Riv. Mat. Univ. Parma (4) **13** (1987), 241-256

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On discrete inequalities in n independent variables (**)

1 - Introduction

The discrete inequalities play an important role not only in the field of finite difference equations and numerical analysis but also in certain areas of engineering, technology, economics and biological sciences. One of the most used result in this direction is the discrete analogue of the celebrated Gronwall-Bellman-Reid inequality [9], [10] and its variants [1], ... [6], [10], [12]. The two and more independent variable generalizations of this inequality has been established recently in [7], [8], [9].

In this paper we shall discuss some new discrete inequalities in n independent variables which are further generalizations of some results we have obtained recently in $[2]_2$ for $n=1$. Some unified results are also presented which covers several results of Pachpatte and Singare [7], [8], [9]. Some applications are also given.

Throughout the paper we shall use the following notations and definitions. Let $N_0=\{0, 1, \dots\}$ and the product $N_0 \times N_0 \times \dots \times N_0$ (n times) be denoted by N_0^n. The expression $u(0)+\sum_{s=0}^{n-1} b(s)$ represents the solution of the linear difference equation $\Delta u(n)=u(n+1)-u(n)$. It is assumed that $\sum_{s=0}^{-1} b(s)=0$. The expression $u(0)\prod_{s=0}^{n-1} c(s)$ represents a solution of the linear difference equation $u(n+1)=c(n)\,u(n)$. It is supposed that $\prod_{s=0}^{-1} c(s)=1$. A point $(x_1^i, \dots, x_n^i)$ in N_0^n is

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(**) Ricevuto: 25-XI-1980.

denoted by x^i. The first difference with respect to the variable x_i of the function on $u(x_1, \ldots, x_n)$ is defined as $\Delta u_{x_i}(x_1, \ldots, x_n) = u(x_1, \ldots, x_{i-1}, x_i+1, x_{i+1}, \ldots, x_n) - u(x_1, \ldots, x_n)$. The second difference with respect to the variables x_i, x_j is defined as $\Delta^2 u_{x_ix_j(x_1,\ldots,x_n)} = \Delta u_{x_i}(x_1, \ldots, x_{j-1}, x_j+1, x_{j+1}, \ldots, x_n) - \Delta u_{x_i}(x_1, \ldots, x_n) = u(x_1, \ldots, x_{i-1}, x_i+1, x_{i+1}, \ldots, x_{j-1}, x_j+1, x_{j+1}, \ldots, x_n) - u(x_1, \ldots, x_{i-1}, x_i+1, x_{i+1}, \ldots, x_n) - u(x_1, \ldots, x_{j-1}, x_j+1, x_{j+1}, \ldots, x_n) + u(x_1, \ldots, x_n)$. The higher order differences are defined analogously. The functions which appear in the inequalities are assumed to be real-valued, nonnegative and defined in N_0^n.

2 - Linear inequalities

Theorem 1. *Let the following inequality be satisfied*

$$u(x) \le \sum_{i=1}^{n} a_i(x_i) + \sum_{r=1}^{m} E^r(x, u) \tag{1}$$

where

$$E^r(x, u) = \sum_{x^1=0}^{x-1} f_{r1}(x^1) \sum_{x^2=0}^{x^1-1} f_{r2}(x^2) \ldots \sum_{x^r=0}^{x^{r-1}-1} f_{rr}(x^r)\, u(x^r)$$

for all $x \in N_0^n$ *and* $a_i(x_i) > 0$, $\Delta a_i(x_i) \ge 0$. *Then*

$$u(x) \le [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{s_1=0}^{x_1-1} [1 + \frac{\Delta a_1(s_1)}{a_1(s_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{r=1}^{m} \Delta E^r_{s_1}(s_1, x_2, \ldots, x_n, 1)] \,. \tag{2}$$

Proof. Let $\phi(x)$ be the right member of (1). Then

$$\Delta\phi_{x_1}(x) = \Delta a_1(x_1) + \sum_{r=1}^{m} \Delta E^r_{x_1}(x, u)$$

and

$$\Delta^n \phi_x(x) = \sum_{r=1}^{m} \Delta^n E^r_x(x, u) \tag{3}$$

also

$$\phi(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = a_i(0) + \sum_{\substack{j=1 \\ j\ne i}}^{n} a_j(x_j) \,. \tag{4}$$

Since $u(x)\le\phi(x)$ and $\phi(x)$ is nondecreasing in x, from (3), we get

$$\Delta^n\phi_x(x)\le\sum_{r=1}^{m}\Delta^n E_x^r(x,\ \phi)\le\sum_{r=1}^{m}\Delta^n E_x^r(x,\ 1)\phi(x)\ . \tag{5}$$

From (5), on using the fact $\phi(x_1,\ \dots,\ x_{n-1},\ x_n+1)\ge\phi(x)$, we obtain

$$\frac{\Delta^{n-1}\phi_{x_1\dots x_{n-1}}(x_1,\ \dots,\ x_{n-1},\ x_n+1)}{\phi(x_1,\ \dots,\ x_{n-1},\ x_n+1)}-\frac{\Delta^{n-1}\phi_{x_1\dots x_{n-1}}(x)}{\phi(x)}\le\sum_{r=1}^{m}\Delta^n E_x^r(x,\ 1)\ .$$

Now keeping $x_1,\ \dots,\ x_{n-1}$ fixed and setting $x_n=s_n$ and summing over $s_n=0,\ 1,\ \dots,\ x_n-1$ in the above inequality, we find

$$\frac{\Delta^{n-1}\phi_{x_1\dots x_{n-1}}(x)}{\phi(x)}\le\sum_{s_n=0}^{x_n-1}\sum_{r=1}^{m}\Delta^n E_{x_1\dots x_{n-1}s_n}^r(x_1,\ \dots,\ x_{n-1},\ s_n,\ 1)=\sum_{r=1}^{m}\Delta^{n-1}E_{x_1\dots x_{n-1}}^r(x,1)\,.$$

Repeating the above arguments successively, to obtain

$$\frac{\Delta\phi_{x_1}(x)}{\phi(x)}\le\frac{\Delta a_1(x_1)}{a_1(x_1)+a_2(0)+\sum_{i=3}^{n}a_i(x_i)}+\sum_{r=1}^{m}\Delta E_{x_1}^r(x,\ 1)\ . \tag{6}$$

From (6), we have

$$\phi(x_1+1,\ x_2,\ \dots,\ x_n)\le[1+\frac{\Delta a_1(x_1)}{a_1(x_1)+a_2(0)+\sum_{i=3}^{n}a_i(x_i)}+\sum_{r=1}^{m}\Delta E_{x_1}^r(x,\ 1)]\phi(x)\ .$$

Now keeping $x_2,\ \dots,\ x_n$ fixed and setting $x_1=s_1$ and summing over $s_1=0,\ 1,\ \dots,\ x_1-1$, in the above inequality, we find from (4)

$$\phi(x)\le[a_1(0)+\sum_{i=2}^{n}a_i(x_i)]\prod_{s_1=0}^{x_1-1}[1+\frac{\Delta a_1(s_1)}{a_1(s_1)+a_2(0)+\sum_{i=3}^{n}a_i(x_i)}+\sum_{r=1}^{m}\Delta E_{s_1}^r(s_1,\ x_2,\ \dots,\ x_n,\ 1)]\,.$$

The result (2) now follows from $u(x)\le\phi(x)$.

Remark 1. There are $n!$ different conclusions possible for the Theorem 1, corresponding to n permutations of $(x_1,\ \dots,\ x_n)$ and corresponding permutations of $a_1,\ \dots,\ a_n$.

Remark 2. For $n=3$, $m=1$, the estimate (2) is same as obtained in [7] (Theorem 1). For $n=3$, $m=2$, $f_{11}=f_{21}$, the estimate (2) is not comparable to as obtained in [7] (Theorem 2). For $n=2$ and m upto 2 some results are given in [8].

Theorem 2. *Let the following inequality be satisfied*

$$u(x) \le a(x) + b(x) \sum_{r=1}^{m} E^r(x, u) \tag{7}$$

for all $x \in N_0^n$, *where*: (i) $a(x) > 0$ *and nondecreasing*, (ii) $b(x) \ge 1$. *Then*

$$u(x) \le a(x)\, b(x) \prod_{s_1=0}^{x_1-1} [1 + \sum_{r=1}^{m} \Delta E_{s_1}^r(s_1, x_2, \dots, x_n, b)] . \tag{8}$$

Proof. From the assumptions on a and b, inequality (7) can be written as

$$v(x) \le 1 + \sum_{r=1}^{m} E^r(x, bv)$$

where $v = u/ab$. The rest of the proof is same as in Theorem 1.

Remark 3. For inequality (1) under the assumptions of Theorem 1 we have from Theorem 2

$$u(x) \le \sum_{i=1}^{n} a_i(x_i) \prod_{s_1=0}^{x_1-1} [1 + \sum_{r=1}^{m} \Delta E_{s_1}^r(s_1, x_2, \dots, x_n, 1)] . \tag{9}$$

Remark 4. There are n different conclusions possible of Theorem 2 and also for (9).

Remark 5. If $a_1 = k$ (constant), then (2) and (9) are same. In the general case (2) and (9) are not comparable. In applications (9) require less work to compute the estimates than (2).

Theorem 3. *Let the inequality* (1) *be satisfied where* $a_i(x_i)$ *be same as in Theorem* 1 *and*

$$f_{ii}(x) = f_i(x) \;\; 1 \le i \le m \qquad f_{i+1,i}(x) = f_{i+2,i}(x) = \dots = f_{m,i}(x) = g_i(x) \tag{10}$$

for all $x \in N_0^n$ *and* $1 \leq i \leq m-1$. *Then*

$$u(x) \leq P_i(x) \qquad i=1,\ 2 \tag{11}$$

where

$$P_1(x) = [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{s_1=0}^{x_1-1} [1 + \frac{\Delta a_1(s_1)}{a_1(s_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{s_2=0}^{x_2-1} \cdots \sum_{s_n=0}^{x_n-1} (\sum_{r=1}^{m} f_r(s) \bigcup_{i=1}^{m-1} g_i(s))] \tag{12}$$

$$P_2(x) = \sum_{i=1}^{n} a_i(x_i) + \sum_{s=0}^{x-1} (f_1(s) \cup g_1(s)) P_1(s)\ . \tag{13}$$

In (12) and (13) the term $\sum_{r=1}^{r_1} f_r(x) \bigcup_{i=1}^{r_2} g_i(x)$ represents the sum of all functions except when $f_k(x) = g_l(x)$ for some $1 \leq k \leq r_1$, $1 \leq l \leq r_2$ then $g_l(x)$ is taken to be zero, also $\bigcup_{i=1}^{0} = 0$.

Proof. Inequality (1), with (10) is equivalent to the following system

$$u_1(x) \leq \sum_{i=1}^{n} a_i(x_i) + \sum_{s=0}^{x-1} [f_1(s) u_1(s) + g_1(s) u_2(s)] \tag{14}$$

$$u_{j-1}(x) = \sum_{s=0}^{x-1} [f_{j-1}(s) u_1(s) + g_{j-1}(s) u_j(s)] \qquad 3 \leq j \leq m \tag{15$_j$}$$

$$u_m(x) = \sum_{s=0}^{x-1} f_m(s) u_1(s)\ . \tag{16}$$

Define $\phi_1(x)$, $\phi_{j-1}(x)$ $(3 \leq j \leq m)$, $\phi_m(x)$ as the right members of (14), (15)$_j$ $(3 \leq j \leq m)$, (16) respectively. Then, we find

$$\Delta^n \phi_{1x}(x) \leq f_1(x) \phi_1(x) + g_1(x) \phi_2(x) \tag{17}$$

$$\Delta^n \phi_{j-1x}(x) \leq f_{j-1}(x) \phi_1(x) + g_{j-1}(x) \phi_j(x) \qquad 3 \leq j \leq m \tag{18$_j$}$$

$$\Delta^n \phi_{mx}(x) \leq f_m(x) \phi_1(x) \tag{19}$$

where $\phi_1(x)$ satisfies (4) and $\phi_j(x)$ $(2 \le j \le m)$, together with all mixed differences upto order $n-1$ are zero at $x_i = 0$ $(1 \le i \le n)$. Adding (17), $(18)_j$ $(3 \le j \le m)$, (19), to find

$$\sum_{r=1}^{m} \Delta^n \phi_{rx}(x) \le \sum_{r=1}^{m} f_r(x)\, \phi_1(x) + \sum_{r=1}^{m-1} g_r(x)\, \phi_{r+1}(x)$$

and hence

$$\sum_{r=1}^{m} \Delta^n \phi_{rx}(x) \le \Big(\sum_{r=1}^{m} f_r(x) \bigcup_{i=1}^{-1} g_i(x)\Big)\Big(\sum_{r=1}^{m} \phi_r(x)\Big)\,.$$

Now following the proof of Theorem 1, we obtain $\sum_{r=1}^{m} \phi_r(x) \le P_1(x)$. Using this in (17), we find

$$\Delta^n \phi_{1x}(x) \le (f_1(x) \cup g_1(x))\, P_1(x)$$

and now once again as in Theorem 1, we get $\phi_1(x) \le P_2(x)$.

Since $u(x) = u_1(x) \le \phi_1(x) \le \sum_{r=1}^{m} \phi_r(x)$, (11) follows.

Remark 6. As in Theorem 1 there are $n!$ different conclusions possible for Theorem 3.

Remark 7. For $n=3$, $m=1$ Theorem 3 is same as given in [7] (Theorem 1). For $n=3$, $m=2$, $f_1 = g_1$ Theorem 3 is same as Theorem 2 of [7]. This also covers some results given in [8] for $n=2$, m upto 2.

Remark 8. $P_1(x)$ and $P_2(x)$ cannot be compared.

Our next result is the discrete analogue of Willet's inequality [11] as discussed in $[2]_2$ for $n=1$.

Theorem 4. *Let the following inequality be satisfied*

$$u(x) \le a(x) + \sum_{i=1}^{m} g_i(x) \sum_{s=0}^{x-1} h_i(s)\, u(s) \tag{20}$$

for all $x \in N_0^n$, *where*: (i) $a(x) > 0$ *and nondecreasing*, (ii) $g_i(x) \ge 1$ *for* $1 \le i \le m$

and nondecreasing for $2 \le i \le m$. *Then*

$$u(x) \le F_m a(x) \tag{21}$$

where

$$F_0 w = w \qquad F_k w = w(F_{k-1} g_k) \prod_{s_1=0}^{x_1-1} [1 + \sum_{s_2=0}^{x_2-1} \dots \sum_{s_n=0}^{x_n-1} h_k(s) F_{k-1} g_k(s)] \tag{22}$$

for $k = 1, \dots, m$.

Proof. The proof is by finite induction. For $m = 1$, we find from Theorem 2

$$u(x) \le a(x) g_1(x) \prod_{s_1=0}^{x_1-1} [1 + \sum_{s_2=0}^{x_2-1} \dots \sum_{s_n=0}^{x_n-1} h_1(s) g_1(s)] = F_1 a(x) .$$

Now, assume that the result is true for some k such that $1 \le k \le m-1$, then for $k+1$ we are given

$$u(x) \le a(x) + g_{k+1}(x) \sum_{s=0}^{x-1} h_{k+1}(s) u(s) + \sum_{i=1}^{k} g_i(x) \sum_{s=0}^{x-1} h_i(s) u(s)$$

and we find

$$u(x) \le F_k a^*(x) \quad \text{where} \quad a^*(x) = a(x) + g_{k+1}(x) \sum_{s=0}^{x-1} h_{k+1}(s) u(s).$$

Thus we find

$$\frac{u}{aF_k g_{k+1}} \le 1 + \sum_{s=0}^{x-1} h_{k+1} F_k g_{k+1} \frac{u}{aF_k g_{k+1}} .$$

Now an application of Theorem 2 yields

$$u(x) \le a(x) F_k g_{k+1}(x) \prod_{s_1=0}^{x_1-1} [1 + \sum_{s_2=0}^{x_2-1} \dots \sum_{s_n=0}^{x_n-1} h_{k+1}(s) F_k g_{k+1}(s)] = F_{k+1} a(x) .$$

Hence the result follows for all m.

Corollary 5. *Let the inequality* (20) *be satisfied for all* $x \in N_0^n$, *where*: (i) $a(x) > 0$ *and nondecreasing*, (ii) $g_i(x) \ge 1$ *for all* $1 \le i \le m$. *Then*

$$u(x) \le a(x) \prod_{i=1}^{m} g_i(x) \prod_{s_1=0}^{x_1-1} [1 + \sum_{s_2=0}^{x_2-1} \dots \sum_{s_n=0}^{x_n-1} \sum_{r=1}^{m} h_r(s) \prod_{i=1}^{m} g_i(s)] .$$

Proof. Inequality (20) can be written as

$$u(x) \le a(x) + \prod_{i=1}^{m} g_i(x) \sum_{s=0}^{x-1} \left(\sum_{i=1}^{m} h_i(s)\right) u(s)$$

and now, the result follows from Theorem 2.

3 - Nonlinear inequalities

Our first result here is the n independent variable generalization of Theorem 6 and Corollary 7 as given in [2]$_2$. We shall consider the following inequality

$$u(x) \le p(x) \left[c + \sum_{r=1}^{m} H^r(x, u)\right] \tag{23}$$

where

$$H^r(x, u) = \sum_{x^1=0}^{x-1} f_{r1}(x^1)\, u^{\alpha_{r1}}(x^1) \ldots \sum_{x^r=0}^{x^{r-1}-1} f_{rr}(x^r)\, u^{\alpha_{rr}}(x^r)$$

and α_{ri} $(1 \le i \le r,\ 1 \le r \le m)$, are nonnegative real numbers, the constant $c > 0$. We shall denote $\alpha_r = \sum_{i=1}^{r} \alpha_{ri}$ and $\alpha = \max_{1 \le r \le m} \alpha_r$.

Theorem 6. *Let the inequality* (23) *be satisfied for all* $x \in N_0^n$. *Then*

$$u(x) \le cp(x) \prod_{s_1=0}^{x_1-1} \left[1 + \sum_{r=1}^{m} \Delta H_{s_1}^r(s_1, x_2, \ldots, x_n, p)\, c^{\alpha_r - 1}\right] \qquad \text{if } \alpha = 1 \tag{24}$$

$$u(x) \le p(x) \left[c^{1-\alpha} + (1-\alpha) \sum_{r=1}^{m} H^r(x, p)\, c^{\alpha_r - \alpha}\right]^{1/1-\alpha} \qquad \text{if } \alpha < 1\,. \tag{25}$$

Proof. Define $\phi(x) = c + \sum_{r=1}^{m} H^r(x, u)$, then since $\phi(x)$ is nondecreasing, we find

$$\Delta^n \phi_x(x) \le \sum_{r=1}^{m} \Delta^n H_x^r(x, p\phi) \le \sum_{r=1}^{m} \Delta^n H_x^r(x, p)\, \phi^{\alpha_r}(x)\,.$$

Thus, it follows from $\phi(x) \geq c$ that

$$\Delta^n \phi_x(x) \leq [\sum_{r=1}^{m} \Delta^n H_x^r(x,\ p)\, c^{\alpha_r - \alpha}]\, \phi^\alpha(x)\ .$$

Hence, we find

$$\frac{\Delta^{n-1}\phi_{x_1 \dots x_{n-1}}(x_1,\ \dots,\ x_{n-1},\ x_n+1)}{\phi^\alpha(x_1,\ \dots,\ x_{n-1},\ x_n+1)} - \frac{\Delta^{n-1}\phi_{x_1 \dots x_{n-1}}(x)}{\phi^\alpha(x)} \leq \sum_{r=1}^{m} \Delta^n H_x^r(x,\ p)\, c^{\alpha_r - \alpha}\ .$$

Keeping $x_1,\ \dots,\ x_{n-1}$ fixed and setting $x_n = s_n$ and summing over $s_n = 0,\ 1,\ \dots,\ x_n - 1$, we obtain

$$\frac{\Delta^{n-1}\phi_{x_1 \dots x_{n-1}}(x)}{\phi^\alpha(x)} \leq \sum_{r=1}^{m} \Delta^{n-1} H_{x_1 \dots x_{n-1}}^r(x,\ p)\, c^{\alpha_r - \alpha}\ .$$

Repeating the above procedure, we obtain

$$\frac{\Delta\phi_{x_1}(x)}{\phi^\alpha(x)} \leq \sum_{r=1}^{m} \Delta H_{x_1}^r(x,\ p)\, c^{\alpha_r - \alpha}\ .$$

Now, since

$$\frac{\Delta\phi_{x_1}^{1-\alpha}(x)}{1-\alpha} = \int_{x_1}^{x_1+1} \frac{d\phi(s_1,\ x_2,\ \dots,\ x_n)}{\phi^\alpha(s_1,\ x_2,\ \dots,\ x_n)} \leq \frac{\Delta\phi_{x_1}(x)}{\phi^\alpha(x)}$$

we find

$$\Delta\phi_{x_1}^{1-\alpha}(x) \leq (1-\alpha) \sum_{r=1}^{m} \Delta H_{x_1}^r(x,\ p)\, c^{\alpha_r - \alpha}\ .$$

Hence, we obtain

$$\phi(x) \leq [c^{1-\alpha} + (1-\alpha) \sum_{r=1}^{m} H^r(x,\ p)\, c^{\alpha_r - \alpha}]^{1/1-\alpha}\ .$$

Since $u(x) \leq p(x)\,\phi(x)$, (25) follows. The case $\alpha = 1$ follows as in Theorem 1.

Next we shall require following class of functions.

Def. A function w:$[0,\ \infty)$ is said to belong to the class S if: (i) $w(u)$ is positive, nondecreasing and continuous for $u \geq 0$, (ii) $(1/v)\,w(u) \leq w(u/v)$ for $u \geq 0$, $v \geq 1$.

Theorem 7. *Let the following inequality be satisfied*

$$u(x) \leq a(x) + \sum_{r=1}^{m} E^r(x,\ u) + \sum_{i=1}^{l} g_i(x) \sum_{s=0}^{x-1} h_i(s)\, W_i(u(s)) \tag{26}$$

for all $x \in N_0^n$, *where*: (i) $a(x) \geq 1$ *and nondecreasing*, (ii) $g_i(x) \geq 1$ $1 \leq i \leq l$, (iii) $W_i \in S$, $1 \leq i \leq l$. *Then*

$$u(x) \leq a(x)\, \psi(x)\, e(x) \prod_{i=1}^{l} F_i(x) \tag{27}$$

where

$$e(x) = \prod_{i=1}^{l} g_i(x) \qquad \psi(x) = \prod_{s_1=0}^{x_1-1} [1 + \sum_{r=1}^{m} \Delta E_{s_1}^r(s_1,\ x_2,\ \ldots,\ x_n,\ e)]$$

$$F_k(x) = G_k^{-1}[G_k(1) + \sum_{s=0}^{x-1} h_k(s)\, \psi(s)\, e(s) \prod_{j=1}^{k-1} F_j(s)] \qquad F_0(x) = 1 \qquad 1 \leq k \leq l$$

$$G_k(\theta) = \int_{\theta_0}^{\theta} \frac{\mathrm{ds}}{W_k(s)} \qquad 0 < \theta_0 \leq \theta$$

as long as

$$G_k(1) + \sum_{s=0}^{x-1} h_k(s)\, \psi(s)\, e(s) \prod_{j=1}^{k-1} F_j(s) \in \mathrm{Dom}(G_k^{-1})\,.$$

Proof. Since $g_i \geq 1$, $1 \leq i \leq l$, we find from (26)

$$\frac{u(x)}{e(x)} \leq a^*(x) + \sum_{r=1}^{m} E^r\left(x,\ \frac{eu}{e}\right) \qquad \text{where} \qquad a^*(x) = a(x) + \sum_{i=1}^{l} \sum_{s=0}^{x-1} h_i(s)\, W_i(u(s))\,.$$

Since $a^*(x)$ is nondecreasing, from Theorem 2 it follows that

$$\frac{u(x)}{e(x)} \leq a^*(x)\, \psi(x)$$

and hence on using the definition of class S

$$v(x) \leq 1 + \sum_{i=1}^{l} \sum_{s=0}^{x-1} h_i(s)\, e(s)\, \psi(s)\, W_i(v(s))$$

where
$$v(x)=\frac{u(x)}{a(x)\psi(x)e(x)}.$$

Thus it is sufficient to show that $v(x)\le\prod_{i=1}^{l}F_i(x)$ and this we shall prove by finite induction. For $l=1$, we have

$$v(x)\le 1+\sum_{s=0}^{x-1}h_1(s)e(s)\psi(s)W_1(v(s)).$$

Let $\phi_1(x)$ be the right member of the above inequality, then on using nondecreasing nature of W_1, we find

$$\Delta^n\phi_{1x}(x)\le h_1(x)e(x)\psi(x)W_1(\phi_1(x))$$

and hence

$$\frac{\Delta^{n-1}\phi_{1x_1\dots x_{n-1}}(x_1,\dots,x_{n-1},x_n+1)}{W_1(\phi_1(x_1,\dots,x_{n-1},x_n+1))}-\frac{\Delta^{n-1}\phi_{1x_1\dots x_{n-1}}(x)}{W_1(\phi_1(x))}\le h_1(x)e(x)\psi(x).$$

Keeping $x_1,\dots,x_{n-1}$ fixed and setting $x_n=s_n$ and summing over $s_n=0,1,\dots,x_n-1$, we obtain

$$\frac{\Delta^{n-1}\phi_{1x_1\dots x_{n-1}}(x)}{W_1(\phi_1(x))}\le\sum_{s_n=0}^{x_n-1}h_1(x_1,\dots,x_{n-1},s_n)e(x_1,\dots,x_{n-1},s_n)\psi(x_1,\dots,x_{n-1},s_n).$$

Repeating the procedure, we obtain

$$\text{(28)}\qquad \frac{\Delta\phi_{1x_1}(x)}{W_1(\phi_1(x))}\le\sum_{s_2=0}^{x_2-1}\cdots\sum_{s_n=0}^{x_n-1}h_1(x_1,s_2,\dots,s_n)e(x_1,s_2,\dots,s_n)\psi(x_1,s_2,\dots,s_n).$$

Now, from the definition of G_1, we have

$$\text{(29)}\qquad G_1(\phi_1(x_1+1,x_2,\dots,x_n))-G_1(\phi_1(x))$$
$$=\int_{x_1}^{x_1+1}\frac{d\phi_1(s,x_2,\dots,x_n)}{W_1(\phi_1(s,x_2,\dots,x_n))}\le\frac{\Delta\phi_{x_1}(x)}{W_1(\phi_1(x))}.$$

Using (28) in (29) and summing over with respect to x_1 from 0 to $x_1 - 1$, we obtain

$$\phi_1(x) \le G_1^{-1}[G_1(1) + \sum_{s=0}^{x-1} h_1(s)\, e(s)\, \psi(s)]$$

and hence $v(x) \le F_1(x)$. Now assume that the result is true for some k such that $1 \le k \le l-1$, then we are given

$$v(x) \le [1 + \sum_{s=0}^{x-1} h_{k+1}(s)\, e(s)\, \psi(s)\, W_{k+1}(v(s))] + \sum_{i=1}^{k}\sum_{s=0}^{x-1} h_i(s)\, e(s)\, \psi(s)\, W_i(v(s)) \,.$$

Since the term inside the bracket is nondecreasing, we find

$$v(x) \le [1 + \sum_{s=0}^{x-1} h_{k+1}(s)\, e(s)\, \psi(s)\, W_{k+1}(v(s))] \prod_{i=1}^{k} F_i(x) \qquad \text{or}$$

$$\frac{v(x)}{\prod_{i=1}^{k} F_i(x)} \le 1 + \sum_{s=0}^{x-1} h_{k+1}(s)\, e(s)\, \psi(s) \prod_{i=1}^{k} F_i(s)\, W_{k+1} \frac{v(s)}{\prod_{i=1}^{k} F_i(s)}$$

and from this $v(s) \le \prod_{i=1}^{k+1} F_i(x)$ follows on using the same arguments as for $l=1$. Thus the result is true for all k.

Theorem 8. *In addition to the hypothesis of Theorem 7 let $g_i(x)$, $1 \le i \le l$, be nondecreasing. Then*

$$u(x) \le a(x)\, \psi_1(x) \prod_{i=1}^{l} F_i(x)$$

where

$$\psi_1(x) = \prod_{s_1=0}^{x_1-1} [1 + \sum_{r=1}^{m} \Delta E_{s_1}^{r}(s_1,\ x_2,\ \ldots,\ x_n,\ 1)]$$

$$F_k(x) = g_k(x)\, G_k^{-1}[G_k(1) + \sum_{s=0}^{x-1} h_k(s)\, \psi_1(s)\, g_k(s) \prod_{i=1}^{k-1} F_i(s)] \qquad F_0(x) = 1,\ 1 \le k \le l$$

as long as $$G_k(1) + \sum_{s=0}^{x-1} h_k(s)\, \psi_1(s)\, g_k(s) \prod_{i=1}^{k-1} F_i(s) \in \mathrm{Dom}(G_k^{-1}) \,.$$

Theorem 9. *Let the following inequality be satisfied*

$$u(x) \le a(x) + \sum_{r=1}^{m} E^r(x,\ u) + \sum_{i=1}^{l} E^i(x,\ W(u)) \tag{30}$$

where: (i) $a(x) \ge 1$ *and nondecreasing*, (ii) $W \in S$. *Then*

$$u(x) \le a(x)\,\psi_1(x)\,G^{-1}[G(1) + \sum_{i=1}^{l} E^i(x,\ \psi_1)] \tag{31}$$

where $\psi_1(x)$ *is same as in Theorem* 8 *and the term inside the bracket of* (31) $\in \mathrm{Dom}(G^{-1})$.

The proofs of Theorem 8 and Theorem 9 are similar to the proof of Theorem 7.

Theorem 10. *Let the inequality* (30) *be satisfied, where*: (i) $a(x)$ *is positive and nondecreasing*, (ii) W *is positive, continuous, nondecreasing and submultiplicative. Then*

$$u(x) \le a(x)\,\psi_1(x)\,G^{-1}[G(1) + \sum_{i=1}^{l} E^i(x,\ \frac{W(a\psi_1)}{a})] \tag{32}$$

where $\psi_1(x)$ *is same as in Theorem* 8 *and the term inside the bracket of* (32) $\in \mathrm{Dom}(G^{-1})$.

Proof. Inequality (30) can be written as

$$u(x) \le a^*(x) + \sum_{r=1}^{m} E^r(x,\ u) \qquad \text{where} \qquad a^*(x) = a(x) + \sum_{i=1}^{l} E^i(x,\ W(u))\,.$$

Since $a^*(x)$ is nondecreasing, from Theorem 2 we find

$$u(x) \le a^*(x)\,\psi_1(x)\,.$$

Now, since $a(x)$ is positive and nondecreasing, we obtain

$$\frac{u(x)}{a(x)\,\psi_1(x)} \le 1 + \sum_{i=1}^{l} E^i(x,\ W(\frac{u}{a\psi_1}\,a\psi_1)/a)\,. \tag{33}$$

Let $\psi(x)$ be the right side of (33), then

$$\Delta^n \phi_x(x) = \sum_{i=1}^{l} \Delta^n E_x^i(x,\ W(\frac{u}{a\psi_1} a\psi_1)/a)\ .$$

now using the fact that W is nodecreasing and submultiplicative, we get

$$\frac{\Delta^n \phi_x(x)}{W(\phi(x))} \leq \sum_{i=1}^{l} \Delta^n E_x^i(x,\ W(a\psi_1)/a)\ .$$

Using the same arguments as in Theorem 7, we find

$$\phi(x) \leq G^{-1}[G(1) + \sum_{i=1}^{l} E^i(x,\ W(a\psi_1)/a]$$

and from this the result follows.

4 - Some applications

The results obtained in **2** and **3** can be directly used to prove the uniqueness and continuous dependence for the solutions of discrete versions of hyperbolic partial differential equations involving n independent variables of more general type than given in [7], [8] and [9], since the arguments are similar the details are not repeated here. To show the importance of our results we shall provide an upper bound on the solutions of difference equation of the form

$$\Delta^n u_x(x) = F(x,\ u(x),\ \sum_{s=0}^{x-1} K(x,\ s,\ u(s))) \tag{34}$$

together with the given suitable boundary conditions $u(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$, $1 \leq i \leq n$.

The function F and K are defined on their respective domains of definitions and

$$|F(x,\ u(x),\ v(x))| \leq f_{11}(x)\,|u(x)| + f_{12}(x)\,|v(x)| \tag{35}$$

$$|K(x,\ s,\ u(s))| \leq f_{22}(s)\,|u(s)| \tag{36}$$

where f_{11}, f_{12}, f_{22} are same as appear in (1).

Any solution $u(x)$ of (34) satisfying the boundary conditions is also a solution of the Volterra difference equation

$$u(x) = g(x) + \sum_{x^1=0}^{x-1} F(x^1,\ u(x^1),\ \sum_{x^2=0}^{x^1-1} K(x^1,\ x^2,\ u(x^2))) \tag{37}$$

where $g(x)$ takes care of the boundary conditions.

Using (35), (36) in (37), to obtain

$$|u(x)| \leq |g(x)| + \sum_{x^1=0}^{x-1} [f_{11}(x^1)\,|u(x^1)| + f_{12}(x^1) \sum_{x^2=0}^{x^1-1} f_{22}(x^2)\,|u(x^2)|]\ .$$

If $|g(x)| \leq a(x)$, where $a(x)$ is same as in Theorem 2, we find from (8)

$$|u(x)| \leq a(x) \prod_{x_1^1=0}^{x_1-1} [1 + \sum_{x_2^1=0}^{x_2-1} \cdots \sum_{x_n^1=0}^{x_n-1} [f_{11}(x^1) + \sum_{x^2=0}^{x^1-1} f_{22}(x^2)]]\,. \tag{38}$$

If $|g(x) \leq \sum_{i=1}^{n} a_i(x_i)$ where $a_i(x_i)$ are same as in Theorem 1, we find from (2)

$$|u(x)| \leq [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{x_1^1=0}^{x_1-1} [1 + \frac{\Delta a_1(x_1^1)}{a_1(x_1^1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{x_2^1=0}^{x_2-1} \cdots \sum_{x_n^1=0}^{x_n-1} (f_{11}(x^1) + f_{12}(x^1) \sum_{x^2=0}^{x^1-1} f_{22}(x^2))] \tag{39}$$

also, in case $f_{11}(x) = f_{12}(x)$, from Theorem 3 it follows that

$$|u(x)| \leq P_i(x) \qquad i = 1,\ 2 \tag{40}$$

where

$$P_1(x) = [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{x_1^1=0}^{x_1-1} [1 + \frac{\Delta a_1(x_1^1)}{a_1(x_1^1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{x_2^1=0}^{x_2-1} \cdots \sum_{x_n^1=0}^{x_n-1} (f_{11}(x^1) + f_{22}(x^1))]$$

$$P_2(x) = \sum_{i=1}^{n} a_i(x_i) + \sum_{x^1=0}^{x-1} f_{11}(x^1)\, P_1(x^1)\ .$$

The estimate (39) cannot be obtained from (38) except when $|g(x)| = \text{const}$, also (40) cannot be obtained from (39). For $n = 3$, (40) is same as obtained in [7]. It appears that in general it is not possible to compare any one of the estimates obtained here, however for a particular situation we have more flexibility to use these results.

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Summary

The paper deals with some new discrete inequalities in n independent variables. Some applications are also given.

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