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Hölder regularity of solutions of some degenerate-elliptic obstacle problems (**)

1 - Introduction

The Hölder continuity of solutions of variational inequalities arising from obstacle problems is dealt with by a number of recent works, both in the elliptic and in the parabolic case. Among these let us mention the papers by Biroli [1] and by Caffarelli and Kinderlehrer [2]: their approach is based on potential theory [5] (1).

Herewith, by using the same potential methods, we obtain a De Giorgi-Nash-Moser result for the solutions of some variational inequalities, which arise from obstacle problems and are related to a class of linear degenerate-elliptic operators with discontinuous coefficients.

Boundary value problems for degenerate-elliptic operators have been studied by many authors in the natural framework of weighted Sobolev spaces. The first paper on this subject is due to Murthy and Stampacchia [8]. They extend to a suitable class of the above mentioned operators the well known Hölder regularity results due to De Giorgi, Nash and Moser.

In a recent work [3] Fabes, Kenig and Serapioni choose their weights in the Muckenhoupt class A_2. They consider the corresponding class of degenerate-elliptic operators and extend the usual Hölder regularity results to the solutions of the Dirichlet problems related to these operators.

We also choose our weights out of the class A_2 and consider the corresponding degenerate-elliptic operators. We prove that the solutions of the obstacle problems associated to these operators are Hölder continuous if the

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(1) More refined results in the elliptic case were obtained by Frehse and Mosco who also used some elements of potential theory [4].

obstacle has the same property. Moreover their Hölder exponents are the minimum between the Hölder exponents of the obstacle and of the solution of the Dirichlet problem related to the same operator, the latter exponent being obtained in [3].

1.1 - Notations

We let Ω be a bounded, connected, open subset of $\boldsymbol{R}^n$, with a «smooth» boundary $\partial\Omega$. More precisely we assume that Ω is of *class* S [6], i.e. there exist two numbers $0<\sigma<1$, $\varrho^*>0$ such that $\forall x_0 \in \partial\Omega$, $\forall \varrho<\varrho^*$

$$|B_\varrho(x_0) - \Omega(x_0, \varrho)| \geqslant \sigma |B_\varrho(x_0)| \tag{1.1}$$

where $B_\varrho(x_0) = \{x \in \boldsymbol{R}^n : |x - x_0| < \varrho\}$, $\Omega(x_0, \varrho) = \Omega \cap B_\varrho(x_0)$ and $|\cdot|$ stands for the n-dimensional Lebesgue measure of a subset of $\boldsymbol{R}^n$.

In order to introduce the weighted Sobolev space framework in which we shall set our obstacle problems, let us recall the definition of *Muckenhoupt's* A_2 *weight class*

$$A_2 = \{w : \boldsymbol{R}^n \to [0, +\infty) : w \quad \frac{1}{w} \in L^1_{\text{loc}}(\boldsymbol{R}^n);\ \exists c(w) > 0 \quad \text{s.t.}$$

$$\forall B_\varrho(x_0) \subset \boldsymbol{R}^n \qquad \frac{w(B_\varrho(x_0))}{|B_\varrho(x_0)|} \frac{(1/w)(B_\varrho(x_0))}{|B_\varrho(x_0)|} \leqslant c(w)\}$$

where $w(B_\varrho(x_0)) = \|w\|_{L^1(B_\varrho(x_0))}$ $\quad \frac{1}{w}(B_\varrho(x_0)) = \|\frac{1}{w}\|_{L^1(B_\varrho(x_0))}$.

We also recall that, by choosing weights w in A_2, the following inner products are well defined on the weighted Hilbert spaces $H^1(\Omega; w)$ and respectively $H^1_0(\Omega; w)$ [3]

$$(u|v)_{H^1(\Omega; w)} = (u|v)_{L^2(\Omega; w)} + (\operatorname{grad} u|\operatorname{grad} v)_{(L^2(\Omega; w))^n}$$

$$= \int_\Omega u(x)\, v(x)\, w(x)\, dx + \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v\, w(x)\, dx$$

$$(u|v)_{H^1_0(\Omega; w)} = (\operatorname{grad} u|\operatorname{grad} v)_{(L^2(\Omega; w))^n} .$$

Now let $a(x) = [a_{ij}(x)]$ $(i, j = 1, \dots, n)$ be an $n \times n$ symmetric, non negative matrix defined on Ω the elements of which $a_{ij} : \Omega \to \boldsymbol{R}$ are measurable functions. Then, if $\lambda(x)$, $\Lambda(x) : \Omega \to [0, +\infty)$ denote respectively the minimum and maximum eigenvalues of $a(x)$, we obtain

$$0 \leqslant \lambda(x)|\xi|^2 \leqslant a(x)\,\xi \cdot \xi \leqslant \Lambda(x)|\xi|^2 \qquad \forall \xi \in \boldsymbol{R}^n, \text{ a.e. in } \Omega\,.$$

We shall assume that $\Lambda(x) \leqslant c\lambda(x)$ a.e. in $\Omega (c \geqslant 1)$ and that $\lambda \in A_2$. Therefore we shall choose λ as a weight factor and set $w = \lambda$. Owing to the condition

$$\text{(1.2)} \qquad \frac{1}{c} w(x)|\xi|^2 \leqslant a(x)\,\xi \cdot \xi \leqslant c w(x)|\xi|^2 \qquad \forall \xi \in \boldsymbol{R}^n, \text{ a.e. in } \Omega\,,$$

the second order, linear operator, in divergence form,

$$L = -\,\partial_i (a_{ij}(x)\,\partial_j \cdot) = -\operatorname{div}(a(x) \operatorname{grad} \cdot)\ (^2)$$

associated to the matrix $a(x)$ is degenerate-elliptic.

We point out that $L \in \mathcal{L}(H_0^1(\Omega; w), H^{-1}(\Omega; w))$ where $H^{-1}(\Omega; w) = (H_0^1(\Omega; w))'$, since $(a(x)/w(x)) \in (L^\infty(\Omega))^{n \times n}$ by (1.2). The latter statement can be shown by considering the bilinear form related to L

$$a_L\colon H_0^1(\Omega; w) \times H_0^1(\Omega; w) \to \boldsymbol{R}$$

$$(u, v) \mapsto a_L(u, v) = {}_{H^{-1}(\Omega; w)}\langle Lu, v \rangle_{H_0^1(\Omega; w)} = \int_\Omega a_{ij}(x)\,\partial_j u \partial_i v \,\mathrm{d}x\,.$$

Owing to (1.2), a_L is well-defined, continuous: $\forall u, v \in H_0^1(\Omega; w)$

$$|a_L(u, v)| \leqslant n^2 \|\frac{a}{w}\|_{(L^\infty(\Omega))^{n \times n}} \|u\|_{H_0^1(\Omega; w)} \|v\|_{H_0^1(\Omega; w)}\ (^3)$$

and coercive: $\forall v \in H_0^1(\Omega; w)$, $a_L(v, v) \geqslant (1/c)\|v\|^2_{H_0^1(\Omega; w)}$. The weighted Sobolev space framework thus enables us to recover the usual properties of bilinear forms needed in the variational approach.

1.2 - Problem setting and results

Let $f = [f_i]$ $(i = 1, \ldots, n)$ where $f_i\colon \Omega \to \boldsymbol{R}$ are measurable functions such that $|f|/w \in L^2(\Omega; w)$. It follows that $\operatorname{div} f = \partial_i f_i \in H^{-1}(\Omega; w)$, where we have set

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(2) Here and in the sequel ∂_i stands briefly for $\partial/\partial x_i$ $(i = 1, \ldots, n)$. Moreover the usual summation convention is understood whenever some indexes are repeated.

(3) We set $\|\frac{a}{w}\|_{(L^\infty(\Omega))^{n \times n}} = \max\{\|\frac{a_{ij}}{w}\|_{L^\infty(\Omega)}\colon i, j = 1, \ldots, n\}$.

$\forall v = H^1_0(\Omega; w)$

$$_{H^{-1}(\Omega;w)}\langle -\operatorname{div} f, v\rangle_{H^1_0(\Omega;w)} = \int_\Omega f(x)\cdot \operatorname{grad} v \, dx \, .$$

Let us consider an obstacle $\psi \in L^\infty(\Omega)$; we associate to ψ the convex set

$$K^\psi = \{v \in H^1_0(\Omega; w) \colon v \leqslant \psi \text{ a.e. in } \Omega\}$$

and assume that $K^\psi \neq \Phi$. We could easily show that K^ψ is a closed, convex set.

We study the obstacle problem

$$\text{(1.3)} \qquad a_L(u, v-u) \geqslant {}_{H^{-1}(\Omega;w)}\langle -\operatorname{div} f, v-u\rangle_{H^1_0(\Omega;w)} \quad \forall v \in K^\psi, \;\; u \in K^\psi.$$

It follows from the general theory of variational inequalities that problem (1.3) has a unique solution $u \in K^\psi$. We are going now to investigate the Hölder regularity of this solution. To this end let us recall the Hölder continuity result obtained in [3] for the solution of the Dirichlet problem

$$\text{(1.4)} \qquad a_L(u_0, v) = {}_{H^{-1}(\Omega;w)}\langle -\operatorname{div} f, v\rangle_{H^1_0(\Omega;w)} \quad \forall v \in H^1_0(\Omega; w), \qquad u_0 \in \mathrm{H}^1_0(\Omega; w)\, .$$

Theorem 1. *Let $u_0 \in H^1_0(\Omega; w)$ be the solution of* (1.4). *Assume that* (1.1) *holds and $(|f|/w) \in L^p(\Omega; w))$, where $p > 2n - \varepsilon$. Then there exist two numbers $M > 0$, $0 < \gamma < 1$, which do not depend on u_0 but only on the fixed parameters in the problem, such that $u_0 \in C^\gamma(\bar{\Omega})$; moreover the following estimate holds: $\forall x_0 \in \bar{\Omega}$, $\forall \rho$ small enough*

$$\omega(\rho) \leqslant M \left\| \frac{|f|}{w} \right\|_{L^p(\Omega;w)} \rho^\gamma$$

where $\omega(\rho) = \operatorname{osc}(u_0; \Omega(x_0, \rho)) = \sup\limits_{\Omega(x_0,\rho)} u_0 - \inf\limits_{\Omega(x_0,\rho)} u_0$ (4) .

Our Hölder regularity result for the solution of the obstacle problem (1.3) is summarised by the following theorem.

(4) We define

$$\sup_{\Omega(x_0,\rho)} u_0 = \inf\{k \in \boldsymbol{R} \colon u_0 \leqslant k \text{ on } \Omega(x_0, \rho) \text{ in the sense of } H^1(\Omega; w)\}$$

$$\inf_{\Omega(x_0,\rho)} u_0 = -\sup_{\Omega(x_0,\rho)} (-u_0) \text{ ([3], [6]).}$$

Theorem 2. *Assume that the same hypotheses as in Theorem* 1 *hold. Let the obstacle* ψ *satisfy* $\psi \in C^{\alpha}(\bar{\Omega})$ $(0<\alpha<1)$ *and* $\psi|_{\partial\Omega} \geqslant 0$. *Then the solution* u *of* (1.3) *is in* $C^{\beta}(\bar{\Omega})$, *where* $\beta = \alpha\Lambda\gamma$, i.e. β *is the minimum between the Hölder exponents of the obstacle and of the solution of* (1.4).

Remark 1. Theorem 2 can be proved, without loss of generality, in the homogeneous case $f \equiv 0$.

In order to show this statement we notice that, if $u \in K^{\psi}$ and $u_0 \in H_0^1(\Omega; w)$ denote the solutions of problems (1.3) and respectively (1.4), then $u^* = u - u_0$ is in $K^{\psi - u_0}$ and solves the following homogeneous obstacle problem

$$a_L(u^*, v - u^*) \geqslant 0 \qquad \forall v \in K^{\psi - u_0}, \quad u^* \in K^{\psi - u_0}. \tag{1.5}$$

Now, if we assume that Theorem 2 holds in the case of the homogeneous problem (1.5), we can state that the solution u^* is in $C^{\alpha\Lambda\gamma}(\bar{\Omega})$, since $\psi - u_0 \in C^{\alpha\Lambda\gamma}(\bar{\Omega})$. Hence we deduce that $u = u^* + u_0$ is in $C^{\alpha\Lambda\gamma}(\bar{\Omega})$.

In order to carry out the proof of Theorem 2, where we set $f \equiv 0$, we need some lemmas.

Lemma 1. *Let* $\psi \in L^{\infty}(\Omega(x_0, R))$, $g \in H^1(\Omega(x_0, R); w)$, *where* $x_0 \in \bar{\Omega}$ *and*

$$K_g^{\psi} = \{v \in H^1(\Omega(x_0, R); w) \colon v - g \in H_0^1(\Omega(x_0, R); w), \quad v \leqslant \psi \text{ a.e. in } \Omega(x_0, R)\}$$

a convex set such that $K_g^{\psi} \neq \Phi$. *Consider the following problem: find* $u \in K_g^{\psi}$ *such that*

$$_{H^{-1}(\cdot\,;w)}\langle Lu, v - u\rangle_{H_0^1(\cdot\,;w)} \geqslant 0 \qquad \forall v \in K_g^{\psi}\,. \tag{1.6}$$

Then the mapping $S\colon L^{\infty}(\Omega(x_0, R)) \ni \psi \mapsto S(\psi) = u \in K_g^{\psi}$ *is well defined and is a contraction on* $L^{\infty}(\Omega(x_0, R))$ *in the sense that*

$$\|S(\psi_1) - S(\psi_2)\|_{L^{\infty}} \leqslant \|\psi_1 - \psi_2\|_{L^{\infty}} \qquad \forall \psi_1, \psi_2 \in L^{\infty}(\Omega(x_0, R)) \tag{1.7}$$

holds.

Lemma 2. *Assume that* (1.1) *holds. Let* $x_0 \in \bar{\Omega}$ *and let* $\tilde{u} \in H^1(\Omega(x_0, R); w)$ *be a weak solution of* $L\tilde{u} = 0$ *in* $\Omega(x_0, R)$, *which vanishes on* $\partial\Omega \cap B_R(x_0)$ *in the sense of* $H^1(\Omega(x_0, R);\ w)$ [3], [6]. *Then, there exists a number* $0 < \eta < 1$, *which does not depend on* $\tilde{u}$ *and* $x_0 \in \bar{\Omega}$, *such that the following relationship holds*:

$\forall\rho \leq \rho_0$ *small enough*

$$(1.8) \qquad \omega(\rho) \leq \eta\omega(8\rho) \qquad \textit{where } \omega(\rho) = \operatorname{osc}(\tilde{u}; \Omega(x_0, \rho)) .$$

Lemma 3. *Let* $\omega\colon (0, \nu\rho_0] \ni \rho \mapsto \omega(\rho) \in [0, +\infty)$ *be an increasing function which fulfils the following condition*

$$(1.9) \qquad \omega(\rho) \leq \eta\omega(\nu\rho) + H\rho^{\alpha} \qquad \forall 0 < \rho \leq \rho_0$$

where $0 < \eta < 1$, $\nu > 1$, $0 < \alpha < 1$, $H \geq 0$ *are some fixed parameters. Then, if* $H = 0$, *there exist* $0 < \gamma < 1$, $K > 0$ *such that*

$$(1.10) \qquad \omega(\rho) \leq K\rho^{\gamma} \qquad \forall 0 < \rho \leq \rho_0$$

more precisely γ *satisfies* $\gamma < -\log\eta/\log\nu$.

If $H > 0$, *there exist* $0 < \beta < 1$, $\tilde{K} > 0$ *such that*

$$(1.11) \qquad \omega(\rho) \leq \tilde{K}\rho^{\beta} \qquad \forall 0 < \rho \leq \min\{\rho_0, \nu\}$$

more precisely β *satisfies* $\beta = \alpha \Lambda \gamma$, *where* $\gamma < -\log\eta/\log\nu$.

Remark 2. This lemma will also enable us to obtain the relationship between the Hölder exponents of the solutions of problems (1.3) and respectively (1.4) mentioned in Theorem 2.

2 – Proof of Lemma 1.

We notice that the bilinear form related to L

$$a_L\colon H^1(\Omega(x_0, R); w) \times H^1(\Omega(x_0, R); w) \to \mathbf{R}$$

$$(u, v) \mapsto a_L(u, v) = \int_{\Omega(x_0, R)} a_{ij}(x)\, \partial_j u \partial_i v \, dx$$

is continuous on $H^1(\cdot\,; w) \times H^1(\cdot\,; w)$ and coercive on $H^1_0(\Omega(x_0, R); w) \supset K^{\psi}_g - K^{\psi}_g$. Therefore the general theory of variational inequalities ensures that the mapping S is well defined.

The relationship (1.7) can be proved via a penalization method analogous to the one used by Mignot and Puel in Theorem 1.5 of [7]. To this end we need to introduce the convex set

$$K^{\psi - g} = \{\zeta \in H^1_0(\Omega(x_0, R); w)\colon \zeta \leq \psi - g \quad \text{a.e. in } (\Omega(x_0, R)\} .$$

Then, if we let $u=\xi+g$, we get $\xi\in K^{\psi-g}$ and problem (1.6) transforms into the equivalent one, to be solved in $K^{\psi-g}$

$$a_L(\xi,\zeta-\xi)\geq {}_{H^{-1}(\cdot;w)}\langle -Lg,\zeta-\xi\rangle_{H^1_0(\cdot;w)}\quad \forall\zeta\in K^{\psi-g};\ \xi\in K^{\psi-g}.$$

Proof of Lemma 2. This lemma is proved in Lemma 2.3.11 and in Lemma 2.4.5 of [3]. It turns out from the proofs that ρ_0 must be chosen such that $8\rho_0<\min\{\operatorname{dist}(x_0,\partial\Omega),R\}$, if $x_0\in\Omega$; $8\rho_0<\min\{\rho^*,R\}$, where ρ^* is defined by (1.1) if $x_0,\in\partial\Omega$.

Proof of Lemma 3. This lemma is proved in Lemma 7.5 and in Lemma 7.6 of [9]$_1$. Nevertheless we shall write the proof extensively because it stresses the relationships among the involved parameters.

Let us choose γ such that $0<\gamma<1$ and $\eta\nu^\gamma<1$; therefore γ satisfies: $0<\gamma<\min\{1,-\log\eta/\log\nu\}$. Let us also define $\beta=\alpha\Lambda\gamma$; it follows that also β satisfies $\eta\nu^\beta<1$.

In order to prove (1.11) we set $\tilde\rho=\min\{\rho_0,\nu\}$ and $K=\sup\{\frac{\omega(\rho)}{\rho^\beta}:\ \frac{\tilde\rho}{\nu}\leq\rho\leq\tilde\rho\}$. We obtain $\omega(\rho)\leq K\rho^\beta\quad\forall\frac{\tilde\rho}{\nu}\leq\rho\leq\tilde\rho$ and hence $\omega(\nu\rho)\leq K\nu^\beta\rho^\beta\quad\forall\frac{\tilde\rho}{\nu^2}\leq\rho\leq\frac{\tilde\rho}{\nu}$.

Then (1.9) implies $\omega(\rho)\leq(K\eta\nu^\beta+H)\rho^\beta\quad\forall\frac{\tilde\rho}{\nu^2}\leq\rho\leq\tilde\rho/\nu$, since $\rho^\alpha\leq\rho^\beta$ holds in this interval.

By repeating this procedure, we obtain in any interval $\frac{\tilde\rho}{\nu^{l+1}}\leq\rho\leq\frac{\tilde\rho}{\nu^l}\quad l=1,2,\ldots$

$$\omega(\rho)\leq[K(\eta\nu^\beta)^l+H\sum_{s=0}^{l-1}(\eta\nu^\beta)^s]\rho^\beta\leq(K+\frac{H}{1-\eta\nu^\beta})\rho^\beta,$$

since we have assumed $\eta\nu^\beta<1$.

Hence (1.11) follows, where $\tilde K=K+H/(1-\eta\nu^\beta)$. In the case $H=0$ we have (1.10).

Proof of Theorem 2. According to Remark 1, we carry out the proof in the homogeneous case $f\equiv 0$. Therefore we consider the problem: find $u\in K^\psi$ such that

(2.1) $$a_L(u,v-u)\geq 0\quad \forall v\in K^\psi,\qquad \psi\in C^\alpha(\bar\Omega)\quad \psi|_{\partial\Omega}\geq 0.$$

Let $x_0\in\bar\Omega$ and let $\tilde u\in H^1(\Omega(x_0,R);w)$ be the weak solution of the Dirichlet problem

$$ {}_{H^{-1}(\cdot\,;w)}\langle L\tilde{u}, \varphi \rangle_{H^1_0(\cdot\,;w)} = 0 \qquad \forall \varphi \in H^1_0(\Omega(x_0,R);\ w), \qquad \tilde{u} - u \in H^1_0(\Omega(x_0,R); w)\ . $$

We point out that, if $x_0 \in \partial\Omega$ or $R > \operatorname{dist}(x_0, \partial\Omega)$, then $\tilde{u}$ vanishes on $\partial\Omega \cap B_R(x_0)$ in the sense of $\mathrm{H}^1(\Omega(x_0,R); w)$.

Now we try to estimate $\|u - \tilde{u}\|_{L^\infty(\Omega(x_0,R))}$. To this end we notice that

$$ \begin{aligned} u|_{\partial(\Omega(x_0,R))} &\leqslant \mathscr{A}\,(\psi; \Omega(x_0,R)) + \operatorname{osc}(\psi; \Omega(x_0,R)) \\ &\leqslant \mathscr{A}\,(\psi; \Omega(x_0,R)) + 2\|\psi\|_{C^\alpha(\bar{\Omega})} R^\alpha \end{aligned} $$

where $\|\psi\|_{C^\alpha(\bar{\Omega})} = \sup\{\dfrac{|\psi(x) - \psi(x')|}{|x - x'|^\alpha}:\ x, x' \in \bar{\Omega},\ x \neq x'\}$ and $\mathscr{A}\,(\psi; \Omega(x_0,R))$ denotes the average of ψ over $\Omega(x_0,R)$, i.e.

$$ \mathscr{A}\,(\psi; \Omega(x_0,R)) = \frac{1}{|\Omega(x_0,R)|} \int_{\Omega(x_0,R)} \psi(x)\,\mathrm{d}x\ . $$

Therefore, as an application of the weak maximum principle (Theorem 2.2.2 in [3]), we get

$$ \tilde{u} \leqslant \tilde{\psi} = \mathscr{A}\,(\psi; \Omega(x_0,R)) + 2\|\psi\|_{C^\alpha(\bar{\Omega})} R^\alpha \qquad \text{a.e. in } \Omega(x_0,R)\ . $$

It follows that $\tilde{u}$ belongs to the convex set

$$ K^{\tilde{\psi}}_u = \{v \in H^1(\Omega(x_0,R); w):\ v - u \in H^1_0(\Omega(x_0,R); w),\ v \leqslant \tilde{\psi} \text{ a.e. in } \Omega(x_0,R)\} $$

and satisfies in $K^{\tilde{\psi}}_u$ the variational inequality

$$ {}_{H^{-1}(\cdot\,;w)}\langle L\tilde{u}, v - \tilde{u} \rangle_{H^1_0(\cdot\,;w)} \geqslant 0 \qquad \forall v \in K^{\tilde{\psi}}_u \qquad \tilde{u} \in K^{\tilde{\psi}}_u\ . $$

Now, by applying Lemma 1, we obtain the estimate

$$ \begin{aligned} \|u - \tilde{u}\|_{L^\infty(\Omega(x_0,R))} &= \|S(\psi) - S(\tilde{\psi})\|_{L^\infty(\Omega(x_0,R))} \leqslant \|\psi - \tilde{\psi}\|_{L^\infty(\Omega(x_0,R))} \\ &\leqslant 2\operatorname{osc}(\psi; \Omega(x_0,R)) \leqslant 4\|\psi\|_{C^\alpha(\bar{\Omega})} R^\alpha \end{aligned} $$

and hence

$$ \begin{aligned} \operatorname{osc}(u; \Omega(x_0,R)) &\leqslant \operatorname{osc}(\tilde{u}; \Omega(x_0,R)) + 2\|u - \tilde{u}\|_{L^\infty(\Omega(x_0,R))} \\ &\leqslant \operatorname{osc}(\tilde{u}; \Omega(x_0,R)) + 8\|\psi\|_{C^\alpha(\bar{\Omega})} R^\alpha. \end{aligned} \tag{2.2} $$

We stress that (2.2) holds in particular $\forall R \leq R_0$, where we choose

$$R_0 < \operatorname{dist}(x_0, \partial\Omega) \text{ if } x_0 \in \Omega, \qquad R_0 < \rho^* \text{ with } \rho^* \text{ given by (1.1) if } x_0 \in \partial\Omega .$$

Let us recall at this point Lemma 2, where we take $\rho_0 = R/8$. We obtain $\forall \rho \leq R/8$ and $\forall R \leq R_0$

$$\text{(2.3)} \qquad \operatorname{osc}(u; \Omega(x_0, \rho)) \leq \operatorname{osc}(\tilde{u}; \Omega(x_0, \rho)) + 8\|\psi\|_{C^\alpha(\bar{\Omega})} R^\alpha \qquad \text{(by (1.8))}$$

$$\leq \eta \operatorname{osc}(\tilde{u}; \Omega(x_0, 8\rho)) + 8\|\psi\|_{C^\alpha(\bar{\Omega})} R^\alpha \qquad \text{(by (2.2))}$$

$$\leq \eta \operatorname{osc}(u; \Omega(x_0, 8\rho)) + 8(\eta + 1)\|\psi\|_{C^\alpha(\bar{\Omega})} R^\alpha .$$

If we let $\rho = R/8$ in (2.3), then we have $\forall R \leq R_0$

$$\text{(2.4)} \qquad \operatorname{osc}(u; \Omega(x_0, \frac{R}{8})) \leq \eta \quad \operatorname{osc}(u; \Omega(x_0, R)) + H(\frac{R}{8})^\alpha .$$

Therefore we are entitled to apply Lemma 3 and state that the solution u of the obstacle problem (2.1) is in $C^\beta(\bar{\Omega})$, where $0 < \beta < 1$ is suitable.

From the inequalities (1.8) and (2.4), which characterise $\operatorname{osc}(\tilde{u}; \Omega(x_0, R))$ and $\operatorname{osc}(u; \Omega(x_0, R))$ respectively, we also obtain, via Lemma 3, the relationship $\beta = \alpha \Lambda \gamma$ between the Hölder exponent β of u and the Hölder exponent γ of any solution $\tilde{u}$ of $L\tilde{u} = 0$ which either is of local type or vanishes on a part of $\partial\Omega$.

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Sunto

Si considerano problemi con ostacolo associati ad una classe di operatori ellittici degeneri con coefficienti discontinui. Si estende alle soluzioni di tali problemi il risultato di hölderianità ottenuto in [3] *per le soluzioni dei problemi di Dirichlet associati alla stessa classe di operatori.*

Si dimostra in particolare che gli esponenti di Hölder delle soluzioni dei problemi con ostacolo sono il minimo tra l'esponente di Hölder dell'ostacolo e quello della soluzione del corrispondente problema di Dirichlet.

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