Riv. Mat. Univ. Parma (4) **12** (1986), 111-120

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On the Banach spaces of large density character (**)

Introduction

In what follows B is a Banach space and dens (B) is the *density character* of B, that is the smallest cardinal number $\aleph$ for which B has a dense subset of cardinality $\aleph$.

We say that a *sequence* $(x_n)_{n=1}^m$ of B $(1 \leqslant m \leqslant \infty)$ is *orthogonal* if, for every finite sequence $(a_n)_{n=1}^p$ of numbers,

$$(1) \qquad \| \sum_{n=1}^{p} a_n x_n \| \geqslant \max \{ \| \sum_{n \in F} a_n x_n \| \quad F \text{ subset of } (n)_{n=1}^p \}.$$

Usually these sequences are called *unconditional basic*, with unconditional basis constant equal to one.

Moreover we say that two *subspaces* X and Y of B are *orthogonal* if (x, y) is orthogonal, for every x of X and y of Y.

The existence of unconditional basic sequences and the existence of infinite dimensional orthogonal subspaces are two famous open problems of the Functional Analysis.

The only tool of the constructions of this Note is the Hahn-Banach theorem; moreover the idea is that the large density character helps the construction of «nice» sequences.

1 concerns the orthogonal subspaces and there is proved the following theorem.

I. *Let $\aleph$ be a transfinite cardinal number and let U, V be subspaces of B with*

$$\text{dens}(U) \geqslant 2^{(2^{\aleph})} \qquad \text{dens}(V) > 2^{(2^{(2^{\aleph})})}.$$

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Then there exist two orthogonal subspaces X and Y of U and V respectively, so that

$$\operatorname{dens}(X) = 2^{(2^{\aleph})} \qquad \operatorname{dens}(Y) = \aleph .$$

2 concerns orthogonal sequences and there is proved the following theorem

II. *There exists a cardinal number $\aleph^*$ so that every B, with* $\operatorname{dens}(B) \geq \aleph^*$, *has finite arbitrarily long orthogonal sequences.*

The following question is open.

Problem 1. Does there exist a cardinal number $\aleph^*$ so that every B, with $\operatorname{dens}(B) \geq \aleph^*$, has an infinite orthogonal sequence?

3 concerns monotone basic sequences and there is proved the following theorem.

III. *If* $\operatorname{dens}(B) > 2^{\aleph}$, *where $\aleph$ is a transfinite cardinal number, there exists in B a transfinite basic monotone sequence of type* $(\aleph)$.

Where *type* $(\aleph)$ is the first ordinal of cardinality $\aleph$.

This theorem is connected with preceding results in the literature (see the end of 3).

We point out that

Corollary. *Every B with* $\operatorname{dens}(B) > \aleph_c$ *contains an infinite basic monotone sequence.*

Where $\aleph_c$ is the cardinality of continuum.

The following question is still open

Problem 2. Does there exist in every separable infinite dimensional B an infinite monotone basic sequence?

Notations and definitions. If I is a set card (I) is the cardinality of I; moreover if $I \subseteq B$ we use $[I]$ for $\overline{\operatorname{span}}\,(I)$. If X is a linear subspace of B, DIM (X) is the algebraical dimension of X (that is the cardinality of an Hamel basis) and COD (X) is the codimension of X in B.

Moreover $S(B)$ is the *unit sphere* of B $(= \{x \in B;\ \|x\| = 1\})$ and if $F \subseteq B^*$ (the dual of B) we set

$$F_{\perp} = \{x \in B;\ f(x) = 0 \text{ for every } f \text{ of } F\} .$$

The following standard definitions can be found in [4] and $[6]_1$.

A sequence (x_n) of B is said to be *basic* if it is *basis* of $[x_n]$, that is if $x = \sum_{n=1}^{\infty} a_n x_n$ with (a_n) unique for every x of $[x_n]$; is the same as saying that there exists $1 \leq K < \infty$ (the *basis constant*) so that

$$\| \sum_{n=1}^{m} a_n x_n \| \leq K \| \sum_{n=1}^{m+p} a_n x_n \| \qquad \text{for every } (a_n)_{n=1}^{m+p} \text{ of numbers.}$$

A basic sequence (x_n) is said to be *monotone* if the basis constant is equal to one; (x_n) is said to be *unconditional basic* if every permutation of (x_n) is basic.

We use the following consequence of the Hahn-Banach theorem.

I* *There exists a map $G: B \to B^*$ so that, for every x of B, $(G(x))(x) = \|x\|$, $\|G(x)\| = 1$.*

We fix now a map G which will be unique in what follows.

We use «subspace» for «closed linear subspace», otherwise we say «linear subspace».

1 - Orthogonal subspaces

In what follows $\aleph_0$ is the cardinality of numerable.

Lemma 1. *If $\aleph$ is a transfinite cardinal number, $F \subset B^*$ with* $\text{card}(F) = \aleph$ *and* $\text{dens}(B) > 2^{\aleph}$, *it follows that* $\text{dens}(F_\perp) = \text{dens}(B)$ *and* $\text{COD}(F_\perp, B) \leq 2^{\aleph}$.

Proof. $F_\perp$ is closed subspace of B, since intersection of closed sets. Let W be a linear subspace of B such that

$$B = F_\perp + W \qquad F_\perp \cap W = \{0\}. \tag{2}$$

If C is the complex field, call τ the map $B \to C^F$ so defined

$$\tau(x) = \{f(x);\ f \in F\} \quad \text{for every } x \text{ of } B.$$

Obviously τ is linear hence, if 0 is the null element of C^F (that is $0(f) = 0$ for every f of F) and if $x_1, x_2 \in B$ with $\tau(x_1) = \tau(x_2)$, it follows that $\tau(x_1 - x_2) = 0$.

On the other hand $\tau(x) = 0$ implies $x \in F_\perp$; hence, by (2),

$$x_1, x_2 \in W \quad \text{with} \quad \tau(x_1) = \tau(x_2) \quad \text{implies} \quad x_1 - x_2 \in F_\perp \cap W = \{0\}.$$

That is $\tau|_W$ is injective, hence card $(W) \leqslant \text{card}(C^F)$, therefore

$$\text{DIM}(W) \leqslant \text{card}(W) \leqslant (\text{card}(C))^{\text{card}(F)} = (\aleph_c)^{\aleph} = (2^{\aleph_0})^{\aleph} = 2^{\aleph}.$$

That is, by (2), $\text{COD}(F_\perp, B) = \text{DIM}(W) \leqslant 2^{\aleph}$.
By (2) it also follows that

$$\text{dens}(B) \leqslant \text{dens}(F_\perp) + \text{dens}(W) \leqslant \text{dens}(F_\perp) + 2^{\aleph}; \tag{3}$$

indeed if DIM (W) is finite $\text{dens}(W) = \aleph_0$, while if DIM (W) is infinite $\text{dens}(W) \leqslant \text{DIM}(W)$ (see [3] Corollary 2.4).

By hypothesis $\text{dens}(B) > 2^{\aleph}$, hence by (3) it follows $\text{dens}(F_\perp) = \text{dens}(B)$; which completes the proof of Lemma 1.

Now we prove an algebraical lemma.

Lemma 2. *Let U, V, W be linear subspaces of a linear space L such that* $L = U + V$, $U \cap V = \{0\}$; $\text{DIM}(L) > \max\{\aleph_c, \text{DIM}(U)\}$, $\text{DIM}(V) = \text{DIM}(W)$. *Then it follows that* $\text{DIM}(W \cap V) = \text{DIM}(L)$.

Proof. Firstly we notice that

$$\text{card}(U) = \max\{\text{DIM}(U); \aleph_c\}; \tag{4}$$

indeed if $(x_i)_{i=1}^n \subset L$, $\text{card}(\text{span}(x_i)_{i=1}^n) = \aleph_c$; moreover the cardinality of the finite subsets of an infinite Hamel basis of U is DIM (U).

By hypothesis there exists (w_i, u_i, v_i) in L so that

$$\begin{aligned} &(w_i)_{i\in I} \text{ is Hamel basis of } W; \\ &w_i = u_i + v_i \quad \text{with } u_i \in U \text{ and } v_i \in V, \text{ for } i \in I. \end{aligned} \tag{5}$$

By hypothesis $\text{DIM}(V) = \text{DIM}(L)$, hence

$$\text{card}(I) > \text{card}(U); \tag{6}$$

indeed by (4), (5) and by hypothesis it follows that

$$\text{card}(I) = \text{DIM}(W) = \text{DIM}(L) > \max\{\text{DIM}(U); \aleph_c\} = \text{card}(U).$$

By (5) and (6) there exists a subset $\bar{I}$ of I and $\bar{w}$, $\bar{u}$, $\bar{v}$ in L, so that

$$\begin{aligned}&\bar{w} \cup (w_i)_{i\in\bar{I}} \subset (w_i)_{i\in I} \quad \text{with card}\,(\bar{I}) = \text{card}\,(I);\\ &\bar{w} = \bar{u} + \bar{v} \quad \text{with} \quad \bar{u} \in U\,,\ \bar{v} \in V\,;\\ &u_i = \bar{u} \quad \text{that is} \quad w_i = \bar{u} + v_i \quad \text{for} \quad i \in \bar{I}\,.\end{aligned} \tag{7}$$

By (5) and (7) there follow two facts: (i) $(w_i - \bar{w}_i)_{i\in I}$ is linearly independent in W; (ii) $w_i - \bar{w} = v_i - \bar{v}$, that is $w_i - \bar{w} \in W \cap V$ for $i \in \bar{I}$.

Hence $V \cap W$ has a linearly indepedent subset of cardinality equal to $\text{DIM}\,(L)$; which completes the proof of Lemma 2.

Proof of Th. I. Let U_1 be a subspace of U with

$$\text{dens}\,(U_1) = 2^{(2^{\aleph})}. \tag{8}$$

By Theorem I* we set

$$F_1 = G\,(U_1)\,. \tag{9}$$

Then we have that

$$\text{card}\,(F_1) \leqslant 2^{(2^{\aleph})}; \tag{10}$$

indeed by (8), (9) and by [3] (Lemma 2) we have that

$$\text{card}\,(F_1) \leqslant \text{card}\,(U_1) = (\text{dens}\,(U_1))^{\aleph_0} = (2^{(2^{\aleph})})^{\aleph_0} = 2^{(2^{\aleph})}\,.$$

By Lemma 1 and by (10) it follows that

$$\text{COD}\,(B_0 \cap (F_1)_\perp, B_0) \leqslant 2^{(2^{(2^{\aleph})})} \qquad \text{where } B_0 = \overline{U_1 + V}\,.$$

Hence by [3], by Lemma 2 and by hypothesis,

$$\text{DIM}\,(V \cap (F_1)_\perp) = \text{DIM}\,(V)\,.$$

Since $V \cap (F_1)_\perp$ is a Banach space, by hypothesis there exists a subspace Y with $\text{dens}\,(Y) = \aleph$.

Call $F_2 = G\,(Y)$ and proceed as for F_1, as regards U_1 instead of V, then by Lemma 1 we get that

$$\text{dens}\,(X) = \text{dens}\,(U_1) = 2^{(2^{\aleph})} \qquad \text{where } X = U_1 \cap (F_2)_\perp.$$

Hence both for X ad Y the density character is as required, moreover by construction it is

$$X \subset (G(Y))_{\perp} \qquad Y \subset (\mathrm{G}(X))_{\perp}.$$

Therefore by Theorem I* and by (1) X and Y are orthogonal; which completes the proof of Theorem I.

2 - Orthogonal sequences

The assertion that the large density character helps the construction of sequences with good properties, follows from the following facts: (i) for every positive integer n there exists a Banach space X, with $\mathrm{DIM}(X) = 2n$, so that every basis of X has basis constant $\geqslant \sqrt{n}$ [7]; (ii) in every infinite dimensional Banach space there exists an *asymptotically monotone* basic sequence (x_n) (that is there exists a sequence (K_m) with $K_m \to 1$ so that, for every m and for every $(a_n)_{n=1}^{m+p}$

$$\| \sum_{n=1}^{m} a_n x_n \| \leqslant K_m \| \sum_{n=1}^{m+p} a_n x_n \|,$$

see [4] or [6]$_1$).

Lemma 3. *Let* $(x_n)_{n=1}^{p}$ $(1 \leqslant p \leqslant \infty)$ *be a sequence of* B *such that for every finite subsequence* $(x_n)_{n=1}^{m}$

$$x_k \in (G([x_n]_{n=1, n \neq k}^{m}))_{\perp} \qquad \text{for } 1 \leqslant k \leqslant m.$$

Then $(x_n)_{n=1}^{p}$ *is orthogonal.*

Proof. It is obvious by (1), indeed for every $(a_n)_{n=1}^{m} = (a_{n_k})_{k=1}^{q} \cup (a_{n'_k})_{k=1}^{q'}$ of numbers, we have

$$\| \sum_{n=1}^{m} a_n x_n \| \geqslant \| \sum_{k=1}^{q} a_{n_k} x_{n_k} + \sum_{k=2}^{q'} a_{n'_k} x_{n'_k} \| \geqslant \ldots \geqslant \| \sum_{k=1}^{q} a_{n_k} x_{n_k} \|.$$

Proof of Theorem II. Set

$$\aleph^* = \sum_{i=0}^{\infty} \aleph_i \qquad \text{where} \qquad \aleph_{i+1} = 2^{(2^{\aleph_i})} \qquad \text{for} \qquad i \geqslant 0. \tag{11}$$

Suppose $\operatorname{dens}(B) \geq \aleph^*$ and fix a positive integer p; it is sufficient to pick up the sequence of Lemma 3.

By the proof of Theorem I there exist x_1 of B and a subspace B_1 of B so that

$$\operatorname{dens}(B_1) = \aleph_{p-1} \qquad x_1 \in S((G(B_1))_\perp).$$

Fix m with $1 \leq m < p-1$ and suppose to have $(x_n)_{n=1}^m$ of B and a subspace B_m of B so that

$$\text{(12)} \qquad \operatorname{dens}(B_m) = \aleph_{p-m} \qquad x_n \in S((G([x_k]_{k=1,k\neq n}^m + B_m))_\perp) \quad \text{for} \quad 1 \leq n \leq m.$$

Then there exists B_{m+1} so that

$$B_{m+1} \text{ is subspace of } B_m \qquad \operatorname{dens}(B_{m+1}) = \aleph_{p-(m+1)}.$$

Hence by [3] by (11) and (12) and by Lemma 1 we have that

$$\operatorname{card}(B_{m+1}) = (\operatorname{dens}(B_{m+1}))^{\aleph_0} = \operatorname{dens}(B_{m+1})$$

$$\operatorname{dens}(B_m \cap (G(B_{m+1} + [x_n]_{n=1}^m))_\perp) = \aleph_{p-m},$$

that is, if we pick up $x_{m+1} \in S(B_m \cap (G(B_{m+1} + [x_n]_{n=1}^m))_\perp)$, we have (12) for $m+1$ instead of m.

Suppose to have (12) for $m = p-1$, since $\operatorname{dens}(B_{p-1}) = \aleph_1$, we pick up an arbitrary x_p of $S(B_{p-1}) \cap (G([x_n]_{n=1}^{p-1}))_\perp$ and we are done; which completes the proof of Theorem II.

3 - Monotone basic sequences

We already pointed out ((i) of 2) that in general a finite dimensional Banach space does not admit a monotone basis.

Proof of Theorem III. Fix x_1 in $S(B)$. Suppose to have a monotone basic sequence $(x_n)_{n<\beta} \subset S(B)$ where β is an ordinal $<$ type $(\aleph)$.

Suppose $\operatorname{card}(\beta)$ finite or numerable, let $(v_k)_{k=1}^\infty$ be a dense numerable subset of $[x_n]_{n<\beta}$, then by Lemmas 1 and 2,

$$\operatorname{COD}((G(v_k)_{k=1}^\infty)_\perp, B_1) \leq 2^{\aleph_0}$$

for every subspace B_1 of B with $\text{dens}(B_1) > \aleph_c$; therefore we can pick up $x_\beta \in S(B_1) \cap (G(v_k)_{k=1}^{\infty})_\perp$.

Suppose now $\text{card}(\beta) > \aleph_0$, then (by [3] Lemma 2) $\text{card}(\beta) = \text{card}([x_n]_{n<\beta})$ hence by Lemma 1 and by hypothesis

$$\text{COD}((G([x_n]_{n<\beta}))_\perp, B) \leqslant 2^{\text{card}(\beta)},$$

therefore we can pick up again $x_\beta \in S(B) \cap (G([x_n]_{n<\beta}))_\perp$.

This completes the proof of Theorem III.

We point out that preceding construction is not valid in general for the separable case, as it appears from the following Proposition. If X and Y are subspaces of B we say that X is *orthogonal* to Y if $\|x+y\| \geqslant \|x\|$ for every x of X and y of Y.

Proposition. *There are two elements y and z in C_{oH1}, so that $[y, z]$ is never orthogonal to $[x]$ for every $x \neq 0$ of C_{oH1}.*

Proof. Set

$$y = \sqrt{t} \qquad z = 1 - t \quad \text{for} \quad 0 \leqslant t \leqslant 1. \tag{13}$$

Let $x \in C_{oH1}$ with

$$x = x(t) \qquad \|x\| = 1 = |x(\bar{t})| = x(\bar{t}) \quad \text{with} \quad 0 \leqslant \bar{t} \leqslant 1.$$

If $\bar{t} = 0$ it is easy to see that there exists $\varepsilon > 0$ so that

$$\|z - \eta x\| < 1 \quad \text{for every} \quad 0 < \eta < \varepsilon.$$

If $\bar{t} = 1$ in the same way we check that $[y]$ is not ortogonal to $[x]$. Suppose $0 < \bar{t} < 1$ and set

$$\bar{u} = y + (1/2\sqrt{\bar{t}})\, z.$$

By (13) $\bar{u}'(t) = 0$ only for $t = \bar{t}$, with $\bar{u}''(\bar{t}) < 0$, moreover $(\bar{u}(\bar{t}))^2 = (1/4)(\sqrt{\bar{t}} + 1/\sqrt{\bar{t}})^2 = 1/2 + (\bar{t} + 1/\bar{t})/4 > 1$ for $0 < \bar{t} < 1$, while $\bar{u}(1) = 1$ and $(\bar{u}(0))^2 = 1/(4\bar{t}) < (\bar{u}(\bar{t}))^2$; therefore $\bar{u}(t)$ has only one maximum for $t = \bar{t}$. Hence again it is easy to see that there exists $\varepsilon > 0$ so that

$$\|\bar{u} - \eta x\| < 1 \quad \text{for every} \quad 0 < \eta < \varepsilon.$$

This completes the proof of the Proposition.

Remark. In preceding Proposition, for every definition of G in Theorem I*, $G([y, z])$ is always *total* on $C_{o\vdash 1}$ (that is $(G([y, z]))_{\perp} = \{0\}$).

In particular we can use the following definition of G for every $\bar{u}$ of (14)

$$(G(\overline{u}))(x) = x(\overline{t})) \quad \text{for every} \quad x \text{ of } C_{o\vdash 1}.$$

In this case $G([y, z])$ becomes 1-*norming* on $C_{o\vdash 1}$ (that is, for every x of $C_{o\vdash 1}$

$$\|x\| = \sup\{|f(x)|/\|f\|; \quad f \in G([y, z])\}).$$

Finally we notice that Theorem III is connected with a results of Bessaga ([1] Prop. 2, see also $[6]_2$, p. 599 Th. 17.10), moreover with results of Reif ($[5]_1$ Th. 1 and $[5]_2$ Prop. 5) and of John-Zizler ($[2]_1$ Prop. 2 and $[2]_2$ Prop. 9).

All these results concern Theorem III for every infinite density character, but only for weakly compacted generated (in particular reflexive) Banach spaces.

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Sommario

Se B è uno spazio di Banach, con dimensione infinita sufficientemente grande, mediante il teorema di Hahn-Banach si possono ottenere in B sottospazi ortogonali con dimensione infinita, successioni basiche monotone infinite e successioni ortogonali finite di lunghezza arbitraria.

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