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G-structures on the frame bundle of second order (**)

Introduction

The purpose of this paper is to describe a procedure which allows the construction of a great variety of G-structures on F^2M, the frame bundle of second order of an n-dimensional manifold M endowed with a connection ω of order 2. The basic idea of this procedure is the following: model structures on $\boldsymbol{R}^n$, $gl(n)$ and $S^2(n)$ are considered and, by means of the connection ω, are translated to F^2M. A similar treatment for the frame bundle FM of M and a linear connection on M was considered by Cordero and one of us in [**1**] and by Terrier in [**12**] in the case of almost complex structures. Actually, both constructions are related.

The paper is structured as follows. In **1** and **2**, we recall, for later use, the definition and properties of the frame bundle of the second order F^2M and of connections of order 2 on M. We notice that standard horizontal vector fields for connections of order 2 can be introduced and that they are essential in our study. In **3**, we consider the absolute parallelism on F^2M associated to ω and the corresponding trivialization of the frame bundle FF^2M, which permits to get, for any Lie subgroup $G \subset Gl(n+n^2+n^2(n+1)/2)$, the ω-associated G-structure P on F^2M. When G is assumed to be the isotropy group of a point $\tilde{u} \in \tilde{F}$ with respect to a linear representation $\tilde{\varrho}$ on a vector space $\tilde{F}$, then P can be defined by a differentiable tensor $\tilde{t}$ on FF^2M of type $(\tilde{\varrho}, \tilde{F})$. And when appropriate and particular choices of $\tilde{F}$, $\tilde{u}$ and $\tilde{\varrho}$ are made, we can give a precise description of the tensor field on F^2M associated to $\tilde{t}$. In **4** and **5**, we study the particular case of tensor fields of type (1, 1) on F^2M and

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some examples are given. Finally, in **6**, we discuss the case of tensor fields of type (0, 2); particularly, almost symplectic and Riemannian structures deserve our special attention.

1 - The frame bundle of the second order

In this section, we recall, for later use, the definition and some properties of the frame bundle of order 2. Details can be found in [**4**], [**6**], [**8**] and [**10**]. Let M be an n-dimensional manifold. If U and V are two neighborhoods of the origin 0 of $\boldsymbol{R}^n$, two mappings $f\colon U \to M$ and $g\colon V \to M$ are said to define the same r-jet at 0 if they have the same partial derivatives up to order r at 0. The r-jet given by f is denoted by $j_0^r(f)$. If f is a diffeomorphism of a neighborhood of 0 onto an open subset of M, then the r-jet $j_0^r(f)$ at 0 is called an r-frame at $x = f(0)$. Clearly, a 1-frame is an ordinary linear frame. The set of r-frames of M, denoted by F^rM, is a principal bundle over M with natural projection π^r, $\pi^r(j_0^r(f)) = f(0)$, and with structure group $G^r(n)$ which will be described next.

Let $G^r(n)$ be the set of r-frames $j_0^r(g)$ at $0 \in \boldsymbol{R}^n$, where g is a diffeomorphism from a neighborhood of 0 in $\boldsymbol{R}^n$ onto a neighborhood of 0 in $\boldsymbol{R}^n$. Then $G^r(n)$ is a group with multiplication defined by the composition of jets, i.e.,

$$j_0^r(g)\cdot j_0^r(g') = j_0^r(g\circ g')\ .$$

The group $G^r(n)$ acts on F^rM on the right by

$$j_0^r(f)\cdot j_0^r(g) = j_0^r(f\circ g) \qquad \text{for } j_0^r(f)\in F^rM,\ j_0^r(g)\in G^r(n)\ .$$

Clearly, F^1M is the bundle of the linear frames over M with group $G^1(n) = Gl(n)$ and projection $\pi^1 = \pi$. From now on, we shall consider only F^1M and F^2M and denote by π_1^2 the natural projection $F^2M \to FM$, $\pi_1^2(j_0^2(f)) = j_0^1(f)$.

For any coordinate system in M, (U, x^i), we consider the induced coordinate systems $\{FU, (x^i, X_j^i)\}$ and $\{F^2U; (x^i, X_j^i, X_{jk}^i)\}$ in FM and F^2M, respectively, where $X_{jk}^i = X_{kj}^i$.

We have a natural isomorphism $G^2(n) \cong Gl(n)\times S^2(n)$, where $S^2(n)$ is the set of symmetric bilinear forms on $\boldsymbol{R}^n$, multiplication on the right being given by $(A, \alpha)\cdot(B, \beta) = (AB, \alpha\circ(B, B) + A\circ\beta)$. Then, the Lie algebra $g^2(n)$ of $G^2(n)$ can be identified with $gl(n)\oplus S^2(n)$, with a bracket product given by

$$[(A, \alpha), (B, \beta)]=([A, B], A\circ\beta-\beta\circ(I, A)-\beta\circ(A, I)-(B\circ\alpha-\alpha\circ(I, B)-\alpha\circ(B, I))),$$

where I is the unit matrix.

With these identifications, the adjoint representations of $G^2(n)$ in $a(n) = \boldsymbol{R}^n \oplus gl(n)$ is given by

$$Ad^{(2)}(A, \alpha)(v, B) = \left(Av, \bar{\alpha}(v)A^{-1} + ABA^{-1}\right),$$

where $\bar{\alpha}\colon \boldsymbol{R}^n \to gl(n)$ is the linear map defined by $\bar{\alpha}(v)(w) = \alpha(v, w)$, and the adjoint representation of $G^2(n)$ in $g^2(n) = gl(n) \oplus S^2(n)$ is given by

$$Ad(A, \alpha)(B, \beta) = (ABA^{-1}, \alpha\circ(A^{-1}, BA^{-1})$$

$$+ \alpha\circ(BA^{-1}, A^{-1}) - ABA^{-1}\circ\alpha\circ(A^{-1}, A^{-1}) + A\circ\beta\circ(A^{-1}, A^{-1}))\,.$$

From now on, we shall denote by $\{E_i\}$, $\{E^i_j\}$ and $\{E^i_{jk}\}$, $i, j, k = 1, \ldots, n$, $E^i_{jk} = E^i_{kj}$, the natural bases of $\boldsymbol{R}^n$, $gl(n)$ and $S^2(n)$, respectively. Since $G^2(n)$ acts on F^2M on the right, every element (A, α) of the Lie algebra $g^2(n)$ of $G^2(n)$ induces a vector field $\lambda(A, \alpha)$ on F^2M, called the *fundamental vector field* corresponding to (A, α). So, the vertical subspace at any point $p \in F^2M$ can be decomposed as $\lambda(gl(n))_p \oplus \lambda(S^2(n))_p$.

The canonical form θ of F^2M is an $a(n)$-valued 1-form of type $Ad^{(2)}G^2(n)$ and satisfies $\theta(\lambda(A, \alpha)) = A$. Let $\theta = \theta_{-1} + \theta_0$ be the decomposition of θ; then, θ_{-1} is an $\boldsymbol{R}^n$-valued 1-form and θ_0 a $gl(n)$-valued 1-form on F^2M. We have

$$\theta_{-1}(\lambda(A, \alpha)) = 0\,, \qquad \theta_0(\lambda(A, \alpha)) = A\,.$$

Moreover,

$$\theta_{-1} = (\pi^2_1)^*\bar{\theta}\,, \tag{1.1}$$

where $\bar{\theta}$ is the canonical form of FM. With respect to the natural bases, we shall put

$$\theta_{-1} = \theta^i E_i\,, \qquad \theta_0 = \theta^i_j E^j_i\,.$$

2 - Connections of order 2

A connection Γ in the bundle F^2M of 2-frames of M is called *a connection of order* 2 on M. Let ω be the connection form of the connection Γ. Then ω can be decomposed as $\omega = \omega_0 + \omega_1$, where ω_0 is a $gl(n)$-valued 1-form an ω_1 an $S^2(n)$-valued 1-form on F^2M. Since $\omega(\lambda(A, \alpha)) = (A, \alpha)$, we have

$$\omega_0(\lambda(A, \alpha)) = A\,, \qquad \omega_1(\lambda(A, \alpha)) = \alpha\,.$$

With respect to the natural bases, we write

$$\omega_0 = \omega^i_j E^j_i\,, \qquad \omega_1 = \omega^i_{jk} E^{jk}_i\,, \qquad \text{where } \omega^i_{jk} = \omega^i_{kj}\,.$$

For any 2-frame p on M, $(\theta_{-1})_p$ gives a linear isomorphism of the horizontal subspace H_p at p onto $\boldsymbol{R}^n$. Thus, we associate with each $\xi \in \boldsymbol{R}^n$ a horizontal vector field $C(\xi)$ on F^2M as follows. For each $p \in F^2M$, $C(\xi)_p$ is the unique horizontal vector at p such that $(\theta_{-1})_p C(\xi)_p = \xi$.

We call $C(\xi)$ the *standard horizontal vector field* on F^2M corresponding to ξ.

The following proposition is easily proved.

Proposition 2.1. *The standard horizontal vector fields have the following properties*

(1) $\quad R_{(A,\alpha)}\, C(\xi) = C(A^{-1}\xi) \quad$ *for* $(A, \alpha) \in G^2(n)$ *and* $\xi \in \boldsymbol{R}^n$.

In particular,

$$R_\alpha\, C(\xi) = 0 \qquad \text{for } \alpha \in S^2(n)\,.$$

(2) *If* $\xi \neq 0$, *then* $C(\xi)$ *never vanishes.*

Proof. (1) follows from the fact that if X is a horizontal vector at p, then $R_{(A,\alpha)}X$ is a horizontal vector at $p(A, \alpha)$ and θ is of type $Ad^{(2)}G^2(n)$. To prove (2), assume that $C(\xi)_p = 0$ at some point $p \in F^2M$. Then $0 = (\theta_{-1})_p\, C(\xi)_p = \xi$.

Proposition 2.2. *If* $\lambda(A, \alpha)$ *is the fundamental vector field corresponding to* $(A, \alpha) \in g^2(n)$ *and if* $C(\xi)$ *is the standard horizontal vector field corresponding to* $\xi \in \boldsymbol{R}^n$, *then* $[\lambda(A, \alpha), C(\xi)] = C(A\xi)$.

In particular, $[\lambda A, C(\xi)] = C(A\xi)$, $[\lambda\alpha, C(\xi)] = 0$.

Proof. Let (a_t, α_t) be the 1-parameter subgroup of $G^2(n)$ generated by (A, α), $a_t = \exp tA$. Then

$$[\lambda(A, \alpha), C(\xi)] = \lim_{t\to 0} \frac{1}{t}\,[C(\xi) - R_{(a_t,\alpha_t)}C(\xi)]$$

$$= \lim_{t\to 0} \frac{1}{t}\,[C(\xi) - C(a_t^{-1}\xi)]\,, \qquad \text{by Proposition 2.1}\,.$$

Since $\xi \to C(\xi)_p$ is a linear isomorphism of $\boldsymbol{R}^n$ onto the horizontal subspace H_p at p, we have

$$\lim_{t\to 0}\frac{1}{t}\,[C(\xi) - C(a_t^{-1}\xi)] = C\bigl(\lim_{t\to 0}\frac{1}{t}\,(\xi - a_t^{-1}\xi)\bigr) = C(A\xi)\,.$$

Let $\Theta = D\theta$, $\Omega = D\omega$ be the torsion and curvature forms of Γ, respectively. Then Θ is a tensorial 2-form of type $Ad^{(2)}G^2(n)$ and Ω a tensorial 2-form of type $AdG^2(n)$. Consequently, θ (resp., Ω) can be decomposed as $\Theta = \Theta_{-1} + \Theta_0$ (resp., $\Omega = \Omega_0 + \Omega_1$).

A simple calculation shows that

$$\Theta_{-1} = D\theta_{-1}\,, \qquad \Theta_0 = D\theta_0\,, \qquad \Omega_0 = D\omega_0\,, \qquad \Omega_1 = D\omega_1\,.$$

Theorem 2.3. (Structure equations) *Let ω, Θ and Ω be the connection form, the torsion form and the curvature form of a connection Γ of order 2 on M, respectively. Then, we have:*

1st structure equation

$$d\theta_{-1}(X, Y) = -\tfrac{1}{2}\{\omega_0(X)\,\theta_{-1}(Y) - \omega_0(Y)\,\theta_{-1}(X)\} + \Theta_{-1}(X, Y)\,;$$

2nd structure equation

$$d\omega(X, Y) = -\tfrac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y)\,,$$

where $X, Y \in T_p(F^2M)$, $p \in F^2M$.

Proof. The second structure equation can be found in [7] for an arbitrary connection on a principal bundle. To prove the first structure equation, we consider the following three special cases:

(1) X and Y are horizontal. In this case, $\omega_0(X) = \omega_0(Y) = 0$ and the equality reduces to the definition of Θ_{-1}.

(2) X and Y are vertical. Let $X = \lambda(A, \alpha)$, $Y = \lambda(B, \beta)$ at p. Then

$$\begin{aligned}2\,d\theta_{-1}(X, Y) &= \lambda(A, \alpha)\,\theta_{-1}(\lambda(B, \beta)) - \lambda(B, \beta)\,\theta_{-1}(\lambda(A, \alpha)) - \theta_{-1}[\lambda(A, \alpha), \lambda(B, \beta)]\\ &= \theta_{-1}(\lambda[(A, \alpha), (B, \beta)]) = 0\,.\end{aligned}$$

On the other hand, $\theta_{-1}(\lambda(A, \alpha)) = \theta_{-1}(\lambda(B, \beta)) = 0$ and $\Theta_{-1}(\lambda(A, \alpha), \lambda(B, \beta)) = 0$.

(3) X is horizontal and Y is vertical. We choose $(A, \alpha) \in g^2(n)$ and $\xi \in \boldsymbol{R}^n$

such that $X = C(\xi)_p$, $Y = \lambda(A, \alpha)_p$. Then

$$2\,\mathrm{d}\theta_{-1}(X, Y) = C(\xi)\,\theta_{-1}(\lambda(A, \alpha)) - \lambda(A, \alpha)\,\theta_{-1}(C(\xi)) - \theta_{-1}([C(\xi), \lambda(A, \alpha)])$$

$$= \theta_{-1}(C(A\xi)) = A\xi$$

by Proposition 2.2. On the other hand, $\Theta_{-1}(X, Y) = 0$, $\omega_0(X)\theta_{-1}(Y) = 0$ and

$$\omega_0(Y)\,\theta_{-1}(X) = \omega_0(\lambda(A, \alpha))\,\theta_{-1}(C(\xi)) = A\xi.$$

With respect to the natural bases, we write

$$\theta_{-1} = \theta^i E_i\,, \qquad \theta_0 = \theta^i_j E^i_j\,, \qquad \Theta_{-1} = \Theta^i E_i\,, \qquad \Theta_0 = \Theta^i_j E^i_j\,,$$

$$\omega_0 = \omega^i_j E^i_j\,, \qquad \omega_1 = \omega^i_{jk} E^i_{jk}\,, \qquad \Omega_0 = \Omega^i_j E^i_j\,, \qquad \Omega_1 = \Omega^i_{jk} E^i_{jk}\,,$$

where $\omega^i_{jk} = \omega^i_{kj}$, $\Omega^i_{jk} = \Omega^i_{kj}$. Then the structure equations can be written as

(1) $\mathrm{d}\theta^i = -\,\omega^i_j \wedge \theta^j + \Theta^i$,

(2) $\mathrm{d}\omega^i_j = -\,\omega^i_k \wedge \omega^k_j + \Omega^i_j$, $\quad \mathrm{d}\omega^i_{jk} = -\,\omega^i_r \wedge \omega^r_{jk} + \omega^r_k \wedge \omega^i_{jr} + \omega^r_j \wedge \omega^i_{rk} + \Omega^i_{jk}$.

We also write the structure equations in the following simplified form

(1) $\mathrm{d}\theta_{-1} = -\,\omega_0 \wedge \theta_{-1} + \Theta_{-1}\,,$ $\qquad$ (2) $\mathrm{d}\theta = -\,\omega \wedge \omega + \Omega\,.$

Corollary 2.4. *For a connection of order* 2 *on* M, *we have*

$$[C_1, C_2] = -\,2C\Theta_{-1}(C_1, C_2) - 2\lambda\Omega(C_1, C_2)\,,$$

for any standard horizontal vector fields C_1, C_2 *on* F^2M.

Proof. By using the first structure equation, we have

$$\mathrm{d}\theta_{-1}(C_1, C_2) = \tfrac{1}{2}\{C_1\theta_{-1}(C_2) - C_2\,\theta_{-1}(C_1) - \theta_{-1}[C_1, C_2]\}$$

$$= -\,\tfrac{1}{2}\,\theta_{-1}[C_1, C_2] = \Theta_{-1}(C_1, C_2)\,.$$

On the other hand, from the second structure equation, we obtain

$$\mathrm{d}\omega(C_1, C_2) = \tfrac{1}{2}\{C_1\omega(C_2) - C_2\omega(C_1) - \omega[C_1, C_2]\} = -\,\tfrac{1}{2}\,\omega[C_1, C_2] = -\,\Omega(C_1, C_2).$$

Taking into account that $C\theta_{-1} + \lambda\omega = 1$, we have

$$[C_1, C_2] = C\theta_{-1}[C_1, C_2] + \lambda\omega[C_1, C_2] = -2\Theta_{-1}(C_1, C_2) - 2\lambda\Omega(C_1, C_2) .$$

Let Γ be a connection of order 2 on M. Since the natural projection π_1^2: $F^1M \to FM$ is a homomorphism of principal bundles over the identity of M and the associate homomorphism is the natural projection $G^2(n) \to Gl(n)$, the connection Γ defines a connection in FM, that is, a linear connection $\bar{\Gamma}$ on M. We call $\bar{\Gamma}$ the *linear connection on M induced* by Γ. If $\bar{\omega}$, $\bar{\Omega}$ are the connection and curvature forms of $\bar{\Gamma}$, then

$$(2.1) \qquad (\pi_1^2)^* \bar{\omega} = \omega_0 , \qquad (\pi_1^2)^* \bar{\Omega} = \Omega_0 .$$

Let $\bar{\lambda}A$ (resp., $B(\xi)$) be the fundamental vector field (resp., the standard horizontal vector field with respect to $\bar{\Gamma}$) corresponding to $A \in gl(n)$ (resp., $\xi \in \boldsymbol{R}^n$). A simple calculation shows that $\pi_1^2\lambda(A, \alpha) = \bar{\lambda}A$ (resp., $\pi_1^2 C(\xi) = B(\xi)$).

Moreover, if $\bar{\Theta}$ is the torsion form of $\bar{\Gamma}$, it is easy to verify that

$$(2.2) \qquad (\pi_1^2)^* \bar{\Theta} = \Theta_{-1} .$$

3 - G-structures on the frame bundle of the second order

Now, suppose we are given a connection Γ of order 2 on M with connection form ω. Then, the family of vector fields $\{CE_i, \lambda E_j^i, \lambda E_{jk}^i\}$ defines an absolute parallelism on F^2M associated to the connection Γ. This parallelism allows to define a trivialization of FF^2M, the frame bundle of F^2M,

$$\tau : F^2M \times Gl(N) \to FF^2M ,$$

where $N = n + n^2 + \frac{1}{2}n^2(n+1)$, by setting $\tau(p, A) = \tilde{p}_0 A$, $p \in F^2M$, $A \in Gl(N)$, $\tilde{p}_0$ being the linear frame of $T_p(F^2M)$ given by $\tilde{p}_0 = \{(CE_i)_p, (\lambda E_j^i)_p, (\lambda E_{jk}^i)_p\}$.

Then, given a Lie subgroup $G \subset Gl(N)$, we may consider a G-structure on the manifold F^2M given by the principal bundle $P_G = \tau(F^2M \times G)$.

Def. 3.1. P_G will be called the *ω-associated G-structure on F^2M*.

Let $\bar{\Gamma}$ be the linear connection on M indiced by Γ; then the family of vector fields $\{BE_i, \bar{\lambda}E_j^i\}$ defines an absolute parallelism on FM which permits to define a trivialization of FFM, the frame bundle of FM, $\bar{\tau}: FM \times Gl(\bar{N})$

$\to FFM$, where $\bar{N} = n + n^2$, in such a way that the following diagram is commutative

$$\begin{array}{ccc} F^2M \times Gl(N) & \xrightarrow{\tau} & FF^2M \\ \downarrow & & \downarrow \\ FM \times Gl(\bar{N}) & \xrightarrow{\bar{\tau}} & FFM \end{array}$$

where the vertical arrows are the natural ones.

Then, if $\bar{G} \subset Gl(\bar{N})$ is the natural projection of $G \subset Gl(N)$, the G-structure $P_G = \tau(F^2M \times G)$ on F^2M projects onto the $\bar{G}$-structure $P_{\bar{G}} = \bar{\tau}(FM \times \bar{G})$ on FM, called *$\bar{\omega}$-associated* in [**1**].

Remark. Obviously, if P_G is integrable, so is $P_{\bar{G}}$.

Let $\tilde{F}$ be a finite dimensional real vector space, $\tilde{\varrho}\colon Gl(N) \to Gl(\tilde{F})$ a linear representation and $G_{\tilde{u}} \subset Gl(N)$ the isotropy group of $\tilde{u} \in \tilde{F}$. Then, we define a differentiable tensor $\tilde{t}\colon FF^2M \to \tilde{F}$ of type $(\tilde{\varrho}, \tilde{F})$ by setting

$$\tilde{t}(\tilde{p}) = \tilde{\varrho}(A^{-1})\tilde{u} \qquad \tilde{p} \in F^2M, \tag{3.1}$$

where $A \in Gl(N)$ is the unique element such that $\tilde{p} = \tilde{p}_0 A$. Obviously, $\tilde{t}$ takes its values in $\tilde{F}_{\tilde{u}} = \{\tilde{\varrho}(\tilde{A})\tilde{u} / A \in Gl(N)\}$ and therefore $\tilde{t}^{-1}(\tilde{u}) \subset FF^2M$ is a principal subbundle with structure group $G_{\tilde{u}}$. Moreover, since $\tilde{t}^{-1}(\tilde{u}) = \tilde{P}_{G_{\tilde{u}}}$, we have proved

Theorem 3.2. *With the notations above, the ω-associated $G_{\tilde{u}}$-structure on F^2M is defined by the differentiable tensor $\tilde{t}$ given by* (3.1).

Sections **4**, **5** and **6** of this paper are mainly devoted to the construction and study of some particular cases of this general situation.

4 - G-structures on F^2M defined by tensor fields of type (1, 1)

Let us consider $\tilde{F} = \mathrm{Hom}(\mathbf{R}^N, \mathbf{R}^N) \cong (\mathbf{R}^N)^* \otimes \mathbf{R}^N$ and let $\tilde{\varrho}\colon Gl(N) \to Gl(\tilde{F})$ be the canonical linear representation given by $\tilde{\varrho}(A) = A\tilde{u}A^{-1}$, $\tilde{u} \in \tilde{F}$, $A \in Gl(N)$. Then, there exists a one-to-one correspondence between $G_{\tilde{u}}$-structures on F^2M which are defined by differentiable tensors $\tilde{t}$ on FF^2M of type $(\tilde{\varrho}, \tilde{F})$ and $F_{\tilde{u}}$-valued, $G_{\tilde{u}}$ being the isotropy group of $\tilde{u} \in \tilde{F}$, and the tensor fields $\tilde{J}$ on F^2M of type (1, 1) given by

$$\tilde{J}_p(X) = \tilde{p}\tilde{t}(\tilde{p})\tilde{p}^{-1}(X), \quad X \in T_pF^2M, \quad p \in F^2M, \quad \tilde{p} \in \pi^{-1}(p). \tag{4.1}$$

In the sequel, we shall be interested only in those G-structures on F^2M defined by tensor fields of type (1, 1) and which can be modelled on some special structures on the vector spaces $\boldsymbol{R}^n$, $gl(n)$ and $S^2(n)$ and which are, at the same time, ω-associated in the sense described above.

Then, if we consider the canonical isomorphism of vector spaces $\boldsymbol{R}^N \cong \boldsymbol{R}^n \times \boldsymbol{R}^{n^2} \times \boldsymbol{R}^{n^2(n+1)/2} \cong \boldsymbol{R}^n \times gl(n) \times S^2(n)$, put $F = (\boldsymbol{R}^n)^* \otimes \boldsymbol{R}^n$, $F' = (gl(n))^* \otimes gl(n)$, $F'' = (S^2(n))^* \otimes S^2(n)$, denote the canonical representations by $\varrho\colon Gl(n) \to Gl(F)$, $\varrho'\colon Gl(n^2) \to Gl(F')$, $\varrho''\colon Gl(n^2(n+1)/2) \to Gl(F'')$ and the canonical injection by $j\colon Gl(n) \times Gl(n^2) \times Gl(n^2(n+1)/2) \to Gl(N)$, we have $\tilde{\varrho} \circ j = \varrho \oplus \varrho' \oplus \varrho''$. Consequently, if $u \in F$, $u' \in F'$, $u'' \in F''$ and if we take $\tilde{u} = u + u' + u'' \in \tilde{F}$, we have $F_u \oplus F'_{u'} \oplus F''_{u''} \subset \tilde{F}_{\tilde{u}}$ and $j(G_u \times G_{u'} \times G_{u''}) \subset G_{\tilde{u}}$.

Now, let $\tilde{t}$ be the differentiable tensor given by (3.1) which defines the ω-associated $G_{\tilde{u}}$-structure on F^2M; then, we have

Theorem 4.1. *For any $X \in T_\varrho F^2M$ and any $p \in F^2M$*

$$\tilde{J}_\varrho(X) = \big(Cu((\theta_{-1})_\varrho X)\big)_\varrho + \big(\lambda u'((\omega_0)_\varrho X)\big)_\varrho + \big((\lambda u''((\omega_1)_\varrho X)\big)_\varrho \qquad \textit{that is}$$

$$\tilde{J} = Cu\theta_{-1} + \lambda u'\omega_0 + \lambda u''\omega_1 \,. \tag{4.2}$$

Proof. The result follows directly from (4.1) taking into account that $\tilde{p}_0(\xi, 0, 0) = (0\xi)_p$, $\tilde{p}_0(0, A, 0) = (\lambda A)_p$, $\tilde{p}_0(0, 0, \alpha) = (\lambda\alpha)_p$, $\xi \in \boldsymbol{R}^n$, $A \in gl(n)$, $\alpha \in S^2(n)$, and that $\tilde{t}(\tilde{p}_0) = \tilde{u} = u + u' + u''$.

We remark that J is 0-deformable in the sense of [9] because, for any $p \in F^2M$, $\tilde{J}_p$ is expressed with respect to the linear frame $\tilde{p}_0$ at p by the matrix

$$\begin{pmatrix} u & 0 & 0 \\ 0 & u' & 0 \\ 0 & 0 & u'' \end{pmatrix},$$

which does not depend on p. Moreover, $\operatorname{rank} \tilde{J} = \operatorname{rank} u + \operatorname{rank} u' + \operatorname{rank} u''$. As well known, we have

Theorem 4.2. *The ω-associated $G_{\tilde{u}}$-structure on F^2M defined by $\tilde{J}$ is integrable if and only if its first structure tensor vanishes identically.*

Next, we shall determine the Nijenhuis torsion $N_{\tilde{J}}$ of $\tilde{J}$ since the vanishing of $N_{\tilde{J}}$ is a necessary, and, in some cases, sufficient condition for the integrability of $\tilde{J}$. To carry on with our work, we need

Def. 4.3. For each $\eta \in \Lambda^2(F^2M, V)$, a 2-form on F^2M valued in a vector space V, and $J_V\colon V \to V$ an endomorphism of V, let $\tilde\eta \in \Lambda^2(F^2M, V)$ be defined by

$$\tilde\eta(X, Y) = 2\{-\eta(\tilde J X, \tilde J Y) + J_V\eta(\tilde J X, Y) + J_V\eta(X, \tilde J Y) - J_V^2\eta(X, Y)\},$$

X and Y being arbitrary vector fields on F^2M.

Then, we have

Theorem 4.4. *Let ω be a connection of order 2 on M and let $\tilde J$ be the tensor field of type $(1, 1)$ on F^2M given by (4.2). Then*

$$N_{\tilde J} = C\,\widetilde{d\theta_{-1}} + \lambda\,\widetilde{d\omega_0} + \lambda\,\widetilde{d\omega_1}\,, \tag{4.3}$$

where $\widetilde{d\theta_{-1}} \in \Lambda^2(F^2M, \boldsymbol{R}^n)$, $\widetilde{d\omega_0} \in \Lambda^2(F^2M, gl(n))$, $\widetilde{d\omega_1} \in \Lambda^2(F^2M, S^2(n))$ are defined with respect to $u \in F$, $u' \in F'$, $u'' \in F''$.

Proof. It is sufficient to compute $N_{\tilde J}$ in the following three cases:

(1) $X = C\xi_1$, $Y = C\xi_2$, $\xi_1, \xi_2 \in \boldsymbol{R}^n$; (2) $X = C\xi$, $Y = \lambda(A, \alpha)$, $\xi \in \boldsymbol{R}^n$, $A \in gl(n)$, $\alpha \in S^2(n)$; (3) $X = \lambda(A, \alpha)$, $Y = \lambda(B, \beta)$, $A, B \in gl(n)$, $\alpha, \beta \in S^2(n)$. The result follows by a straightforward computation using Proposition 2.4, Def. 4.3 and (4.2).

According to the structure equations of Theorem 2.3, (4.3) can be written equivalently as

$$N_{\tilde J} = C(\tilde\Theta_{-1} - \widetilde{\omega_0\wedge\theta_{-1}}) + \lambda(\tilde\Omega_0 - \widetilde{\omega_0\wedge\omega_0}) + \lambda(\tilde\Omega_1 - \widetilde{\omega_0\wedge\omega_1} - \widetilde{\omega_1\wedge\omega_0})\,. \tag{4.4}$$

Now, let $\bar\omega$ be the linear connection on M induced from ω. In [**1**], by a device similar to that used here, a $G_{u+u'}$-structure $\bar\omega$-associated on FM is constructed and the corresponding tensor field of type $(1, 1)$ on FM is given by $\bar J = Bu\bar\theta + \bar\lambda u'\bar\omega$. A simple calculation proves

Proposition 4.5. *The following diagram is commutative*

$$\begin{array}{ccc} T_p(F^2M) & \xrightarrow{\ \tilde J_p\ } & T_p(F^2M) \\ \downarrow & & \downarrow \\ T_{\bar p}(FM) & \xrightarrow{\ \bar J_{\bar p}\ } & T_{\bar p}(FM) \end{array}$$

where $\bar p = \pi_1^2(p)$.

Corollary 4.6. *Under the same hypothesis as above,* $N_{\tilde{J}} = 0$ *implies* $N_{\bar{J}} = 0$.

Proof. This follows taking into account (1.1), (2.1), (2.2), (4.3), Proposition 4.5 and the expression of the Nijenhuis torsion of $\bar{J}$, $N_{\bar{J}} = B\,\mathrm{d}\bar{\theta} + \bar{\lambda}\,\mathrm{d}\bar{\omega}$ given in [1].

The situation described above can be set within the following most general framework. Let us consider $H = (gl(n) \oplus S^2(n))^* \otimes (gl(n) \oplus S^2(n))$ and denote by $\sigma\colon Gl(N') \to Gl(H)$ the canonical representation, where we set $N' = n^2 + n^2(n+1)/2$, and by $j'\colon Gl(n) \times Gl(N') \to Gl(N)$ the canonical injection; we have $\tilde{\varrho} \circ j' = \varrho \oplus \sigma$. Consequently, if $u \in F$, $v \in H$ and if we take $\tilde{u} = u + v$, we have $F_u \oplus H_v \subset \tilde{F}_{\tilde{u}}$ and $j'(G_u \times G_v) \subset G_{\tilde{u}}$. Similarly to Theorem 4.1, we have

Theorem 4.7. *The tensor field of type* (1,1) *on* F^2M *given by* (3.1) *satisfies*

$$\tilde{J}_p(X) = \big(Cu((\theta_{-1})_p X)\big)_p + \big(\lambda v(\omega_p(X))\big)_p\,,$$

$X \in T_p(F^2M)$, $p \in F^2M$, *that is*

$$\tilde{J} = Cu\theta_{-1} + \lambda v\omega\,. \tag{4.5}$$

As above, the tensor field $\tilde{J}$ is 0-deformable and $\operatorname{rank}\tilde{J} = \operatorname{rank} u + \operatorname{rank} v$. Once again, we have

Theorem 4.8. *The Nijenhuis torsion* $N_{\tilde{J}}$ *of* $\tilde{J}$ *is given by*

$$N_{\tilde{J}} = C\,\widetilde{\mathrm{d}\theta}_{-1} + \lambda\,\widetilde{\mathrm{d}\omega}\,, \tag{4.6}$$

where $\widetilde{\mathrm{d}\theta}_{-1} \in \Lambda^2(F^2M, \boldsymbol{R}^n)$, $\widetilde{\mathrm{d}\omega} \in \Lambda^2(F^2M, gl(n) \oplus S^2(n))$ *are defined with respect to* $u \in F$, $v \in H$, *respectively.*

We remark that (4.6) can be written equivalently as

$$N_{\tilde{J}} = C(\tilde{\Theta}_{-1} - \widetilde{\omega_0 \wedge \theta_{-1}}) + \lambda(\tilde{\Omega} - \widetilde{\omega \wedge \omega})\,. \tag{4.7}$$

Now, if we take into account that $F' \oplus F''$ can be considered as a subspace of H, putting $v = u' + u''$, where $u' \in F'$, $u'' \in F''$, the previous construction is obtained again. Actually, Theorems 4.1 and 4.4 are a consequence of Theorems 4.6 and 4.7, respectively.

5 - Polynomial structures on F^2M

Of all the G-structures which can be defined on a manifold M by a tensor field φ of type (1, 1), those named polynomial structures have always deserved special attention in the literature. In this section, according to the general construction described in the preceding sections, we shall study the existence and some properties of those polynomial structures which can be defined on F^2M starting from a connection ω of order 2 on M and from algebraic models in $\mathbf{R}^n$, $gl(n)$ and $S^2(n)$.

Def. 5.1. A polynomial structure of degree k on a manifold M is a *tensor field* φ of type (1,1) and constant rank r which satisfyies the algebraic equation

$$(5.1)\qquad Q(\varphi) \equiv \varphi^k + a_k \varphi^{k-1} + \ldots + a_2 \varphi + a_1 I = 0 ,$$

where I is the identity tensor field and $\varphi^{k-1}(x), \ldots, \varphi(x)$ and I are linearly independent for any $x \in M$. Q is said the *structure polynomial.*

Now, let us consider $u \in F$, $u' \in F''$, $u'' \in F''$ of ranks r, r' and r'', respectively, satisfying

$$(5.2)\qquad Q(u) \equiv u^k + a_k u^{k-1} + \ldots + a_2 u + a_1 I = 0 ,$$

$$(5.3)\qquad Q'(u') \equiv (u')^{k'} + a'_{k'}(u')^{k'-1} + \ldots + a'_2 u' + a'_1 I = 0 ,$$

$$(5.4)\qquad Q''(u'') \equiv (u'')^{k''} + a''_{k''}(u'')^{k''-1} + \ldots + a''_2 u'' + a''_1 I = 0 ,$$

where I is the unit matrix and $u^{k-1}, \ldots, u, I$ (resp. $(u')^{k'-1}, \ldots, u', I$ and $(u'')^{k''-1}, \ldots, u''$ and I) are linearly independent. Then, $\tilde{u} = u + u' + u'' \in \tilde{F}$ has rank $r + r' + r''$ and satisfies

$$(5.5)\qquad \tilde{Q}(\tilde{u}) \equiv \tilde{u}^k + \tilde{a}_{\tilde{k}} \tilde{u}^{\tilde{k}-1} + \ldots + \tilde{a}_2 \tilde{u} + \tilde{a}_1 I = 0 ,$$

where $\tilde{Q}$ is the least common multiple of Q, Q' and Q'', and $\tilde{u}^{\tilde{k}-1}, \ldots, \tilde{u}, I$ are linearly independent.

Then, given a connection ω of order 2 on M, the ω-associated $G_{\tilde{u}}$-structure on F^2M is determined by a tensor field $\tilde{J}$ on F^2M of type (1, 1) which, indeed, defines on F^2M a polynomial structure of structural polynomial $\tilde{Q}$ and with constant rank $r + r' + r''$. Observe that all the results in section 4 can be applied to this polynomial structure, but they can be significantly strengthened if some additional assumptions on the nature of $u' \in F'$ and $u'' \in F''$ or on the polynomials Q, Q' and Q'' are made. For example, suppose that $u' \in F'$ and

$u'' \in F''$ are given by $u'(A) = NA$ and $u''(\alpha) = N \circ \alpha$, respectively, $N \in gl(n)$ being a fixed matrix; then the following lemma is immediate

Lemma 5.2. *If* $\operatorname{rank} N = r$, *then* $\operatorname{rank} u' = rn$ *and* $\operatorname{rank} u'' = r(n(n+1))/2$.

Theorem 5.3. *Let*: (1) $u \in F$ *with* $\operatorname{rank} u = s$ *and satisfying* (5.2); (2) $u' \in F'$ *and* $u'' \in F''$ *given by* $u'(A) = NA$ *and* $u''(\alpha) = N \circ \alpha$, *respectively, for a fixed matrix* $N \in gl(n)$ *with rank* $N = r$ *and satisfying* (5.3). *Then*

(i) $\tilde{u} = u' + u'' \in \tilde{F}$ *has rank* $s + rn + rn(n+1)/2$ *and satisfies* (5.5), $\tilde{Q}$ *being the least common multiple of* Q *and* Q'.

(ii) *If* ω *is a connection of order 2 on* M, *the tensor field* $\tilde{J}$ *of type* (1, 1) *given by* (4.2) *defines a polynomial structure on* F^2M *of constant rank* $s + rn + rn(n+1)/2$ *and structure polynomial* $\tilde{Q}$.

(iii) $N_{\tilde{J}} = C\widetilde{d\theta_{-1}} + \lambda\bar{\Omega}$.

Proof. (i) and (ii) are trivial. In order to prove (iii), it suffices to recall formulas (4.3) and (4.4) and check by direct computation the vanishing of $\widetilde{\omega_0 \wedge \omega_0}$, $\widetilde{\omega_0 \wedge \omega_1}$ and $\widetilde{\omega_1 \wedge \omega_0}$.

The remainder of this section is devoted to the description of some interesting examples.

Example 1. An $f(3,1)$-structure on $F^2M(\dim M = n = 2m)$.

If we take $u = 0$ and $N = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$, then $\operatorname{rank} u' = n^2$, $\operatorname{rank} u'' = n^2(n+1)/2$, and $Q(u) \equiv u = 0$, $Q'(u') \equiv (u')^2 + I = 0$, $Q''(u'') \equiv (u'')^2 + I = 0$. Therefore $\tilde{u} = 0 + u' + u''$ satisfies $\tilde{Q}(\tilde{u}) \equiv \tilde{u}^3 + \tilde{u} = 0$ and $\operatorname{rank} \tilde{u} = n^2 + n^2(n+1)/2$. Consequently, if ω is a connection of order 2 on M, the tensor field $\tilde{J}$ is given by $\tilde{J} = \lambda u' \omega_0 + \lambda u'' \omega_1$ and satisfies $\tilde{J}^3 + \tilde{J} = 0$ with $\operatorname{rank} \tilde{J} = n^2 + n^2(n+1)/2$. Hence, $\tilde{J}$ defines on F^2M an $f(3,1)$-structure of rank $n^2 + n^2(n+1)/2$. Actually, we have $N_{\tilde{J}} = \lambda\Omega$; then, from the results in [**5**], we have

Proposition 5.4. *The* $f(3,1)$-*structure on* F^2M *defined by* $\tilde{J}$ *as above is integrable if and only if* ω *is flat.*

If we consider the canonical projection operators associated to $\tilde{J}$, given by $l_1 = -\tilde{J}^2$, $l_2 = \tilde{J}^2 + I$, then $L_1 = \operatorname{Im} l_1$ is the vertical distribution on F^2M and $L_1 = \operatorname{Im} l_2$ is the horizontal distribution of ω. Obviously, L_1 is always integrable, its integrable manifolds being the fibres of F^2M; L_2 is integrable if and only if ω is flat and $\tilde{J}$ is always partially integrable since $\operatorname{Im} \tilde{J} = L_1$. Moreover, it is no hard to see that $\tilde{J}$ actually defines a framed $f(3,1)$-structure on F^2M. Indeed $CE_1, \ldots, CE_n$ span L_2 and $l_2 = \sum_i CE_i \otimes \theta^i$; being $\theta_{-1} = \theta^i E_i$. Nevertheless, the structure is not normal in general as can be easily proved. Now, we must also remark that the f (3, 1)-structure $\bar{J}$ on FM obtained by projecting $\tilde{J}$, is, in fact, the $f(3, 1)$-structure defined by Okubo in [**11**]$_1$.

Example 2. An $f(3, -1)$-structure on F^2M.

Take $u = 0$ and $N = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$, $p + q = n$. Then rank $u' = n^2$, rank $u'' = n^2 \cdot (n+1)/2$ and the structural polynomials are $Q(u) \equiv u = 0$, $Q'(u') \equiv (u')^2 - I = 0$ and $Q''(u'') \equiv (u'')^2 - I = 0$. Therefore, $\tilde{u} = 0 + u' + u''$ satisfies $\tilde{Q}(\tilde{u}) \equiv \tilde{u}^3 - \tilde{u} = 0$ and rank $\tilde{u} = n^2 + n^2(n+1)/2$. Now, if ω is a connection of order 2 on M, the tensor field $\tilde{J}$ is given by $\tilde{J} = \lambda u'\omega_0 + \lambda u''\omega_1$ and satisfies $\tilde{J}^3 - \tilde{J} = 0$, with rank $n^2 + n^2(n+1)/2$. Theorem 5.4 is still valid as well as the other results in Example 1, only making the appropriate changes. As above, $\bar{J}$ defines the $f(3, -1)$-structure on FM considered by Okubo in [**11**]$_2$.

Example 3. An $f(4, 2)$-structure on $F^2M(\dim M = n = 2m)$. Take $u = \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$. Then rank $u = m$, rank $u' = n^2$, rank $u'' = n^2 \cdot (n+1)/2$ and $Q(u) \equiv u^2 = 0$, $Q'(u') \equiv (u')^2 + I = 0$, $Q''(u'') \equiv (u'')^2 + I = 0$. Therefore, $\tilde{u} = u + u' + u''$ satisfies $\tilde{Q}(\tilde{u}) \equiv \tilde{u}^4 + \tilde{u}^2 = 0$ and rank $\tilde{u} = m + n^2 + (n^2(n+1)/2)$. So, if ω is a connection of order 2 on M, $\tilde{J}$ given by (4.2) defines on F^2M an $f(4, 2)$-structure of rank $m + n^2 + (n^2(n+1)/2)$. Here, $N_{\tilde{J}}$ is given by Theorem 5.3. Let $l_1 = -\tilde{J}^2$, $l_2 = \tilde{J}^2 + I$ the canonical projection operators; then $L_1 = \operatorname{Im} l_1$ is the vertical distribution and $L_2 = \operatorname{Im} l_2$ is the horizontal distribution. With the terminology of [**3**], this structure is easily shown to be always c-partially integrable and t-partial and partial integrabilities are found to be equivalent and they are verified if and only if $\tilde{\Theta}_{-1}(l_2X, l_2Y) = 0$, $\tilde{\Omega}(l_2X, l_2Y) = 0$.

Example 4. An $f(4, -2)$-structure on $F^2M(\dim M = n = 2m)$. Take u as in Example 3 and N as in Example 2. Then, for $\tilde{u} = u + u' + u''$, we have $\tilde{Q}(\tilde{u}) \equiv \tilde{u}^4 - \tilde{u}^2 = 0$. If ω is a connection of order 2 on M, $\tilde{J}$ defines on F^2M an $f(4, -2)$-structure of rank $m + n^2 + n^2(n+1)/2$. Similar results to those in Example 3 are valid.

Example 5. A family of examples.

Let $u \in F$ with rank $u = r$ and satisfying (5.2). If we take $N = u$, then u' and u'' also satisfy (5.2) and rank $u' = rn$, rank $u'' = rn(n+1)/2$. Obviously, $Q = Q' = Q'' = \tilde{Q}$ and, for a given connection ω of order 2 on M, $\tilde{J}$ defines on F^2M a polynomial structure of rank $r(1 + n + n(n + 1)/2)$ and structural polynomial Q. A direct computation shows that $N_{\tilde{J}}$ becomes simpler $N_{\tilde{J}} = C\tilde{\Theta}_{-1} + \lambda\tilde{\Omega}$.

Actually, with the appropriate choice of u, we obtain almost tangent, almost product or almost complex structures on F^2M.

6 - G-structures on F^2M defined by tensor fields of type (0, 2)

Let $F = V^* \otimes V^*$, $V = \mathbf{R}^N \simeq \mathbf{R}^n \times gl(n) \times S^2(n)$, and $\tilde{\varrho}: Gl(N) \to Gl(\tilde{F})$ the canonical linear representation given by $(\tilde{\varrho}(A)\tilde{u})(\xi, \xi') = \tilde{u}(A^{-1}\xi, A^{-1}\xi')$, $A \in Gl(N)$, $\tilde{u} \in \tilde{F}$ and $\xi, \xi' \in V$. It is well known that there exists a one-to-one correspondence between the $G_{\tilde{u}}$-structures on F^2M defined by $\tilde{t}^{-1}(\tilde{u})$, $\tilde{t}: FF^2M \to \tilde{F}$ being an $\tilde{F}_{\tilde{u}}$-valued differentiable tensor of type $(\tilde{\varrho}, \tilde{F})$, $G_{\tilde{u}}$ the isotropy group of $\tilde{u} \in \tilde{F}$, and the tensor fields $\tilde{\varphi}$ on F^2M of type (0, 2) given by

$$\tilde{\varphi}_p(X, Y) = \tilde{t}(\tilde{p})(\tilde{p}^{-1}X, \tilde{p}^{-1}Y) \qquad X, Y \in T_pF^2M,\ \tilde{p} \in \pi^{-1}(p),\ p \in F^2M . \tag{6.1}$$

Now, let us consider $F = (\mathbf{R}^n)^* \otimes (\mathbf{R}^n)^*$, $F' = (gl(n))^* \otimes (gl(n))^*$, $F'' = (S^2(n)_2)^* \otimes (S^2(n))^*$ and $\varrho: Gl(n) \to Gl(F)$, $\tilde{\varrho}: Gl(n^2) \to Gl(F')$, $\varrho'': Gl(n^2(n + 1)/2) \to Gl(F'')$ the canonical representations. If for any $u \in F$, $u' \in F'$, $u'' \in F''$ we put $\tilde{u} = u + u' + u'' \in \tilde{F}$, we have $F_u \otimes F'_{u'} \otimes F''_{u''} \subset \tilde{F}_{\tilde{u}}$ and $j(G_u \times G_{u'} \times G_{u''}) \subset G_{\tilde{u}}$, j being the canonical injection.

For a given connection ω of order 2 on M, we consider the ω-associated $G_{\tilde{u}}$-structure, which, according to Theorem 3.2, is defined by an $\tilde{F}_{\tilde{u}}$-valued differentiable tensor on FF^2M of type $(\tilde{\varrho}, \tilde{F})$; the corresponding tensor field $\tilde{\varphi}$ on F^2M is described as follows

Theorem 6.1. *We have*

$$\tilde{\varphi}_p(X, Y) = u((\theta_{-1})_pX, (\theta_{-1})_pY) + u'((\omega_0)_pX, (\omega_0)_pY) + u''((\omega_1)_pX, (\omega_1)_pY)$$

$X, Y \in T_pF^2M$, $p \in F^2M$, *that is*

$$\tilde{\varphi} = u(\theta_{-1}, \theta_{-1}) + u'(\omega_0, \omega_0) + u''(\omega_1, \omega_1) . \tag{6.2}$$

Proof. The proof is similar to that of Theorem 4.1.

Next, we shall apply this construction to two particular cases.

I - *Almost symplectic structures on* F^2M(dim $M = n = 2m$).

Let $u \in F$, $u' \in F'$ and $u'' \in F''$ be antisymmetric and of maximal rank. Hence, the corresponding tensor field $\tilde{\varphi}$ given by (6.2) is actually a differentiable 2-form of maximal rank. Therefore, we have

Theorem 6.2. *$\tilde{\varphi}$ defines on F^2M an almost symplectic structure.*

We remark that, in general, if rank $u = r$, rank $u' = r'$ and rank $u'' = r''$, then rank $\tilde{\varphi} = r + r' + r''$, that is, $\tilde{\varphi}$ defines on F^2M an almost pre-symplectic structure of rank $r + r' + r''$.

In order to characterize the integrability of these structures, we introduce the following definition.

Def. 6.3. Let $u \in F$, $u' \in F'$, $u'' \in F''$ and ω a connection of order 2 on M. Then:

(1) if $\eta \in \Lambda^2(F^2M, \boldsymbol{R}^n)$, $\tilde{\eta} \in \Lambda^3(F^2M, \boldsymbol{R})$ is the 3-form on F^2M given by $3\tilde{\eta}(X, Y, Z) = u(\eta(X, Y), \theta_{-1}(Z)) + u(\eta(Y, Z), \theta_{-1}(X)) + u(\eta(Z, X), \theta_{-1}(Y))$;

(2) if $\eta \in \Lambda^2(F^2M, gl(n))$, $\tilde{\eta} \in \Lambda^3(F^2M, \boldsymbol{R})$ is the 3-form on F^2M given by $3\tilde{\eta}(X, Y, Z) = u'(\eta(X, Y), \omega_0(Z)) + u'(\eta(Y, Z), \omega_0(X)) + u'(\eta(Z, X), \omega_0(Y))$;

(3) if $\eta \in \Lambda^2(F^2M, S^2(n))$, $\tilde{\eta} \in \Lambda^3(F^2M, \boldsymbol{R})$ is the 3-form on F^2M given by $3\tilde{\eta}(X, Y, Z) = u''(\eta(X, Y), \omega_1(Z)) + u''(\eta(Y, Z), \omega_1(X)) + u''(\eta(Z, X), \omega_1(Y))$, where X, Y, Z are arbitrary vector fields on F^2M.

Theorem 6.4. $d\tilde{\varphi} = 2\,\widetilde{d\theta_{-1}} + 2\,\widetilde{d\omega_0} + 2\,\widetilde{d\omega_1}$.

Proof. It suffices to check the identity in the basis cases, as in the proof of Theorem 4.4.

Now, keeping in mind the usual terminology of symplectic manifolds theory, we have

Proposition 6.5. *Let H_p be the horizontal subspace of ω at $p \in F^2M$ and denote by V_p the vertical subspace at the same point. Then, with respect to the almost symplectic structure defined by $\tilde{\varphi}$ on F^2M, we have:*

(1) H_p and V_p are symplectic subspaces of T_pF^2M and φ-orthogonal. Moreover, if we take into account that V_p can be decomposed as $V_p = \lambda(gl(n))_p \oplus \lambda(S^2(n))_p$, then both subspaces are also symplectic and $\tilde{\varphi}$-orthogonal;

(2) the fibres of $F^2M \to M$ and $F^2M \to FM$ are almost symplectic manifolds;

(3) if ω is flat, the integral manifolds of the horizontal distribution are almost symplectic manifolds.

Proof. (1) is a direct consequence of (6.2); (2) and (3) follow from (1).

Now, let $\bar{\omega}$ be the linear connection on M induced by a connection ω of order 2 on M. Then, the almost symplectic structure $\bar{\omega}$-associated on FM in the sense of [**1**] is defined by the differentiable 2-form $\bar{\varphi}=u(\bar{\theta},\bar{\theta})+u'(\bar{\omega},\bar{\omega})$. We have

Proposition 6.6. *If $\tilde{\varphi}$ is symplectic, so is $\bar{\varphi}$.*

Proof. This follows taking into account (1.1), (2.1), (2.2), Theorem 6.4 and the expression $d\bar{\varphi} = 2\,\widetilde{d\bar{\theta}} + 2\,\widetilde{d\bar{\omega}}$ obtained in [**1**].

Proposition 6.6 follows directly from the Remark in **3**.

II - *Riemannian structures on F^2M.*

Let $u \in F$, $u' \in F'$ and $u'' \in F''$ be symmetric and positive definite; then, given a connection ω of order 2 on M, the associated tensor field $\tilde{\varphi}$ on F^2M of type $(0, 2)$ is also symmetric and positive definite; thus, we have

Theorem 6.7. *$\tilde{\varphi}$ defines a Riemannian structure on F^2M.*

We remark that the horizontal and vertical distributions are mutually orthogonal.

Now, we shall consider the canonical flat connection $\overline{\nabla}$ on F^2M induced by the absolute parallelism defined by ω. Then

$$\overline{\nabla}_X Y = \sum_i (Xa^i)(CE_i) + \sum_{i,j} (Xa^i_j)(\lambda E^j_i) + \sum_{i,j,k} (Xa^i_{jk})(\lambda E^{jk}_i)$$

$$Y = \sum_i a^i(CE_i) + \sum_{i,j} a^i_j(\lambda E^j_i) + \sum_{i,j,k} a^i_{jk}(\lambda E^{jk}_i)\,, \qquad a^i_{jk} = a^i_{kj} \qquad \text{on } F^2M\,.$$

By a simple calculation, we get

Proposition 6.8. *$\overline{\nabla}$ is a metric connection, that is, $\overline{\nabla}\tilde{\varphi} = 0$.*

Consequently, if $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{\varphi}$, then $\tilde{\nabla}$ and $\overline{\nabla}$ are equal if and only if the torsion tensor $\overline{T}$ of $\overline{\nabla}$ vanishes. But one easily obtains $\overline{T} = 2C\,d\theta_{-1} + 2\lambda\,d\omega$; then, $\overline{T}$ is never identically zero; thus it is always $\overline{\nabla} \neq \tilde{\nabla}$.

To end this section, let us go back to Examples in **5**.

Examples 1 and 2 (continuation). Consider the $f(3, 1)$-structure $\tilde{J}$ on F^2M given in Example 1. A Riemannian metric $\tilde{g}$ on F^2M is said compatible or adapted to $\tilde{J}$ if $\tilde{g}(X, \tilde{J}Y) + \tilde{g}(\tilde{J}X, Y) = 0$.

Let u' (resp., u'') be a hermitian inner product on $gl(n)$ (resp. $S^2(n)$) with respect to the almost complex structure on $gl(n)$ (resp., $S^2(n)$) which we have considered in Example 1. Then, the Riemannian metric $\tilde{\varphi}$ given by (6.2) is adapted to $\tilde{J}$.

A similar result can be obtained for the $f(3, -1)$-structure given in Example 2, only changing the hermitian condition with respect to the almost complex structures on $gl(n)$ and $S^2(n)$ by the condition of being compatible with the almost producr structures there.

Examples 3 and 4 (continuation). To obtain a Riemannian metric on F^2M adapted to the $f(4, -2)$-structure J on F^2M given in Example 3, it suffices to consider the inner products on $\boldsymbol{R}^n$, $gl(n)$ and $S^2(n)$ adapted to the corresponding almost tangent and almost complex structures on $\boldsymbol{R}^n$, $gl(n)$ and $S^2(n)$, respectively. Similar considerations can be made for Example 4, only making the appropriate changes.

Example 5 (continuation). As in the Examples above, choosing the appropriate inner products on $\boldsymbol{R}^n$, $gl(n)$ and $S^2(n)$, we shall obtain Riemannian metrics on F^2M adapted to the different polynomial structures considered.

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Résumé

Dans cette article, on décrit un procédé qui permet de construir une grande variété de G-structures sur F^2M, espace total du fibré principal des repères d'ordre deux d'une variété différentiable M de dimension n, quand il est muni d'une connexion du deuxième ordre. L'idée essentielle consiste à transporter dans F^2M les structures modèles sur R^n, $gl(n)$ et $S^2(n)$ respectivement, au moyen de la forme de la connexion.

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