Riv. Mat. Univ. Parma (4) **13** (1987), 407-412

DĂNUŢ MARCU (*)

Note on the matroidal families (**)

1 - Introduction

Matroidal families were introduced and studied by Simões-Pereira [5]. Although we know uncountably many matroidal families of simple graphs and infinitely many matroidal families with multigraphs as members, it is an open question how one can find all matroidal families. In this paper, by certain submodular functions on the set of all finite graphs, we characterize the families of connected graphs being matroidal families.

Let $\mathscr{G}$ be the class of all finite graphs, where loops and multiple edges are allowed, and $\mathscr{G}_0 \subset \mathscr{G}$ the class of all finite simple graphs. For a graph $G=(V, E)$ and for $X \subset E$, we denote the subgraph of G, induced by the edges of X, by $G|X$. Further, we denote the number of vertices by $\alpha(G)$, the number of edges by $k(G)$ and the number of components by $\sigma(G)$.

Def. 1.1. Let M be a finite set and $\mathscr{C} \subset 2^M$ a collection of non-empty subsets of M. The pair $(M, \mathscr{C})$ is called a *matroid* on M, if the following axioms hold:

(C_1) No element of $\mathscr{C}$ contains properly another element of $\mathscr{C}$.

(C_2) If $C, C' \in \mathscr{C}$, $C \neq C'$ and $x \in C \cap C'$, then $(C \cup C') - \{x\}$ contains an element C'' of $\mathscr{C}$.

The elements of $\mathscr{C}$ are called the *circuits* of the matroid $(M, \mathscr{C})$.

(*) Indirizzo: Institute of Matematics, Academiei 14, R-70109, Bucharest.
(**) Ricevuto: 13-X-1987.

In this paper, we presuppose a knowledge of matroid theory: our standard reference is Welsh [7].

Def. 1.2. A *matroidal family* of graphs is a non-empty set of finite connected graphs $\mathscr{F}$, such that, given any graph G, the edge-sets of the subgraphs of G isomorphic to members of $\mathscr{F}$ may be regarded as the circuits of a matroid on the edge-set of G, denoted by $\mathscr{M}(G, \mathscr{F})$ and called the $\mathscr{F}$*-matroid* of G.

Analogously, we may define matroidal families of simple graphs.

Simões-Pereira $[5]_1$ introduced matroidal families in 1972 and discovered four of them: $\mathscr{P}_0$, $\mathscr{P}_1$, $\mathscr{P}_2$, $\mathscr{P}_3$. The trivial family $\mathscr{P}_0$ consists only of the complete graph on two vertices. The family $\mathscr{P}_1$, consisting of all cycles, is well-known, because $\mathscr{M}(G, \mathscr{P}_1)$, for any G, is the polygon-matroid of G. The family $\mathscr{P}_2$ consists of all bicycles, that means all graphs homeomorphic to one of the following graphs: two loops at one vertex, three parallel edges and two loops joined by an edge. The $\mathscr{P}_2$-matroids $\mathscr{M}(G, \mathscr{P}_2)$ are called *bicircular matroids* and their structure was analysed by Matthews [3]. Clearly, $\mathscr{P}_i \cap \mathscr{G}_0$ $(i = 0, 1, 2, 3)$ are matroidal families of simple graphs.

Using several methods, Andrae [1], Loréa [2] and Schmidt [4] constructed infinitely many new matroidal families of simple graphs. A generalization of Schmidt's method is given in Walter [6].

Schmidt [4] proved that there are uncountably many matroidal families of simple graphs. Nevertheless, it is an open question, how one can find all matroidal families. In order to solve this problem, we use a generalization of Loréa's method [2].

2 - Matroidal functions

Let N be the set of non-negative integers and denote isomorphic graphs by $G \simeq H$, union of graphs by $G \cup H$ and intersection of graphs by $G \cap H$.

Def. 2.1. A map $f\colon \mathscr{G} \to \mathrm{N}$ is said to be a *matroidal function* on $\mathscr{G}$, if the following conditions hold:

(1) $f(G) = 0$ if $E(G) = \emptyset$. (2) $G \simeq H \Rightarrow f(G) = f(H)$.

(3) f is nondecreasing: $G \subset H \Rightarrow f(G) \leqslant f(H)$.

(4) f is submodular: $f(G \cup H) + f(G \cap H) \leq f(G) + f(H)$ for all $G, H \in \mathscr{G}$.

(5) If $G \in \mathscr{G}$ has components $G_1, G_2, \ldots, G_n$, then $f(G) = \sum_{i=1}^{n} f(G_i)$.

Example 2. Let n, r be integers, with $n \geq 0$ and $-n+1 \leq r \leq 1$. Then, $f_{n,r}\colon \mathscr{G} \to \mathbb{N}$ defined by

$$f_{n,r}(G) = \begin{cases} 0 & \text{if } E(G) = \emptyset \\ n \cdot \alpha(G) + \sigma(G) \cdot (r-1) & \text{if } E(G) \neq \emptyset \,. \end{cases}$$

In the same way, we may define matroidal functions on $\mathscr{G}_0$. Clearly, the restriction $f|\mathscr{G}_0$ of a matroidal function f on $\mathscr{G}$ is a matroidal function on $\mathscr{G}_0$. On the other hand, for every matroidal function on $\mathscr{G}_0$ there exists a natural extension on $\mathscr{G}$, which we obtain in the following way: consider the map $P\colon \mathscr{G} \to \mathscr{G}_0$, with $P(G) = G_0$ as the graph one obtains by removing all loops of G and identifying parallel edges. Then, the composition $f \circ P$, with $(f \circ P)(G) = f(G_0)$, is an extension of matroidal function f on $\mathscr{G}_0$ to $\mathscr{G}$. In the sequel, we shall see that there is a nice correspondence between matroidal functions and matroidal families.

Lemma 2.3. *Let $\mathscr{F}$ be a matroidal family. Then, $f(G) = rg\,\mathscr{M}(G, \mathscr{F})$ defines a matroidal function on $\mathscr{G}$, where rg denotes the rank function of the matroid $\mathscr{M}(G, \mathscr{F})$.*

Proof. If $E(G) = \emptyset$, then $rg\,\mathscr{M}(G, \mathscr{F}) = 0$. Obviously, f is nondecreasing and submodular because of the corrresponding properties of rank functions.

Further, isomorphic graphs have isomorphic $\mathscr{F}$-matroids and, hence, their images under f are equal. $\mathscr{F}$-circuits are connected subgraphs of G and the $\mathscr{F}$-matroid of G is the direct sum of the $\mathscr{F}$-matroids of the components of G. So, f satisfies the condition (5) of the Def. 2.1.

3 - Matroidal families induced by matroidal functions

Theorem 3.1. *Let f be a matroidal function on $\mathscr{G}$ and let $\mathscr{L}(f)$ be the set of all $G \in \mathscr{G}$ having the properties:*

(1) $k(G) = f(G) + 1$ (2) *G is minimal with* (1).

If $\mathscr{L}(f)$ *is non-empty, then* $\mathscr{L}(f)$ *is a matroidal family.*

Proof. Let G be an arbitrary graph. Let f': $2^{E(G)} \to \mathbb{N}$ be defined as follows: for each $X \subset E(G)$, $f'(X) = f(G|X)$. The function f' is nondecreasing, submodular and $f'(\emptyset) = 0$. Such a function defines a matroid $\mathscr{M}$ on $E(G)$, whose independent sets are the $X \subset E(G)$ with $|Y| \leq f'(Y)$, for all $Y \subset X$ (see [7]). We have to show that $\mathscr{M}[G, \mathscr{L}(f)] = \mathscr{M}$ or, more precisely, that the circuits of $\mathscr{M}$ are exactly the sets $Z \subset E(G)$, satisfying:

(3.2) $|Z| = f'(Z) + 1 = f(G|Z) + 1$ (3.3) Z is minimal with (3.2).

It suffices to show:

(3.4) If Z is a circuit of $\mathscr{M}$, then it satisfies (3.2).

(3.5) If Z satisfies (3.2), then Z is dependent in $\mathscr{M}$.

Obviously, (3.5) follows immediately from $f'(Z) = |Z| - 1$ and the definition of the independent sets of $\mathscr{M}$. Now, we consider a circuit Z of $\mathscr{M}$; Z being dependent, we have $f'(Z) < |Z|$. Every $Z' \subset Z$, with $Z \neq Z'$, is independent and we get $|Z'| \leq f'(Z') \leq f'(Z) < |Z|$.

If $|Z| = 1$, then $Z' = \emptyset$ and $0 = f'(Z) < 1$. Hence, Z satisfies (3.2).

Suppose $|Z| \geq 2$ and choose $Z' \subset Z$, with $|Z'| = |Z| - 1$. Then, from the above chain of inequalities, by just adding 1, the following hold

$$|Z| = |Z'| + 1 \leq f'(Z') + 1 \leq f'(Z) + 1 \leq |Z| .$$

We obtain $|Z| = f'(Z) + 1$ and so Z satisfies (3.2).

Finally, we have to show that all $G \in \mathscr{L}(f)$ are connected. Suppose that there is a $G \in \mathscr{L}(f)$, with components $G_1, G_2, \ldots, G_n$, $n \geq 2$ and corresponding edge-sets $Z_i = E(G_i)$, $i = 1, 2, \ldots, n$. The definition of component implies that Z_i is a proper subset of $E(G)$, for all $i = 1, 2, \ldots, n$. Since $G \in \mathscr{L}(f)$, all Z_i's are independent sets in $\mathscr{M}[G, \mathscr{L}(f)]$ and we get $f'(Z_i) \geq |Z_i|$ for $i = 1, 2, \ldots, n$ and

$$f(G) = f'[E(G)] = f'(\bigcup_{i=1}^{n} Z_i) = \sum_{i=1}^{n} f'(Z_i) \geq \sum_{i=1}^{n} |Z_i| = k(G) .$$

The last inequality leads to a contradiction to $f(G) + 1 = k(G)$ and finishes the proof.

Recalling Lemma 2.3, we obtain our main result

Theorem 3.6. *Every matroidal family is of the form* $\mathscr{L}(f)$, *where* f *is a matroidal function on* $\mathscr{G}$.

Clearly, the matroidal families of simple graphs are exactly the $\mathscr{L}(f)$, defined by a matroidal function f on $\mathscr{G}_0$.

Example 3.7. The matroidal functions $f_{n,r}$ defined in Example 2.2 induce the matroidal families $\mathscr{L}(f_{n,r})$, which were introduced by Loréa [2] and Schmidt [4], independently, for simple graphs (here, it is possible to choose $-2n+1 \leqslant r \leqslant 1$). It is easy to see that $\mathscr{B}_1 = \mathscr{L}(f_{1,0})$, $\mathscr{B}_2 = \mathscr{L}(f_{1,1})$ and $\mathscr{B}_0 = \mathscr{L}(f_{1,-1})$.

Schmidt [4] proved that there are uncountably many matroidal families of simple graphs. Since distinct matroidal functions f, g on $\mathscr{G}_0$ have distinct extension $f \circ P$ and $g \circ P$ on $\mathscr{G}$, and for every matroidal function f on $\mathscr{G}_0$ the matroidal family $\mathscr{L}(f \circ P)$ contains $\mathscr{L}(f)$, we get the interesting result

Corollary 3.8. *There are uncountably many matroidal families of graphs with multigraphs as members.*

Acnowledgement. I am grateful to the referee for his careful reading of the paper and for the useful suggestions, which lead to better and more complete formulations of the stated material.

References

[1] T. ANDREAE, *Matroidal families of finite non-homeomorphic graphs exist*, J. Graph Theory 2 (1979), 149-153.

[2] M. LORÉA, *On matroidal families*, Discrete Math. 28 (1979), 103-106.

[3] L. R. MATTHEWS, *Bicircular matroids*, Quart. J. Math. Oxford 28 (2) (1977), 213-228.

[4] R. SCHMIDT, *On the existence of uncountably many matroidal families*, Discrete Math. 27 (1979), 93-97.

[5] J. M. S. SIMÕES-PEREIRA: [∘]$_1$ *On subgraphs as matroid cells*, Math. Z. **127** (1972), 315-322; [∘]$_2$ *On matroids on edge sets of graphs with connected subgraphs as circuits*, Proc. Amer. Math. Soc. 38 (1973), 503-506; [∘]$_3$ *On matroids on edge sets of graphs with connected subgraphs as circuits* (II), Discrete Math. **12** (1975), 55-78.

[6] M. WALTER, *Construction of matroidal families by partially closed sets*, Discrete Math. **41** (1982), 309-315.

[7] D. J. A. WELSH, *Matroid Theory*, Academic Press, New York, 1976.

Résumé

Dans ce travail, on caractérise les familles des graphes connexes, qui sont des familles matroidales.

***