Riv. Mat. Univ. Parma (4) **12** (1986), 59-64

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Uniqueness theorems in linear elastodynamics without strain-energy function (**)

1 - Introduction and mathematical preliminaries

In this paper we are concerned with the uniqueness issue associated with the mixed boundary-initial-value problem of linear elastodinamycs in unbounded domains. The special feature of our results consists of the absence of any assumption of symmetry on the elasticity tensor, with the obvious exception of those implied by the theory (minor symmetries).

Assume (1) that an unbounded elastic body $\mathbb{B}$ occupies at instant $t=0$ the region B of $\mathbb{R}^n$ ($n=2, 3$) with smooth boundary $\partial B = \partial_1 B \cup \partial_2 B$ ($\partial_1 B \cap \partial_2 B = \emptyset$). As is known, the mixed boundary-initial-value problem associated with the motion of $\mathbb{B}$ in the time interval $[0, +\infty)$ consists of finding the solution $\boldsymbol{u}(\boldsymbol{x}, t)$ to the system [2]

$$(1)\qquad \begin{aligned} &\rho\ddot{\boldsymbol{u}} = \nabla\cdot\boldsymbol{C}[\nabla\boldsymbol{u}] + \boldsymbol{b} && \text{on } Q = B\times(0, +\infty)\\ &\boldsymbol{u} = \boldsymbol{u}^* && \text{on } \partial_1 B\times(0, +\infty)\\ &\boldsymbol{C}[\nabla\boldsymbol{u}]\,\boldsymbol{n} = \boldsymbol{s}^* && \text{on } \partial_2 B\times(0, +\infty)\\ &\boldsymbol{u} = 0 \quad \dot{\boldsymbol{u}} = \boldsymbol{v}_0 && \text{on } B\times\{0\}\,. \end{aligned}$$

In (1) ρ, $\boldsymbol{C}$, $\boldsymbol{b}$, $\boldsymbol{u}^*$, $\boldsymbol{s}^*$, $\boldsymbol{v}_0$ and $\boldsymbol{n}$ are respectively the mass density, the fourth-order elasticity tensor, the body force, the surface displacement, the surface

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(**) Ricevuto: 12-IX-1984.

(1) Light-face letters indicate scalars; bold face case letters indicate tensors.

traction, the initial velocity field and the outward unit normal to ∂B. Moreover, the dot denotes partial differentiation with respect to time and, denoting by C_{ijhk}, $u_h (i, j, h, k = 1, \ldots, n)$ the components of $\boldsymbol{C}$ and $\boldsymbol{u}$ in a fixed orthonormal reference frame $\{0, \boldsymbol{e}_i\}$ of $\mathbb{R}^n$, $\nabla \cdot \boldsymbol{C}[\nabla \boldsymbol{u}] = \partial_j \{C_{ijhk} \partial_k u_h\} \boldsymbol{e}_i$, where $\partial_k = \partial / \partial x_k$ and summation on repeated indexes is implied. Recall that $C_{ijhk} = C_{jihk} = C_{ijkh}$. Thus, letting $\boldsymbol{A}$ and $\boldsymbol{B}$ denote two second-order tensors, $\boldsymbol{A} \cdot \boldsymbol{C}[\boldsymbol{B}] = \operatorname{sym} \boldsymbol{A} \cdot \boldsymbol{C}[\operatorname{sym} \boldsymbol{B}]$, where $\operatorname{sym} \boldsymbol{A}$, $\operatorname{sym} \boldsymbol{B}$ stand for the symmetric parts of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Here we wish to find sufficient conditions on data $\boldsymbol{C}$ and ρ, and suitable rate of decay at infinity on the relevant fields in order that system (1) admits at most one solution. To prove this it is sufficient to consider the system

$$\rho \ddot{\boldsymbol{u}} = \nabla \cdot \boldsymbol{C}[\nabla \boldsymbol{u}] \quad \text{on } Q \tag{2}$$

$$\boldsymbol{u} = 0 \text{ on } \partial_1 B \times (0, +\infty), \quad \boldsymbol{C}[\nabla \boldsymbol{u}]\boldsymbol{n} = 0 \text{ on } \partial_2 B \times (0, +\infty), \quad \boldsymbol{u} = \dot{\boldsymbol{u}} = 0 \text{ on } B \times \{0\}$$

and to show that (2) has only the identically null solution. Set

$$\Xi = \{\boldsymbol{u} \in [C^2(\overline{Q})]: \boldsymbol{u} \text{ has a Laplace-transform on } \overline{\boldsymbol{B}}\} .$$

Let us recall that a tensor function ψ defined on $\overline{Q}$ has a Laplace transform on $\overline{B}$ if $\exists \xi_0 \geq 0 : \forall \xi \geq \xi_0$ the integral $\hat{\psi}(\boldsymbol{x}, \xi) = \int_0^{+\infty} \exp[-\xi t] \psi(\boldsymbol{x}, t)\, dt$ converges for any $\boldsymbol{x}$ belonging to $\overline{\boldsymbol{B}}$.

Throughout the paper we shall assume that $1/\rho$ is positive and bounded on B. Also we shall require that $\boldsymbol{C}$ is bounded on B and satisfies one of the following well-known conditions

$$(*) \qquad \begin{array}{ll} \text{positive definiteness} & \exists \mu > 0: \ \boldsymbol{A} \cdot \boldsymbol{C}[\boldsymbol{A}] \geq \mu (\operatorname{sym} \boldsymbol{A})^2 \\ \text{strong ellipticity} & \exists \varkappa > 0: \ \boldsymbol{a} \cdot \boldsymbol{C}[\boldsymbol{a} \otimes \boldsymbol{b}]\, \boldsymbol{b} \geq \varkappa a^2 b^2 \end{array}$$

the former for any second-order tensor $\boldsymbol{A}$, the latter for any couple of vectors $\boldsymbol{a}$, $\boldsymbol{b}$.

As far as the regularity assumptions are concerned, we suppose that $\boldsymbol{C}$ and ρ are continuous on $\overline{B}$ and that $\boldsymbol{C}$ is once continuously differentiable on B.

The following symbols will be used

$$\forall \boldsymbol{x} \in \mathbb{R}^n \quad r = |\boldsymbol{x} - 0|, \quad \boldsymbol{e}_r = r^{-1}(\boldsymbol{x} - 0), \quad S_R = \{\boldsymbol{x} \in \mathbb{R}^n : r < R\},$$

$$B_R = B \cap S_R, \quad \Sigma_R = B \cap \partial S_R .$$

2 - Uniqueness theorems

It is well known that, if $\boldsymbol{u}$ is a solution to system (2) belonging to Ξ, its Laplace transform $\hat{\boldsymbol{u}}$ satisfies the system

$$\text{(3)}\qquad \begin{array}{ll} \xi^2 \rho \hat{\boldsymbol{u}} = \nabla \cdot \boldsymbol{C}[\nabla \hat{\boldsymbol{u}}] & \text{on } B \\ \hat{\boldsymbol{u}} = 0 \quad \text{on } \partial_1 B \qquad \boldsymbol{C}[\nabla \hat{\boldsymbol{u}}]\boldsymbol{n} = 0 & \text{on } \partial_2 B \end{array}$$

$\forall \xi \geq \xi_0$. Thus, if we show that $\exists \nu_0 \geq 0$: $\forall \nu \geq \nu_0$ $\hat{\boldsymbol{u}} = 0$ on $\overline{B}$, by the properties of the Laplace transform we can certainly conclude that $\boldsymbol{u} = 0$ on $\overline{Q}$. We shall always follow this procedure in proving our uniqueness theorems.

Theorem 1. *Let* $\boldsymbol{C}$ *be positive definite and let* $\boldsymbol{u}$ *be a solution to system* (2) *belonging to* Ξ. *If*

$$\text{(4)}\qquad \exists m \in N \qquad \exists M, \overline{R} > 0: \quad \rho u^2, (\nabla \boldsymbol{u})^2 \leq M r^m \qquad \forall r > \overline{R}\ ,$$

then $\boldsymbol{u} = 0$ *on* $\overline{Q}$.

Proof. Multiply both sides of $(3)_1$ by $\hat{\boldsymbol{u}}$ and integrate over B_R. Then an integration by parts gives

$$\text{(5)}\qquad \int_{B_R} \{\xi^2 \rho \hat{u}^2 + \nabla \hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla \hat{\boldsymbol{u}}]\}\, \mathrm{d}v = \int_{\Sigma_R} \hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla \hat{\boldsymbol{u}}]\, \boldsymbol{e}_r \, \mathrm{d}a \qquad \forall \xi \geq \xi_0 > 0\ .$$

Since, by virtue of the arithmetic-geometric mean inequality,

$$\hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla \hat{\boldsymbol{u}}]\, \boldsymbol{e}_r \leq \frac{|\boldsymbol{C}|}{2\xi_0 \sqrt{\mu \rho}} \{\xi^2 \rho \hat{u}^2 + \mu (\operatorname{sym} \nabla \boldsymbol{u})^2\}\ ,$$

setting $C = \sup\limits_B \dfrac{|\boldsymbol{C}|}{2\xi_0 \sqrt{\mu\rho}}$ $\qquad \xi^2 \rho \hat{u}^2 + \mu (\operatorname{sym} \nabla \boldsymbol{u})^2 = \eta\ ,$ (5) yields

$$\text{(6)}\qquad \int_{B_R} \eta \, \mathrm{d}v \leq C \int_{\Sigma_R} \eta \, \mathrm{d}a\ .$$

Now, since $\mathrm{d}/\mathrm{d}R \int_{B_R} \eta \, \mathrm{d}v = \int_{\Sigma_R} \eta \, \mathrm{d}a$, setting $\int_{B_R} \eta \, \mathrm{d}v = f(R)$, (6) takes the form $f \leq Cf'$ on $(0, +\infty)$, which, by virtue of a simple integration, leads to $f(R) \geq f(\overline{R}) \exp[CR]$ $\forall R \geq \overline{R} > 0$. Hence the desired result follows by a comparison with (4).

Theorem 2. *Let $\boldsymbol{C}$ be positive definite and let $\boldsymbol{u}$ be a solution to system* (2) *belonging to Ξ. Further, let p be a smooth, increasing and positive function on* $[0, +\infty)$ *such that* $\lim_{r\to+\infty} p(r) = +\infty$. *If* $\exists M, R_0 > 0 : \forall r \geq R_0$ *either*

(7) $\quad p'(\nabla \boldsymbol{u})^2 \leq M r^{1-n} \qquad$ *or* $\qquad$ (8) $\quad p' \rho u^2 \leq M r^{1-n}$,

then $\boldsymbol{u} = 0$ *on* $\overline{Q}$.

Proof. Assume that (7) holds. By virtue of the Schwartz inequality

$$\int_{\Sigma_R} \hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla \hat{\boldsymbol{u}}] \boldsymbol{e}_r \, da \leq C \{\xi^2 \int_{\Sigma_R} \rho \hat{u}^2 \, da \int_{\Sigma_R} (\mathrm{sym}\, \nabla \hat{\boldsymbol{u}})^2 \, da\}^{\frac{1}{2}} \tag{9}$$

where $C = \sup_B \dfrac{|\boldsymbol{C}|}{\xi\sqrt{\rho}}$. Thus, setting $\xi^2 \int_{B_R} \rho \hat{u}^2 \, dv = f(R)$, (5) gives

$$p'(R)\,[f(R)]^2 \leq C f'(R) \qquad \forall R > R_0 \,. \tag{10}$$

(10) implies that $f(+\infty) = \lim_{R\to+\infty} f(R) = 0$. Indeed, if $f(+\infty) > 0$, then $\exists \overline{R} > R_0 : f(R) > 0 \ \forall R > \overline{R}$. Thus, by integrating (10) on $(\overline{R}, R)$, one has

$$\frac{1}{f(\overline{R})} \geq \frac{1}{f(\overline{R})} - \frac{1}{f(R)} \geq p(R) - p(\overline{R}) \geq p(R).$$

But this contradicts the hypothesis $\lim_{r\to+\infty} p(r) = +\infty$. Hence $f(+\infty) = 0$ and since $\hat{\boldsymbol{u}} = 0$, uniqueness is completely proved.

Let now (8) be verified. From (5)-(9) it follows

$$\mu \int_{B_R} (\mathrm{sym}\, \nabla \boldsymbol{u})^2 \, dv \leq C\xi \{ \int_{\Sigma_R} \rho \hat{u}^2 \, da \int_{\Sigma_R} (\mathrm{sym}\, \nabla \boldsymbol{u})^2 \, da\}^{\frac{1}{2}} \,. \tag{11}$$

Setting $\int_{B_R} (\mathrm{sym}\, \nabla \boldsymbol{u})^2 \, dv = f(R)$, (11) takes the form (10). Hence the desired result immediately follows.

Theorem 3. *Let $\partial_2 B = \emptyset$ and assume that $\boldsymbol{C}$ is strongly elliptic and uniformly continuous on $\overline{B}$. If $\boldsymbol{u}$ is a solution to system* (2) *belonging to Ξ and satisfying* (4), *then* $\boldsymbol{u} = 0$ *on* $\overline{Q}$.

Proof. By multiplying both sides of $(3)_1$, by $g\hat{\boldsymbol{u}}$ with $g = e^{-r}$ and integrating over B_R, a simple computation yields

$$(12) \quad \int_{B_R} g\{\xi^2 \rho \hat{u}^2 + \nabla\hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla\hat{\boldsymbol{u}}]\}\, dv = \int_{B_R} g\hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla\hat{\boldsymbol{u}}]\, \boldsymbol{e}_r dv + \int_{\Sigma_R} g\hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla\hat{\boldsymbol{u}}]\, \boldsymbol{e}_r da\ .$$

Since $\lim_{r\to+\infty} \int_{\Sigma_R} g\hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla\hat{\boldsymbol{u}}]\, \boldsymbol{e}_r da = 0$, by letting $R \to +\infty$ in (12), one has

$$(13) \quad \int_B g\{\xi^2 \rho \hat{u}^2 + \nabla\hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla\hat{\boldsymbol{u}}]\}\, dv = \int_B g\hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla\hat{\boldsymbol{u}}]\, \boldsymbol{e}_r dv\ .$$

A lemma of $[4]_1$ implies that $\exists\, \sigma, \delta > 0$:

$$\int_B g\nabla\hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla\hat{\boldsymbol{u}}]\, dv \geq \sigma \int_B g(\nabla\hat{\boldsymbol{u}})^2 dv - \delta \int_B g\hat{u}^2 dv\ .$$

Thus (13) leads to

$$(14) \quad \int_B g\{\xi^2 \rho \hat{u}^2 + \sigma(\nabla\boldsymbol{u})^2\}\, dv \leq \delta \int_B g\hat{u}^2 dv + \int_B g\hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla\boldsymbol{u}]\, \boldsymbol{e}_r dv\ .$$

Now, since

$$g\hat{\boldsymbol{u}} \cdot \boldsymbol{C}[\nabla\boldsymbol{u}]\, \boldsymbol{e}_r \leq g\sigma(\nabla\hat{\boldsymbol{u}})^2 + \frac{|\boldsymbol{C}|^2}{4\sigma\rho}\, g\rho\hat{u}^2\ ,$$

setting $C = \sup_B \dfrac{|\boldsymbol{C}|^2}{4\sigma\rho}$, $h = \sup_B \dfrac{\delta}{\rho}$, (14) yields

$$(\xi^2 - C - h) \int_B g\rho\hat{u}^2 dv = 0\ ,$$

whence the desired result follows, since $\hat{\boldsymbol{u}} = 0\ \forall \xi > \nu_0 = (C + h)^{\frac{1}{2}}$.

3 - Concluding remarks

As far as we are aware, the present paper is to first one concerning uniqueness of solutions to system (1) in unbounded regions, under hypothesis $(*)$ without any simmetry assumption on $\boldsymbol{C}$. A previous uniqueness theorem not requiring symmetry of $\boldsymbol{C}$ is given in [3].

If $\boldsymbol{C}$ satisfies the major symmetry condition $\boldsymbol{A} \cdot \boldsymbol{C}[\boldsymbol{B}] = \boldsymbol{B} \cdot \boldsymbol{C}[\boldsymbol{A}]$, for any couple of second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, strong uniqueness theorems are available $[1]_1$, $[4]_2$, [5], [6], [7]. For example, in $[1]_1$ it is proved that uniqueness holds for solutions to system (1) if the elasticity tensor is positive definite, the density is positive and the acoustic tensor obeys a suitable condition at large distance (2). Further, in $[4]_2$ it is shown that the displacement problem for system (1) ($\partial_2 B = -$) has at most one solution under the same assumption on the acoustic tensor as in $[1]_1$, by requiring that C is strongly elliptic. It is worth remarking that in $[1]_1$, $[4]_2$, [5], [6], [7] the uniqueness is achieved by not assuming any restriction at infinity on the solutions.

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Abstract

We prove three uniqueness theorems for the boundary-initial-value problem of linear elastodynamics, by not requiring that the elasticity tensor by symmetric.

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(2) In $[1]_2$ it is proved that uniqueness fails to hold if such a hypothesis is dropped.