-forms on Ω, it follows that the local extension property of* CR-*functions is valid at every point of* $b\Omega$.

Proof. We shall apply the following result on extension of CR-functions (cf. [6] for a slightly more general version).

Let D be a relatively compact domain in $\mathbb{C}^n$, K a compact subset of bD, and let $\hat{K}$ denote the polynomial hull of K. If $bD \setminus \hat{K}$ is a connected C^1-smooth hypersurface in $\mathbb{C}^n \setminus \hat{K}$, then every CR-function f on $bD \setminus \hat{K}$ has a unique extension $F \in C^0(\bar{D} \setminus \hat{K}) \cap \mathcal{O}(D \setminus \hat{K})$.

Let $z^0 \in b\Omega$ and U be an open neighbourhood of z^0 in $\mathbb{C}^n$. If $B(z^0, r)$ is a small $2n$-ball with center z^0, such that $\bar{B}(z^0, r) \subset U$ and $B(z^0, r) \cap b\Omega$ is connected, let us write

$$D = B(z^0, r) \cap \Omega \qquad K = bB(z^0, r) \cap \bar{\Omega}.$$

We shall prove that $\hat{K}$ is disjoint form $bD \setminus K = B(z^0, r) \cap b\Omega$, provided r has been chosen small enough. On account of the quoted result on extension of CR-functions, it follows at once that every CR-function f on $b\Omega \cap U$ has a unique extension $F \in C^0(\bar{D} \setminus \hat{K}) \cap \mathcal{O}(D \setminus \hat{K})$, thus proving the validity of the local extension property of CR-functions at the given point $z^0 \in b\Omega$.

We assume that r is small enough so that $\bar{D} = \bar{B}(z^0, r) \cap \bar{\Omega}$ is polynomially convex (cf. [4], p. 132) and there exists an $\varepsilon_0 > 0$ such that, if ζ is any point in

$\bar{B}(z^0, r) \cap b\Omega$ and $\vec{\nu}(\zeta)$ is the exterior unit normal to $b\Omega$ at ζ, then

$$\bar{B}(z^0, r) \cap \bar{\Omega} = \bar{D} \subset \Omega + \varepsilon\vec{\nu}(\zeta) \qquad \text{for all } \varepsilon \text{ with } 0 < \varepsilon \leqslant \varepsilon_0 .$$

Now, to prove that $\hat{K}$ does not meet $B(z^0, r) \cap b\Omega$, we use a contradiction argument. Thus suppose there is a point $\zeta \in \hat{K} \cap (B(z^0, r) \cap b\Omega)$. By a result of Pflug [7], we may consider a function $\phi \in L^2(\Omega) \cap \mathcal{O}(\Omega)$ such that

$$\sup_{0<\varepsilon\leqslant\varepsilon_0} |\phi(\zeta - \varepsilon\vec{\nu}(\zeta))| = \infty . \tag{1}$$

Let $B(\zeta, r')$ be a $2n$-ball with center ζ such that $\bar{B}(\zeta, r') \subset B(z^0, r)$, and χ a real-valued C^∞ function on $\mathbb{C}^n$ with $\chi = 1$ on $\bar{B}(\zeta, r'/2)$ and $\operatorname{supp}(\chi) \subset B(\zeta, r')$. Consider the $\bar\partial$-closed $L^2_{(0,1)}$-form on Ω, $\alpha = \phi\bar\partial\chi$ and let S denote its singular support. By assumption the equation $\bar\partial\beta = \alpha$ can be solved by a function $\beta \in L^2(\Omega)$ such that

$$\text{sing supp}(\beta) = \text{sing supp}(\alpha) = S .$$

Clearly S is disjoint from $B(\zeta, r'/2) \cap \bar{\Omega}$ and from $\bar{\Omega} \setminus \bar{B}(\zeta, r')$, and hence β is of class C^h on these two relatively open subsets of $\bar{\Omega}$. It follows in particular that we may find a neighbourhood V_1 of ζ in $\bar{\Omega}$ and a positive constant M_1 such that

$$|\beta| < M_1 \quad \text{on} \quad V_1 . \tag{2}$$

Then, consider the function $g = \phi\chi - \beta$.

This function is holomorphic on Ω and of class C^h on $\bar{\Omega} \setminus \bar{B}(\zeta, r')$; therefore we may find a neighbourhood V_2 of K in $\bar{\Omega}$ and a positive constant M_2 such that

$$|g| < M_2 \quad \text{on} \quad V_2 . \tag{3}$$

For $0 < \varepsilon \leqslant \varepsilon_0$ let $g_\varepsilon \in \mathcal{O}(\Omega + \varepsilon\vec{\nu}(\zeta))$ be given by $g_\varepsilon(z) = g(z - \varepsilon\vec{\nu}(\zeta))$. Since $\bar{D} \subset \Omega + \varepsilon\vec{\nu}(\zeta)$, $g_\varepsilon \in \mathcal{O}(\bar{D})$, and since $\bar{D}$ is polynomially convex and $\zeta \in \hat{K}$, it follows that $|g_\varepsilon(\zeta)| \leqslant \max_K |g_\varepsilon|$. Hence we have, for $0 < \varepsilon \leqslant \varepsilon_0$,

$$|(\phi\chi)(\zeta - \varepsilon\vec{\nu}(\zeta))| \leqslant |\beta(\zeta - \varepsilon\vec{\nu}(\zeta))| + \max_{z\in K} |g(z - \varepsilon\vec{\nu}(\zeta))| .$$

Then, on account of (2) and (3), for all ε small enough so that $\zeta - \varepsilon\vec{\nu}(\zeta) \in V_1$,

$K - \varepsilon\vec{v}(\zeta) \subset V_2$ and $\chi(\zeta - \varepsilon\vec{v}(\zeta)) = 1$, we get

$$|\phi(\zeta - \varepsilon\vec{v}(\zeta))| < M_1 + M_2 .$$

But this contradicts (1).

The argument we used to prove that $\hat{K}$ does not meet $\mathrm{b}D \setminus K$ was suggested by a proof of Sibony [8] that, when $\mathrm{b}\Omega$ is C^∞, the hypoellipticity of the $\bar{\partial}$-problem for $L^2_{(0,1)}$-forms on Ω implies, for every compact set $K \subset \bar{\Omega}$

$$\hat{K}_\infty \cap \mathrm{b}\Omega = K \cap \mathrm{b}\Omega$$

where $\hat{K}_\infty$ means the hull of K with respect to the algebra $A^\infty(\bar{\Omega}) = C^\infty(\bar{\Omega}) \cap \mathcal{O}(\Omega)$ (1). This result was proved before by Catlin in $[2]_1$ and is also stated in $[2]_2$. In fact, we can readily prove that this last condition in by itself sufficient to guarantee the local extension property of CR-functions at every point of $\mathrm{b}\Omega$. More generally, the following holds

Proposition 2. *Let* $\mathrm{b}\Omega$ *be smooth of class* $C^h (1 \leq h \leq \infty)$ *and assume that, for every compact set* $K \subset \bar{\Omega}$ *we have*

$$(*) \qquad \hat{K}_h \cap \mathrm{b}\Omega = K \cap \mathrm{b}\Omega$$

where $\hat{K}_h$ *means the hull of* K *with respect to the algebra* $A^h(\bar{\Omega}) = C^h(\bar{\Omega}) \cap \mathcal{O}(\Omega)$. *Then the local extension property of* CR-*functions is valid at every point of* $\mathrm{b}\Omega$.

Proof. Arguing as in the proof of Proposition 1, it suffices to show that, for $D = B(z^0, r) \cap \Omega$ and $K = \mathrm{b}B(z^0, r) \cap \bar{\Omega}$, the polynomial hull $\hat{K}$ of K does not meet $bD \setminus K$, provided r is small enough.

As a matter of fact, let $\zeta \in \hat{K}$, i.e. $|w(\zeta)| \leq \max_K |w|$ for all $w \in \mathcal{O}(\mathbb{C}^n)$. For each $\psi \in A^h(\bar{\Omega})$ and for $0 < \varepsilon \leq \varepsilon_0$ (ε_0 as in the proof of Proposition 1), let $\psi_\varepsilon \in \mathcal{O}(\Omega + \varepsilon\vec{v}(z^0))$ be given by $\psi_\varepsilon(\zeta) = \psi(z - \varepsilon\vec{v}(z^0))$. Since $\bar{D} \subset \Omega + \varepsilon\vec{v}(z^0)$, it follows that $\psi_\varepsilon \in \mathcal{O}(\bar{D})$, and hence $|\psi_\varepsilon(\zeta)| \leq \max_K |\psi_\varepsilon|$ (because $\bar{D}$ is polynomially convex), and

(1) In [8] this condition is proved to be also sufficient at least for the following weaker hypoellipticity property of $\bar{\partial}$. For every $\bar{\partial}$-closed (0, 1)-form α with coefficients in $L^2_{\mathrm{loc}}(\Omega)$, the equation $\bar{\partial} u = \alpha$ can be solved by an $u \in L^2_{\mathrm{loc}}(\Omega)$ with $\operatorname{sing\,supp}(u) = \operatorname{sing\,supp}(\alpha)$.

since $\lim_{\varepsilon\to 0} \psi_\varepsilon = \psi$ uniformly on $\bar{D}$, taking the limit of this inequality for $\varepsilon \to 0$ gives $|\psi(\zeta)| \leq \max_K |\psi|$. Therefore $\zeta \in \hat{K}_h$, which proves that $\hat{K} \subset \hat{K}_h$. Then, since (*) holds, it follows at once that $\hat{K}$ does not meet $bD \setminus K$.

Note that, unlike for the case $h = \infty$, Proposition 2 might not imply Proposition 1 for $1 \leq h < \infty$. Indeed, as far as we know, the last quoted result of Catlin has not been extended to the case when $b\Omega$ is smooth of class C^h with $1 \leq h < \infty$.

Using the above proposition combined with a further result of Catlin we can readily prove the following particular case of Trépreau's theorem on local extension of CR-functions.

Proposition 3. *Let $n = 2$ and $b\Omega$ be smooth of class C^∞. A necessary and sufficient condition for the local extension property of* CR-*functions to hold at every point of* $b\Omega$ *is not having any complex-analytic curve in* $b\Omega$.

Proof. In fact, when $n = 2$ and $b\Omega$ is C^∞, the condition of non-existence of any complex-analytic curve contained in $b\Omega$ is necessary and sufficient for having $\hat{K}_\infty \cap b\Omega = K \cap b\Omega$ for every compact set $K \subset \bar{\Omega}$ (whereas it is only necessary for general n). This result is proved in $[2]_1$. Hence Proposition 2 implies that the same condition is sufficient for the validity of the local extension property of CR-functions at every point of $b\Omega$. On the other hand there are counterexamples which show that the condition is also necessary (cf. [1]).

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Sunto

Si considerano due possibili proprietà di un dominio pseudoconvesso $\Omega \subset\subset \mathbf{C}^n$, con b$\Omega$ *di classe C^1 almeno: l'ipoellitticità di $\bar{\partial}$ per le $L^1_{(p,1)}$-forme su Ω e l'estendibilità olomorfa locale dei germi di CR-funzioni su* bΩ. *Si mostra che la seconda consegue dalla prima ed ha luogo anche sotto una opportuna ipotesi di «convessità» di* bΩ.

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