Riv. Mat. Univ. Parma (4) **12** (1986), 257-262

L. A. CORDERO and M. DE LEÓN (*)

f-structures on the frame bundle of a riemannian manifold (**)

Introduction

It is well known that the tangent bundle TM of a Riemannian manifold (M, g) possesses an almost Kähler structure which is Kähler if and only if (M, g) is locally Euclidean [2]. On the other hand, Okubo $[4]_1$ constructed on the frame bundle FM of (M, g) an f-structure of rank n^2, when $n = \dim M$ is even.

In this paper we construct an f-structure F_α of rank $2n$ on FM for each α with $1 \leq \alpha \leq n$, α being arbitrary, and then we prove the following

Theorem. (1) *FM is always an fAK-manifold;* (2) *FM is an fK-manifold if and only if (M, g) is locally Euclidean.*

Let us remark that these structures F_α on FM are all different from that considered by Okubo in $[4]_1$.

1 - Preliminaries

Let us summarize all the basic definitions and results which will be used later on. In the following all manifolds, maps, connections and metrics are assumed to be differentiable of class C^∞; indices $i, j, k, \ldots, \alpha, \beta, \gamma, \ldots$ have range in $\{1, \ldots, n\}$ and summation over repeated indices is always implied unless otherwise stated.

Let M be an n-dimensional differentiable manifold, FM its frame bundle and $\pi: FM \to M$ the projection map. For each coordinate system (U, x^i) in M, we put

---

(*) Indirizzo: Departamento de Geometria y Topologia, Facultad de Matematicas, Universidad de Santiago de Compostela, Santiago de Compostela, Spain.

(**) Ricevuto: 28-VIII-1985.

$FU = \pi^{-1}(U)$; a frame $p = (p_\alpha)$ at $x \in U$ can be expressed uniquely in the form $p_\alpha = x^i_\alpha (\partial/\partial x^i)_x$, and the induced coordinate system in FM will be denoted by $\{FU, (x^i, x^i_\alpha)\}$.

Let ∇ be a linear connection on M, with local components Γ^h_{ij}, and let $K: TTM \to TM$ be the connection map corresponding to ∇ [2]; for each $\alpha = 1, \dots, n$ we define a new connection map $K^{(\alpha)}: TFM \to TM$ as follows. Let $\pi^{(\alpha)}: FM \to TM$ be given by $\pi^{(\alpha)}(p) = p_\alpha$, and put $K^{(\alpha)} = K \cdot T\pi^{(\alpha)}$. An easy computation leads to the local equations defining $K^{(\alpha)}$

$$K^{(\alpha)}_p\left(\frac{\partial}{\partial x^i}\right) = \Gamma^h_{ij}(x) x^j_\alpha(p) \left(\frac{\partial}{\partial x^h}\right)_x \qquad K^{(\alpha)}_p\left(\frac{\partial}{\partial x^i_\beta}\right) = \delta^\alpha_\beta \left(\frac{\partial}{\partial x^i}\right)_x \qquad x = \pi(p) .$$

Let X be a vector field on M; then there is exactly one vector field X^H on FM, called *the horizontal lift of* X, and exactly one vector field $X^{(\alpha)}$ on FM for each $\alpha = 1, \dots, n$, called *the* α^{th}*-vertical lift of* X, such that for any $p \in FM$ we have

$$\pi_* X^H_p = X_{\pi(p)} \qquad K^{(\beta)} X^H_p = O_{\pi(p)} \qquad \pi_* X^{(\alpha)}_p = O_{\pi(p)} \qquad K^{(\beta)} X^{(\alpha)}_p = \delta^{\beta\alpha} X_{\pi(p)} .$$

If $X = X^i(\partial/\partial x^i)$ in U, then in FU

$$X^H = X^i \frac{\partial}{\partial x^i} - X^i \Gamma^h_{ij} x^j_\alpha \frac{\partial}{\partial x^h_\alpha} \qquad X^{(\alpha)} = X^i \frac{\partial}{\partial x^i_\alpha}$$

and, if $p \in FM$ is fixed, then the vectors X^H_p, $X^{(\alpha)}_p$ are determined by the value of $X_{\pi(p)}$.

Let F be a tensor field of type (1, 1) on M; for each $\alpha = 1, \dots, n$, define the following vertical (resp. horizontal) vector field $\gamma^\alpha F$ (resp. $\sigma_\alpha F$) on FM: for all $p = (p_\alpha) \in FM$

$$(1.1) \qquad (\gamma^\alpha F)_p = (F(p_\alpha))^{(\alpha)}_p \ \text{(not summed over } \alpha)$$

$$(1.2) \qquad (\sigma_\alpha F)_p = (F(p_\alpha))^H_p .$$

Locally if $\{F^h_j\}$ are the components of F in U, then in FU we have

$$(1.3) \qquad \begin{aligned} \gamma^\alpha F &= x^j_\alpha F^h_j \frac{\partial}{\partial x^h_\alpha} \qquad \text{(not summed over } \alpha) \\ \sigma_\alpha F &= F^h_j x^j_\alpha \left(\frac{\partial}{\partial x^h} - \Gamma^h_{ij} x^i_\beta \frac{\partial}{\partial x^h_\beta}\right) . \end{aligned}$$

Also, we shall denote

$$(1.4) \qquad \gamma F = \sum_{\alpha=1}^n \gamma^\alpha F \qquad \sigma F = \sum_{\alpha=1}^n \sigma_\alpha F .$$

The brackets of vertical and horizontal lifts are expressed by the following formulae

$$[X^{(\alpha)}, Y^{(\beta)}] = 0 \qquad [X^H, Y^{(\alpha)}] = (\nabla_X Y)^{(\alpha)}$$

(1.5)

$$[X^H, Y^H] = [X, Y]^H - \gamma R(X, Y) \qquad [X^{(\alpha)}, \gamma^\beta F] = \delta^{\alpha\beta}(F(X))^{(\beta)}$$

where $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$.

Let (M, g) be a Riemannian space, ∇ an arbitrary and no necessarily metric connection on M. *The diagonal lift g^D of g to FM* with respect to the connection ∇ is a Riemannian metric on FM determined by the identities

$$g^D(X^H, Y^H) = \{g(X, Y)\}^V$$

(1.6)

$$g^D(X^H, Y^{(\alpha)}) = 0 \qquad g^D(X^{(\alpha)}, Y^{(\beta)}) = \delta^{\alpha\beta}\{g(X, Y)\}^V$$

for any vector fields X, Y on M and every α, β. Here $f^V = f \cdot \pi$ denotes the canonical vertical lift to FM of differentiable functions on M.

Bearing in mind the resemblances of g^D to the so called Sasaki metric on the tangent bundle TM, we call it the *Sasaki-Mok metric induced on FM*, since K.P. Mok [3] has been the first one to consider this metric g^D on FM, defining it by means of its local expression. The intrinsic formulae (1.6) have been obtained by ourselves in $[1]_{1,2}$.

2 - *f*-structures on *FM*

Let (M, g) be a Riemannian manifold of dimension n. Define a tensor field F_α, $1 \leq \alpha \leq n$, of type (1, 1) on FM by

$$F_\alpha X^H = -X^{(\alpha)} \qquad F_\alpha X^{(\beta)} = \delta_\alpha^\beta X^H \tag{2.1}$$

for any vector field X on M, where the horizontal lifts are considered with respect to the Levi-Civita connection of g.

Then F_α has constant rank $2n$ and $F_\alpha^3 + F_\alpha = 0$, so F_α defines over FM an *f*-structure of rank $2n$.

The integrability and the partial integrabilities of an *f*-structure have been studied by Yano-Ishihara [7]. In order to study them for the *f*-structure F_α on FM let us state an auxiliary lemma.

Lemma. *Let G be a tensor field of type* (1, 1) *on M. Then*

$$(1) \quad F_\alpha(\gamma(G)) = \sigma_\alpha(G) \qquad (2) \quad F_\alpha(\sigma_\alpha G) = -\gamma^\alpha G\ .$$

Proof. Direct from (1.3), (1.4) and (2.1).

Let $l_\alpha = -F_\alpha^2$, $m_\alpha = F_\alpha^2 + I$ be the projection operators associated to F_α, and let be $L_\alpha = \operatorname{Im} l_\alpha$, $M_\alpha = \operatorname{Im} m_\alpha$ the complementary distributions on M associated to l_α and m_α, respectively. Then $\dim L_\alpha = 2n$ and $\dim M_\alpha = n^2 - n$. Moreover

$$(2.2) \qquad \begin{array}{ll} l_\alpha(X^H) = X^H & l_\alpha(X^{(\beta)}) = \delta_\alpha^\beta X^{(\beta)} \\ m_\alpha(X^H) = 0 & m_\alpha(X^{(\beta)}) = (I - \delta_\alpha^\beta) X^{(\beta)}\ . \end{array}$$

Then $\{X^H, X^{(\alpha)}\}$ span L_α and $\{X^{(\beta)}/\beta \neq \alpha\}$ span M_α. From the Lemma above, we deduce that M_α is always completely integrable; the integrability of L_α is determined in the following proposition.

Proposition 1. *L_α is completely integrable if and only if (M, g) is locally Euclidean.*

The Nijenhuis tensor of F_α behaves as follows

$$(2.3) \qquad \begin{array}{ll} N_{F_\alpha}(X^H, Y^H) = \gamma^\alpha R(X, Y) & \\ N_{F_\alpha}(X^H, Y^{(\beta)}) = \delta_\alpha^\beta \sigma_\alpha R(X, Y) & N_{F_\alpha}(X^{(\beta)}, Y^{(\mu)}) = -\delta_\alpha^\beta \delta_\alpha^\mu \gamma R(X, Y) \end{array}$$

for any vector fields X, Y on M, $1 \leq \beta$, $\mu \leq n$.

From (2.3) and Proposition 1, we easily deduce

Proposition 2. *The following assertions are equivalent:*

(1) *(M, g) is locally Euclidean.* (2) *L_α is completely integrable.*

(3) *$F\alpha$ is partially integrable.* (4) *F_α is integrable.*

Now, let us recall some definitions and results of the theory of f-structures. Let F be an f-structure on a manifold N, and let g be a Riemannian metric on N adapted to F. Let Ω be the fundamental 2-form of F, defined by $\Omega(X, Y) = g(FX, Y)$ for any vector fields X, Y on N (see [6]). Let ∇ be the Levi-

Civita connection of g. Vohra and Singh [5] have introduced the following terminology, by paralleling the usual in Hermitian geometry: N is said

(1) *fAK-manifold* if and only if $d\Omega(FX, FY, FZ)=0$ for any vector fields X, Y, Z on N.

(2) *fH-manifold* if and only if F is partially integrable.

(3) *fK-manifold* if and only if $\nabla_{FX}F=0$ for any vector field X on N.

Then, we have

Proposition 3. *N is an fK-manifold if and only if it is both fAK-manifold and fH-manifold.*

From a geometric point of view, note that N is an fK-manifold if and only if the distribution L is completely integrable and each of its integral manifolds is Kählerian.

Let us go back to the frame bundle FM of the Riemannian manifold (M, g). Consider the diagonal lift g^D of g to FM. Then,

Proposition 4. *g^D is adapted to F_α.*

Proof. In fact, the distribution L_α and M_α are orthogonal with respect to g^D and $g^D(\tilde{X}, F_\alpha\tilde{X})=0$ for any vector field $\tilde{X}$ on FM.

Then, the fundamental 2-form Ω_α of F_α is given by

$$\Omega_\alpha(\tilde{X}, \tilde{Y})=g^D(F_\alpha\tilde{X}, \tilde{Y}) \tag{2.4}$$

for any $\tilde{X}$, $\tilde{Y}$ vector fields on FM. From (1.6) and (2.4) we deduce

$$\Omega_\alpha(X^H, Y^H)=0 \qquad \Omega_\alpha(X^H, Y^{(\beta)})=-\delta^\beta_\alpha\{g(X, Y)\}^V \qquad \Omega_\alpha(X^{(\beta)}, Y^{(\mu)})=0 \tag{2.5}$$

for any vectors fields X, Y on M, $1\leq\beta$, $\mu\leq n$.

A simple computation from (2.5) shows that $d\Omega_\alpha=0$. Therefore, we have

Theorem. (1) *FM is always an fAK-manifold;* (2) *FM is an fK-manifold if and only if (M, g) is locally Euclidean.*

Remark. Let us remark that an obvious modification in (2.1) allows to define on FM a tensor field F'_α of type (1.1), $1\leqslant\alpha\leqslant n$, such that $F'^3_\alpha-F'_\alpha=0$ and with rank $F'_\alpha = 2n$. These $f(3, -1)$-structure on FM are all different from that considered by Okubo in $[4]_2$. We omit the study of these $f(3, -1)$-structures on FM because it is similar to that we have developed for the f-structures $F_{\alpha'}$.

References

[1] L. A. CORDERO and M. DE LEÓN: $[\cdot]_1$ *Lifts of tensor fields to the frame bundle*, Rend. Circ. Mat. Palermo (2) **32** (1983), 236-271; $[\cdot]_2$ *On the curvature of the induced Riemannian metric on the frame bundle of a Riemannian manifold*, J. Math. Pures Appl. **65** (1986), 81-91.

[2] P. DOMBROWSKI, *On the geometry of the tangent bundles*, J. Reine Angew. Math. **210** (1962), 73-88.

[3] K. P. MOK, *On the differential geometry of frame bundles of Riemannian manifolds*, J. Reine Angew. Math. **302** (1978), 16-31.

[4] T. OKUBO: $[\cdot]_1$ *On the differential geometry of frame bundles* $F(X_n)$, $n = 2m$. Mem. Defense Acad. 5 (1965), 1-17; $[\cdot]_2$ *On the differential geometry of frame bundles*, Ann. Mat. Pura Appl. (4) **72** (1966), 29-44.

[5] V. C. VOHRA and K. D. SINGH, *Some structures on an f-structure manifold*, Ann. Polon. Math. **27** (1972), 85-91.

[6] K. YANO, *On a structure defined by a tensor field f of type (1, 1) satisfying* $f^3 + f = 0$, Tensor **14** (1963), 99-109.

[7] K. YANO and S. ISHIHARA, *Tangent and cotangent bundles*, Marcel Dekker, New York, 1973.

Résumé

On construit une famille F_α, $1 \leqq \alpha \leqq n$, *de f-structures sur le fibré des repères linéaires FM d'une variété Riemannienne M à dimension n et on étude l'integrabilité des* $F_\alpha{}'$.

* * *