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On annihilator ideals (IV) (**)

Dedicated to prof. F. Kasch for his sixtieth birthday

Introduction

In this sequel to [**10**]$_{7,8}$ further characteristic properties of von Neumann regular rings are given in terms of annihilators. Strongly regular rings are also characterized. It is proved that a commutative ring A is von Neumann regular iff any one of the following conditions is satisfied: (a) Every essential ideal of A which is an annihilator ideal is p-injective. (b) There exists a faithful cyclic A-module such that the annihilator of each of its elements is p-injective. (c) Every principal ideal of A is a flat annihilator. (d) A contains a maximal ideal such that the annihilator of each of its elements is p-injective. Conditions for rings to be regular with non-zero socle are given. A generalization of right duo rings (introduced in [**9**]) is studied in connection with strongly regular rings.

1 – Throughout, A represents an associative ring with identity and A-modules are unital. Z, J denote respectively the left singular ideal and the Jacobson radical of A. An ideal of A always means a two-sided ideal and A is called right (resp. left) duo if every right (resp. left) ideal of A is an ideal. A right (left) ideal of A is called reduced if it contains no non-zero nilpotent element.

Recall that a left A-module M is p-injective (resp. f-injective) if, for any principal (resp. finitely generated) left ideal I of A, every left A-homomorphism

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of I into M extends to one of A into M. In general, p-injectivity and flatness are distinct concepts in spite of the following characteristic properties of von Neumann regular rings: (1) All modules are flat. (2) All modules are p-injective. However, if K is a maximal left ideal of A which is an ideal, then A/K_A is injective iff A/K_A is p-injective iff ${}_AA/K$ is flat ([**10**]$_5$, Lemma 1). Also, if I is a p-injective left ideal of A, then ${}_AA/I$ is flat. A is called left p-injective if ${}_AA$ is p-injective. A theorem of M. Ikeda-T. Nakayama asserts that A is left p injective iff every principal right ideal of A is a right annihilator ideal.

Our first result is concerned with some new characterizations of commutative regular rings in terms of annihilators.

THEOREM 1. *The following conditions are equivalent for a commutative ring A:*

(1) *A is von Neumann regular.*

(2) *every essential ideal of A which is an annihilator ideal is p-injective.*

(3) *there exists a faithful cyclic A-module C such that the annihilator of any element of C is p-injective.*

(4) *every principal ideal of A is a flat annihilator.*

(5) *A contains a maximal ideal K such that the annihilator of every element of K is p-injective.*

PROOF. (1) implies (2) evidently.

Assume (2). Suppose that $Z \neq 0$. Then there exists $0 \neq z \in Z$ such that $z^2 = 0$ by [**10**]$_6$ (Lemma 7). Now $E = l(z)$ is an essential ideal of A and since E is a p-injective A-module, if i: $Az \to E$ is the inclusion map, there exist $u \in E$ such that $z = i(z) = zu$. Therefore $1 - u \in r(z) = l(z) = E$ implies that $1 \in E$, whence $z = 0$, a contradiction! This proves that $Z = 0$ and since A is a p-injective ring, then (2) implies (3) by [**10**]$_3$ (Theorem 1).

Assume (3). Then A is p-injective ring (because A is the annihilator of 0 in C). Suppose that A contains a non-zero element u such that $u^2 = 0$. If $C = Ac$, then $uc \neq 0$ (because C is faithful) and since $I = l(uc)$ is a p-injective left A-module, the inclusion map $Au \to I$ yields $u = uv = vu$ for some $v \in I$, whence $1 - v \in l(u) \subseteq I$. Thus $1 \in I$ which implies that $uc = 0$, a contradiction! This proves that A is a reduced ring and since A is p-injective, then A is regular and (3) implies (4).

Assume (4). If $b \in A$ such that $b^2 = 0$, then $A/l(b)$ ($\approx Ab$) is a flat A-module and since $b \in l(b)$, then $b = bd$ for some $d \in l(b)$ which yields $1 - d \in r(b) = l(b)$, whence $1 \in l(b)$ and $b = 0$. This proves that A must be reduced and (4) implies (5) by [**10**]$_3$ (Theorem 1).

Assume (5). Let K be the maximal left ideal of A. If $k \in K$ such that $k^2 = 0$, since $l(k)$ is p-injective, then $k = kt$ for some $t \in l(k)$ which implies that $1 \in l(k)$, whence $k = 0$. This proves that K is a reduced ideal. If K is essential in A, then A is also reduced (cf. [**10**]$_6$, Lemma 7). Now suppose that $K \oplus U = A$ for some minimal ideal U of A. If $K = Ae$, $U = A(1-e)$, then for any $0 \neq b \in A$ such that $b^2 = 0$, $b = be + b(1-e)$ and $(be)^2 = 0$ implies $be = 0$. Therefore $U = Ab$ implies $U^2 = 0$, a contradiction! This proves that A is reduced again. In any case, (5) implies (1) by [**10**]$_3$ (Theorem 1).

2 – Following [7], a left A-module M is called *semi-simple* if the intersection of all maximal left submodules is zero. Then A is semi simple iff $J = 0$. A is called a *left V-ring* if every simple left A-module is injective. A well-known result of I. Kaplansky guarantees that a commutative ring is von Neumann regular iff it is a V-ring. For the past few years, von Neumann regular rings, V-rings and generalizations have been extensively studied (cf. for example, [**1**]-[**4**], [**6**]–[**8**]).

The next result is motivated by [**10**]$_4$ (Question).

Proposition 2. *If A is semi-simple left semi-hereditary whose maximal left ideals are f-injective such that there exists either an injective maximal right ideal or a finitely generated f-injective maximal right ideal which is an ideal of A, then A is von Neumann regular with non-zero socle.*

Proof. First suppose there exist a finitely generated f-injective maximal right ideal F of A which is an ideal of A. Then $A = F \oplus U$, where U is minimal right ideal of A. Since A is semi-prime, then F is generated by a central idempotent which implies that ${}_AU$ is a minimal left ideal of A. Since A/F_A is flat, then ${}_AA/F$ is injective and since ${}_AF$ is f-injective, then ${}_AA$ is f-injective. Now A being left semi-hereditary implies that A is von Neumann regular. If we now suppose that A contains an injective maximal right ideal K which is not an ideal of A, then $A = K \oplus V$, where V is a minimal right ideal of A. If I is a finitely generated proper left ideal, M a maximal left ideal containing I, i: $I \to M$ the inclusion map, since ${}_AM$ is f-injective, there exists $u \in M$ such that $i(b) = by$ for all $b \in I$. Thus $1 - u \in r(I)$ which implies that $r(I) \neq 0$. Now if $y \in Y$, the right singular ideal of A, $r(1 - ay) = 0$ for all $a \in A$ which implies that $1 - ay$ is left invertible in A. This proves that $y \in J = 0$, whence $Y = 0$. Since $A = AK$, the maximal right quotient ring Q of A is a projective right A-module by [5] (Theorem 4) and therefore A_A is a direct summand of Q_A which implies that $A = Q$ is right self-injective regular. This proves the proposition.

In [9], weakly right duo rings, which form a non-trivial generalization of right duo rings, are introduced. A is called *weakly right duo* if, for every $a \in A$, there exists a positive integer n such that a ^nA is an ideal of A. We here write «A is WRD» in case A is weakly right duo.

In ring theory, the quotient ring A/J plays an important role (cf. for example, [2] and [4]). We now consider a significant property of WRD rings.

Proposition 3. *If A is a* WRD *ring, then A/J is reduced.*

Proof. Since A is WRD, then it is easily seen that $B = A/J$ is also WRD. Suppose there exists $0 \neq c \in B$ such that $c^2 = 0$. We claim that cB is a right nilideal of B. If not, there is an element cb which is not nilpotent for some $b \in B$. Since B is WRD, there is a positive integer m such that $(cb)^m B$ is an ideal of B. If $m = 1$, cbB is an ideal of B which implies that $(cb)^2 B$ is an ideal of B. Now $(cb)^3 = cb(cb)^2 = c(cb)^2 d$ for some $d \in B$ which yields $(cb)^3 = 0$ (because $c^2 = 0$), contradicting cb non-nilpotent. If $m > 1$, since $(cb)^m B$ is an ideal of B, then $(cb)^{m+1} = cb(cb)^m = c(cb)^m u$ for some $u \in B$, whence $(cb)^{m+1} = 0$ again contradicting cb non-nilpotent. This proves that cB must be a right nilideal which is therefore contained in the Jacobson radical of B. But the Jacobson radical of B ($= A/J$) is zero which yields $c = 0$. This proves that B must be a reduced ring.

Corollary 3.1. *If A is right duo, then A/J is reduced.*

Corollary 3.2. *If A is* WRD *whose simple right modules are flat, then A/J is strongly regular.*

Corollary 3.3. *If A is* WRD *right self-injective, then A/J is a right and left self-injective strongly regular ring* (apply $[2]_2$, Corollary 19.28).

Corollary 3.4. *If A is* WRD *right GQ-injective, then A/J is strongly regular* (apply $[10]_6$, Proposition 1).

Corollary 3.5. *A prime semi-simple* WRD *ring is a right Ore domain.*

Remark 1. *If A* is a WRD ring containing an injective maximal right ideal, then A is right self-injective such that A/J is left and right self-injective strongly regular.

The relationship between right duo rings and WRD rings has an analogue between p-injectivity and GP-injectivity. A right A-module M is called *GP-injective* (generalized *p-injective*) if, for any $a \in A$, there exists a positive integer m such that any right A-homomorphism of $a^m A$ into M extends to one of A into M. Recall that A is π-regular if, any $a \in A$, there exist a positive integer n and an element $u \in A$ such that $a^n = a^n u a^n$.

Proposition 4. *The following conditions are equivalent for a* WRD *ring A:*
(1) *A is π-regular.*
(2) *Every right A-module is GP-injective.*

Proof. Assume (1). If M is a right A-module, $a \in A$, since $a^n = a^n u a^n$ for some positive integer n and $u \in A$, then $a^n A$ is a direct summand of A_A which shows that any right A-homomorphism of $a^n A$ into M extends to one of A into M. Therefore (1) implies (2).

Assume (2). Let $b \in A$. Since A is WRD, there exists a positive integer k such that $B = b^k A$ is an ideal of A. Since B_A is GP-injective, there exists a positive integer n such that if i: $b^{kn} A \to B$ is the inclusion map, then $b^{kn} = i(b^{kn}) = cb^{kn}$ for some $c \in B$. Now $c = b^k d$, $d \in A$, and $b^{kn} = c^n b^{kn} = (b^k d)^n b^{kn}$ and since B is an ideal of A, then $b^{kn} = b^{kn} v b^{kn}$, $v \in A$, which proves that (2) implies (1).

Proposition 3 yields the following nice remark.

Remark 2. If A is WRD π-regular, then A/J is a strongly regular ring.

We now consider another result on annihilator ideals.

Proposition 5. *The following conditions are equivalent:*
(1) *For any $a \in A$, there exist an idempotent e such that $l(a) = l(e)$ and $r(a) = r(e)$.*
(2) *For any $a \in A$, there exist an idempotent u and a non-zero-divisor c such that $a = uc = cu$.*

Proof. Assume (1). If $a \in A$, there exists an idempotent e such that $l(a) = l(e)$ and $r(a) = r(e)$. Then $l(a) = A(1-e)$ implies $a = ea$. Set $c = a + 1 - e$. Then $a = ec$ and if $u \in r(c)$, $au = -(1-e)u$ implies $eau = 0$, whence $au = 0$, yielding $u \in r(a) = r(e) = (1-e)A$. Since $u = eu$, then $u \in eA \cap (1-e)A = 0$, which proves that c is right regular. Similarly, if $v \in l(c)$, since $a = ce$ and $a = ae$, then $va = 0$ implies $v \in A(1-e) \cap Ae = 0$, which proves that c is a non-zer-divisor. Thus (1) implies (2).

Assume (2). If $a \in A$, then $a = ec = ce$ for some idempotent e and non-zero-divisor c. Then $l(a) = l(ec) = l(e)$ and $r(a) = r(ce) = r(e)$ and therefore (2) implies (1).

Proposition 6. *Let A have a classical left quotient ring Q. If, for any $a \in A$, there exists an idempotent e such that $l(a) = l(e)$ and $r(a) = r(e)$, then Q is strongly regular.*

Proof. Let $q = b^{-1} a \in Q$, b, $a \in A$, b non-zero-divisor in A. Then $a = ec = ce$ for some idempotent e and non-zero-divisor c of A by Proposition 5. Therefore $qc^{-1} bq = qc^{-1} a = qe = b^{-1} ae = b^{-1} a = q$ which proves that Q is von Neumann regular. If $t \in Q$ such that $t^2 = 0$, $t = u^{-1} d$, u, $d \in A$, u non-zero-divisor in A, then $dt = ut^2 = 0$. There exist v, $k \in A$, v non-zero-divisor, such that $vd = ku$. Now $kd = kuu^{-1} d = vdu^{-1} d = vdt = 0$ and if $dk = ws = sw$, w idempotent, s non-zero-divisor, then $0 = (dk)^2 = sws$ implies $w = 0$, whence $dk = 0$. Therefore $dvd = dku = 0$ implies $(vd)^2 = 0$ and hence $vd = 0$ which yields $d = 0$. Thus $t = 0$ which proves that Q is strongly regular.

As usual, A is called *fully idempotent* if every ideal of A is idempotent. We now give some characteristic properties of strongly regular rings. Known results (cf. for example [**6**], Theorem 5) are improved. Note that strongly regular rings are left and right duo, left and right V-rings. (For certain generalizations of strongly regular rings, cf. [**1**].)

Proposition 7. *The following conditions are equivalent:*

(1) *A is strongly regular.*

(2) *A is* WRD *such that every cyclic semi-simple right A-module is flat.*

(3) *A is* WRD *such that every cyclic semi-simple left A-module is flat.*

(4) *A is* WRD *such that every factor ring of A is a flat right A-module.*

(5) *A is a* WRD *right V-ring.*

(6) *A is a* WRD *left V-ring.*

(7) *A is a* WRD *ring whose simple right modules are p-injective.*

(8) *A is a* WRD *ring whose simple left modules are p-injective.*

(9) *A is a fully idempotent* WRD *ring.*

(10) *Every simple right A-module is flat and for any $a \in A$, there exists an idempotent e such that $l(a) = l(e)$ and $r(a) = r(e)$.*

(11) *Every non-zero-divisor is invertible in* A *and for any* $a \in A$, *there exists an idempotent* e *such that* $l(a) = l(e)$ *and* $r(a) = r(e)$.

(12) *For any* $a \in A$, *there exist an idempotent* e *and an invertible element* u *such that* $a = eu = ue$.

Proof. It is clear that (1) implies (2) through (6).

Assume (2). Since A/J_A is flat, for any $u \in J$, there exists $v \in J$ such that $u = vu$. Now $w(1-v) = 1$ for some $w \in A$ which implies that $u = w(1-v)u = w(u - vu) = 0$, whence $J = 0$. By Proposition 3, A is reduced and since every simple right A-module is flat, then (2) implies (1).

Similarly, (3) implies (1).

Assume (4). Then $J = 0$ as before. For any $0 \neq b \in A$, there exists a positive integer n such that $A/b^n A_A$ is flat. It follows that $b^n = b^n c b^n$ for some $c \in A$ which yields $b = bdb$, $d \in A$, whence (4) implies (1).

(5) (resp. (6)) implies (7) (resp. (8)) evidently.

Either (7) or (8) implies (9) by $[\mathbf{10}]_1$ (Lemma 1).

Assume (9). Then A is reduced by Proposition 3. For any $a \in A$, there exists a positive integer n such that $a^n A = A a^n A$ and hence $a^n A = (a^n A)^2$ implies $a^n = a^{2n} u$ for some $u \in A$. A is therefore strongly π-regular and since A is reduced, then A is strongly regular and thus (9) implies (10).

If every simple right A-module is flat, then every non-zero-divisor is invertible in A and therefore (10) implies (11).

(11) implies (12) by Proposition 5.

Assume (12). If c is a non-zero-divisor of A, then $c = ev$, where e is idempotent and v is invertible in A. Now $l(e) \subseteq l(c) = 0$ implies $e = 1$, whence $c = v$ is invertible. A is therefore its own classical left quotient ring and (12) implies (1) by Propositions 5 and 6.

Remark 3. The following conditions are equivalent for a WRD ring A: (a) Every principal right ideal of A is projective. (b) Every principal left ideal of A is projective. (c) For any $a \in A$, there is a central idempotent e and a non-zero-divisor c such that $a = ec$.

Remark 4. If A is WRD whose simple left modules are either p-injective or projective, then $Z = 0$ (cf. $[\mathbf{10}]_2$, Proposition 3 and $[\mathbf{10}]_6$, Lemma 7).

The next remark completes $[\mathbf{10}]_7$ (Remark 2).

Remark 5. The following conditions are equivalent: (1) Every factor ring of A is quasi-Frobeniusean. (2) A is right p-injective with maximum condition on right annihilators such that every finitely generated right ideal is principal (cf. $[2]_2$, Proposition 25.4.6 B).

Remark 6. The following conditions are equivalent: (1) A is von Neumann regular. (2) Every left ideal of A which is isomorphic to a direct summand of ${}_AA$ is a direct summand of ${}_AA$ and for any $a \in A$, there exist an idempotent e and a non-zer-divisor c such that $l(a) = l(ec)$.

We add a last remark on commutative self-injective regular rings and raise a few questions.

Remark 7. (1) A commutative ring A is self-injective regular iff every essential ideal which is an annihilator ideal is injective. (2) The following conditions are equivalent for a commutative ring A: (a) A is self-injective regular with non-zero socle. (b) A contains an injective maximal ideal and every principal ideal of A is flat. (c) A contains a non-singular injective maximal ideal.

Questions: (1) Is a right p-injective ring with maximum condition on right annihilators right Artinian? (We know that such rings are semi-primary).

(2) Does Proposition 4 hold for all rings A?

(3) Is A von Neumann regular if every principal left ideal of A is projective and A satisfies any one of the following conditions: (a) A is right p-injective. (b) Every maximal right ideal of A is p-injective?

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Abstract

Further properties of annihilators are studied. Commutative von Neumann regular rings are characterized in terms of annihilators. Weakly right duo rings (which effectively generalize right duo rings) are considered in connection with strongly regular rings and π-regular rings.

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