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M. FERNÁNDEZ and B. R. MOREIRAS (*)

Symmetry properties of the covariant derivative of the fundamental 4-form of a quaternionic manifold (**)

1 - Introduction

The group $Sp(n) \cdot Sp(1)$ is the structure group of a quaternionic manifold. On such a manifold M there is a fundamental (globally defined) 4-form Ω [4]. If Ω is parallel with respect to the Levi-Civita connection, then M is said to be a quaternionic Kähler manifold. In this case the holonomy group of M is a subgroup of $Sp(n) \cdot Sp(1)$ (see [1], $[2]_{2,3}$, [3], [6]).

In this paper we give some theorems, concerning some symmetric properties of the covariant derivative $\nabla\Omega$ of the fundamental 4-form Ω on M (Theorems 4.1, 4.2, 4.3).

2 - Preliminaries

Let (M, g) be a C^∞ $4n$-dimensional Riemannian manifold. (M, g) is said to be a quaternionic manifold if the structural group $O(4n)$ of the frame bundle of M can be reduced to $Sp(n) \cdot Sp(1)$ or, equivalently, if there exists a 3-dimensional vector bundle V consisting of tensors of type (1, 1) over M satisfying the condition that each point $p \in M$ has a neighborhood $\mathscr{U}$ on which there is a local basis $\{I, J, K\}$ of V such that

$$(2.1)\qquad \begin{gathered} I^2 = J^2 = K^2 = -1 \\ IJ = -JI = K \qquad KI = -IK = J \qquad JK = -KJ = I \end{gathered}$$

(*) Indirizzo: Departamento de Geometria y Topología, Facultad de Matemátícas, Univ. de Santiago, Santiago de Compostela, Spain and Univ. Pais Vasco, Bilbao, Spain.

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where 1 denotes the identity tensor field of type (1,1) on M

$$g(IX, IY) = g(JX, JY) = g(KX, KY) = g(X, Y) \tag{2.2}$$

for any vector fields X, Y on $\mathscr{U}$.

Since g is Hermitian with respect to each almost complex structures I, J and K, we can consider their Kähler local 2-forms F_I, F_J and F_K. Then,

$$\Omega = F_I \wedge F_I + F_J \wedge F_J + F_K \wedge F_K$$

is a globally defined 4-form on M (originally defined by Kraines [4]), which is called the fundamental form of the quaternionic manifold M. One easily verifies

$$\begin{aligned}(2.3)\qquad & \Omega(X, Y, Z, U) \\ & = 2 \underset{YZU}{\mathfrak{S}} \{g(IX, Y)\, g(IZ, U) + g(JX, Y)\, g(JZ, U) + g(KX, Y)\, g(KZ, U)\}\end{aligned}$$

for any vector fields X, Y, Z, U on $\mathscr{U}$ and where $\mathfrak{S}$ denotes the cyclic sum. The quaternionic manifold is said to be Kählerian if Ω is parallel with respect to the Levi-Civita connection of g.

3 - Properties of the fundamental 4-form

In this section we study the properties of the fundamental 4-form Ω of a quaternionic manifold (M, g) with respect to the almost complex structures I, J and K.

Let us recall that the Kähler (local) 2-forms F_I, F_J and F_K satisfy

$$F_I(IX, IY) = F_I(X, Y) \quad F_J(JX, JY) = F_J(X, Y) \quad F_K(KX, KY) = F_K(X, Y) \tag{3.1}$$

$$F_I(X, IX) = F_J(X, JX) = F_K(X, KX) = \|X\|^2 \tag{3.2}$$

for X, Y vector fields on $\mathscr{U}$.

A similar condition to (3.1) for Ω is given in the following

Proposition 3.1. *For any vector fields X, Y, Z, U on $\mathscr{U}$ we have*

$$\begin{aligned}(3.3)\qquad & \Omega(IX, IY, IZ, IU) = \Omega(JX, JY, JZ, JU) \\ & = \Omega(KX, KY, KZ, KU) = \Omega(X, Y, Z, U).\end{aligned}$$

Proof. From (2.1), (2.2) and (2.3) one follows that

$$\begin{aligned} &\Omega(IX, IY, IZ, IU) \\ &= 2 \underset{YZU}{\mathfrak{S}} \{g(X, IY)g(Z, IU) + g(JX, Y)g(JZ, U) + g(KX, Y)g(KZ, U)\} \\ &= 2 \underset{YZU}{\mathfrak{S}} \{g(IX, Y)g(IZ, U) + g(JX, Y)g(JZ, U) + g(KX, Y)g(KZ, U)\} \\ &= \Omega(X, Y, Z, U) . \end{aligned}$$

This proves the first equality of (3.3). The remainder identities are proved in a similar way.

The analogous condition to (3.2) for Ω is given in the following

Proposition 3.2. *For any vector field X on $\mathscr{U}$ we have*

$$\Omega(X, IX, JX, KX) = 6\,\|X\|^4 . \tag{3.4}$$

Proof. Direct computations from (2.3).

Finally, we also have

Proposition 3.3. *For any vector fields X, Y, Z, U on $\mathscr{U}$ we have*

$$\underset{YZU}{\mathfrak{S}} \{\Omega(X, IY, JZ, KU) + \Omega(X, IY, JU, KZ)\} = 12 \underset{YZU}{\mathfrak{S}} g(X, Y)g(Z, U) . \tag{3.5}$$

Proof. From (3.4), by polarization, we have

$$\begin{aligned} &\Omega(X + Y, IX + IY, JX + JY, KX + KY) \\ &= 6\,\|X\|^4 + 6\,\|Y\|^4 + 12\,\{\|X\|^2\,\|Y\|^2 + 2g(X, Y)^2 + 2g(X, Y)\,[\|X\|^2 + \|Y\|^2]\} . \end{aligned} \tag{3.6}$$

Using (3.3), (3.6) can be written as follows

$$\begin{aligned} &2\Omega(X, IX, JX, KY) + 2\Omega(X, IY, JY, KY) \\ &+ \Omega(X, IX, JY, KY) + \Omega(X, KX, IY, JY), + \Omega(X, JX, KY, IY) \\ &= 6\,\{\|X\|^2\,\|Y\|^2 + 2g(X, Y)^2 + 2g(X, Y)\,[\|X\|^2 + \|Y\|^2]\} . \end{aligned}$$

If we replace Y and Z in this identity by $Y + Z$ and $Z + U$ we obtain (3.5).

4 - The covariant derivative of the fundamental 4-form

In this section we obtain several symmetry properties for the covariant derivative $\nabla\Omega$ of the fundamental 4-form Ω of the quaternionic manifold (M, g), ∇ being the Levi-Civita connection of (M, g).

It is well known that the covariant derivative ∇F_I of the Kähler 2-form F_I satisfies

$$\nabla_W(F_I)(X, Y) = -\nabla_W(F_I)(IX, IY) \tag{4.1}$$

for any vector fields X, Y, W on $\mathscr{U}$. Analogous equations hold when I is replaced by J or by K.

Furthermore, one easily verifies

$$\begin{aligned}
\nabla_W(F_I)(JX, Y) &= \nabla_W(F_K)(X, Y) + \nabla_W(F_J)(X, IY) = \nabla_W(F_K)(IX, IY) - \nabla_W(F_J)(IX, Y)\\
\nabla_W(F_J)(IX, Y) + \nabla_W(F_J)(X, IY) &= \nabla_W(F_K)(IX, IY) - \nabla_W(F_K)(X, Y)\\
\nabla_W(F_I)(JX, Y) + \nabla_W(F_I)(X, JY) &= \nabla_W(F_K)(IX, IY) + \nabla_W(F_K)(X, Y)\\
\nabla_W(F_I)(JX, Y) - \nabla_W(F_I)(X, JY) &= \nabla_W(F_J)(X, IY) - \nabla_W(F_J)(IX, Y)
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\nabla_W(\Omega)(X, Y, Z, U) = 2 \underset{YZU}{\mathfrak{S}} \{&g(\nabla_W(I)X, Y)g(IZ, U) + g(\nabla_W(I)Z, U)g(IX, Y)\\
&+ g(\nabla_W(J)X, Y)g(JZ, U) + g(\nabla_W(J)Z, U)g(JX, Y) + g(\nabla_W(K)X, Y)g(KZ, U)\\
&+ g(\nabla_W(K)Z, U)g(KX, Y)\} = 2\sum_{IJK}(\nabla_W(F_I)\wedge F_I)(X, Y, Z, U)
\end{aligned} \tag{4.3}$$

for any vector fields W, X, Y, Z, U on $\mathscr{U}$.

Theorem 4.1. *The covariant derivative $\nabla\Omega$ of the fundamental 4-form Ω of (M, g) satisfies*

$$\nabla_W(\Omega)(X, IX, JX, KX) = 0 \tag{4.4}$$

for all W, X vector fields on $\mathscr{U}$.

Proof. From (4.3) one follows that

$$\begin{aligned}
\nabla_W(\Omega)(X, IX, JX, KX) = 2\|X\|^2\{&g(\nabla_W(I)X, IX) + g(\nabla_W(I)JX, KX) + g(\nabla_W(J)X, JX)\\
&+ g(\nabla_W(J)KX, IX) + g(\nabla_W(K)X, KX) + g(\nabla_W(K)IX, JX)\}.
\end{aligned}$$

From (4.1) and (4.2) each summand in this identity is zero which proves (4.4).

Proposition 4.1. *For any vector fields W, X, Y on $\mathscr{U}$ we have*

(i) $$\nabla_W(\Omega)(X, IX, JY, KY) + \nabla_W(\Omega)(JX, KX, Y, IY) = 0$$

(ii) $$\nabla_W(\Omega)(X, IX, JX, KY) + \nabla_W(\Omega)(X, IX, JY, KX) + \nabla_W(\Omega)(X, IY, JX, KX) + \nabla_W(\Omega)(Y, IX, JX, KX) = 0\,.$$

Proof. From (2.1), (2.2), (4.1) and (4.3) we get

$$\nabla_W(\Omega)(X, IX, JY, KY) = 2\{g(X, Y)[g(\nabla_W(J)X, JY) - g(\nabla_W(J)IX, KY) + g(\nabla_W(K)X, KY) + g(\nabla_W(K)IX, JY)] + g(IX, Y)[g(\nabla_W(J)X, KY) + g(\nabla_W(J)IX, JY) - g(\nabla_W(K)X, JY) + g(\nabla_W(K)IX, KY)]\}\,. \tag{4.5}$$

If we replace X and Y in (4.5) by JX and JY and add the result with (4.5) we obtain (i). In order to prove (ii) we first replace Y in (i) by $X+Y$. Then, from (4.4) and (i) we deduce (ii).

Proposition 4.2. *For any vector fields W, X, Y, Z, U on $\mathscr{U}$ we have*

$$\begin{aligned}&\nabla_W(\Omega)(X, Y, Z, U) + \nabla_W(\Omega)(IX, IY, IZ, IU)\\ &= 2 \underset{YZU}{\mathfrak{S}} \{[g(\nabla_W(J)X, Y) - g(\nabla_W(J)IX, IY)]g(JZ, U)\\ &+ [g(\nabla_W(J)Z, U) - g(\nabla_W(J)IZ, IU)]g(JX, Y)\\ &+ [g(\nabla_W(K)X, Y) - g(\nabla_W(K)IX, IY)]g(KZ, U)\\ &+ [g(\nabla_W(K)Z, U) - g(\nabla_W(K)IZ, IU)]g(KX, Y)\}\,.\end{aligned}$$

Similar results can be obtained by cyclic permutations of I, J, K.

Proof. From (2.1), (2.2), (4.1) and (4.3) we have

$$\begin{aligned}\nabla_W(\Omega)(IX, IY, IZ, IU) = -2 \underset{YZU}{\mathfrak{S}} \{&g(\nabla_W(I)X, Y)g(IZ, U)\\ &+ g(\nabla_W(I)Z, U)g(IX, Y) + g(\nabla_W(J)IX, IY)g(JZ, U)\\ &+ g(\nabla_W(J)IZ, IU)g(JX, Y) + g(\nabla_W(K)IX, IY)g(KZ, U)\\ &+ g(\nabla_W(K)IZ, IU)g(KX, Y)\}\,.\end{aligned} \tag{4.6}$$

Now, from (4.3) and (4.6) one gets the formula.

Corollary. *For W, X, Y, Z, U vector fields on $\mathscr{U}$ we have*

$$\nabla_W(\Omega)(X,Y,Z,U) - \nabla_W(\Omega)(IX,IY,IZ,IU)$$
$$= 2 \underset{YZU}{\mathfrak{S}} \{2[g(\nabla_W(I)X,Y)g(IZ,U) + g(\nabla_W(I)Z,U)g(IX,Y)]$$
$$+ [g(\nabla_W(J)X,Y) + g(\nabla_W(J)IX,IY)]g(JZ,U)$$
$$+ [g(\nabla_W(J)Z,U) + g(\nabla_W(J)IZ,IU)]g(JX,Y)$$
$$+ [g(\nabla_W(K)X,Y) + g(\nabla_W(K)IX,IY)]g(KZ,U)$$
$$+ [g(\nabla_W(K)Z,U) + g(\nabla_W(K)IZ,IU)]g(KX,Y)\}.$$

Similar results can be obtained by cyclic permutations of I, J, K.

Proof. It suffices to substract (4.6) from (4.3).

Theorem 4.2. *For any vector fields W, X, Y, Z, V on $\mathscr{U}$ we have*

$$\nabla_W(\Omega)(X,Y,Z,U) + \nabla_W(\Omega)(IX,IY,IZ,IU) \tag{4.7}$$
$$+ \nabla_W(\Omega)(JX,JY,JZ,JU) + \nabla_W(\Omega)(KX,KY,KZ,KU) = 0.$$

Proof. From (2.1), (2.2), (4.1) and (4.3) we obtain

$$\nabla_W(\Omega)(JX,JY,JZ,JU) = -2 \underset{YZU}{\mathfrak{S}} \{g(\nabla_W(I)JX,JY)g(IZ,U) \tag{4.8}$$
$$+ g(\nabla_W(J)Z,U)g(JX,Y) + g(\nabla_W(I)JZ,JU)g(IX,Y) + g(\nabla_W(J)X,Y)g(JZ,U)$$
$$- g(\nabla_W(K)IX,IY)g(KZ,U) - g(\nabla_W(K)IZ,IU)g(KX,Y)\},$$

$$\nabla_W(\Omega)(KX,KY,KZ,KU) = 2 \underset{YZU}{\mathfrak{S}} \{g(\nabla_W(I)JX,JY)g(IZ,U) \tag{4.9}$$
$$+ g(\nabla_W(I)JZ,JU)g(IX,Y) + g(\nabla_W(J)IX,IY)g(JZ,U)$$
$$+ g(\nabla_W(J)IZ,IU)g(JX,Y) - g(\nabla_W(K)X,Y)g(KZ,U)$$
$$- g(\nabla_W(K)Z,U)g(KX,Y)\}.$$

Summing up term to term (4.3), (4.6), (4.8) and (4.9) we obtain (4.7).

Theorem 4.3. *For W, X, Y, Z, U vector fields on $\mathscr{U}$ we have*

$$\nabla_W(\Omega)(X, Y, Z, U) + \nabla_W(\Omega)(KX, KY, KZ, KU) \tag{4.10}$$
$$= \nabla_W(\Omega)(X, Y, KZ, KU) + \nabla_W(\Omega)(X, KY, Z, KU) + \nabla_W(\Omega)(KX, KY, Z, U)$$
$$+ \nabla_W(\Omega)(KX, Y, KZ, U) + \nabla_W(\Omega)(X, KY, KZ, U) + \nabla_W(\Omega)(KX, Y, Z, KU).$$

The result is still true replacing K by I or by J.

Proof. We put

$$I' = aI + bJ \qquad J' = -bI + aJ \qquad K' = K \tag{4.11}$$

where a, b are real numbers such that $a^2 + b^2 = 1$. Then $\{I', J', K'\}$ is also a basis of V on $\mathscr{U}$, and the fundamental 4-form Ω' corresponding to $\{I', J', K'\}$ is equal to Ω. Now, we replace I, J, K in (4.7) by I', J', K'. Then, using (4.11) we obtain

$$0 = \nabla_W(\Omega)(X, Y, Z, U) + \nabla_W(\Omega)(KX, KY, KZ, KU) \tag{4.12}$$
$$+ (a^4 + b^4)\{\nabla_W(\Omega)(IX, IY, IZ, IU) + \nabla_W(\Omega)(JX, JY, JZ, JU)\}$$
$$+ (a^3b - ab^3)\{\nabla_W(\Omega)(IX, IY, IZ, JU) + \nabla_W(\Omega)(IX, IY, IZ, IU)$$
$$+ \nabla_W(\Omega)(IX, JY, IZ, IU) + \nabla_W(\Omega)(JX, IY, IZ, IU)$$
$$- \nabla_W(\Omega)(JX, JY, JZ, IU) - \nabla_W(\Omega)(JX, JY, IZ, JU)$$
$$- \nabla_W(\Omega)(JX, IY, JZ, JU) - \nabla_W(\Omega)(IX, JY, JZ, JU)\}$$
$$+ 2a^2b^2\{\nabla_W(\Omega)(IX, IY, JZ, JU) + \nabla_W(\Omega)(IX, JY, IZ, JU)$$
$$+ \nabla_W(\Omega)(IX, JY, JZ, IU) + \nabla_W(\Omega)(JX, IY, IZ, JU)$$
$$+ \nabla_W(\Omega)(JX, IY, JZ, IU) + \nabla_W(\Omega)(JX, JY, IZ, IU)\}\,.$$

Furthermore, since $a^4 + b^4 = 1 - 2a^2b^2$, using (4.7) we get

$$\nabla_W(\Omega)(IX, IY, IZ, IU) + \nabla_W(\Omega)(JX, JY, JZ, JU) \tag{4.13}$$
$$= \nabla_W(\Omega)(IX, IY, JZ, JU) + \nabla_W(\Omega)(IX, JY, IZ, JU) + \nabla_W(\Omega)(IX, JY, JZ, IU)$$
$$+ \nabla_W(\Omega)(JX, IY, IZ, JU) + \nabla_W(\Omega)(JX, IY, JZ, IU) + \nabla_W(\Omega)(JX, JY, IZ, IU).$$

Replacing X, Y, Z, U in (4.13) by IX, IY, IZ and IU we deduce (4.10).

It is well known that any 2-dimensional almost Hermitian manifold is Kählerian. Actually, we have the following

Proposition 4.3. *Let M be a 4-dimensional quaternionic manifold. Then M is a Kähler quaternionic manifold.*

Proof. Let $\{X, IX, JX, KX\}$ an arbitrary local frame field. Using Theorem 4.1. we get $\nabla_Y(\Omega)(X, IX, JX, KX) = 0$ for any vector field Y on $\mathscr{U}$. Hence the proposition follows.

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Résumé

Nous étudions les propriétés de symétrie de la dérivée covariant $\nabla\Omega$ *de la 4-forme fondamentale Ω d'une variété Riemannienne dont le groupe structurel c'est $Sp(n)\cdot Sp(1)$.*

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