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Non linear vibrations of strings: the F.P.U. problem (**)

1 - Introduction

One of the first examples of a euristic study on a computer of a difficult problem in analisys is the examination, conducted by Fermi, Pasta and Ulam in the 50's, of the problem of propagation given below [4], whose solutions seem appropriate for the description of certain types of non-linear vibrations of strings and crystal lattices

$$(1.1)_1 \qquad u_{tt} - (1 + \varepsilon u_x) u_{xx} = 0 \qquad \forall (x, t) \in \mathcal{D} = [0, 1] \times [0, T] ,$$

$$(1.1)_2 \qquad u(x, 0) = \sin(\pi x) ,$$

$$(1.1)_3 \qquad u_t(x, 0) = 0 ,$$

$$(1.1)_4 \qquad u(0, t) = u(1, t) = 0 .$$

Multiplying equation $(1.1)_1$ by u_t and integrating the result by parts under the assumption of smoothness of the derivatives involved, the following equation is obtained

$$(1.2) \qquad \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{2} \int_0^1 u_t^2 \, \mathrm{d}x + \frac{1}{2} \int_0^1 \left(u_x^2 + \frac{\varepsilon}{3} u_x^3 \right) \mathrm{d}x \right] = 0 ,$$

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which assures that during motion, the total energy is conserved. If the conditions of smoothness are not satisfied (1.2) does not hold. The correct relations in the case when the first derivatives have a jump will be derived in **4**.

When $\varepsilon = 0$ (1.1) goes over into the classical linear problem of vibrating strings. The solution is

$$u(x, t) = \sin(\pi x) \cdot \cos(\pi t)\,, \tag{1.3}$$

and describes the first mode of vibration. Other modes of vibration are not excited in view of the initial conditions even though they would be allowed by the boundary conditions. During motion, all energy is maintained in the mode in which it was initially.

For $\varepsilon > 0$, it can immediately be seen that the solution cannot have the same form (1.3). One can try to express the solution as a series

$$\begin{aligned} &u(x, t) = \sum_{n}^{\infty}{}_1 a_n(t) \sin(n\pi x)\,, \\ &a_1(0) = 1\,, \quad a_n(0) = 0 \qquad \text{for } n = 2, 3, \ldots; \end{aligned} \tag{1.4}$$

then $|a_n(t)|$ could be considered as a measure of the relevance of the n-th mode in that instant. Thus energy appears to flow in time into higher modes.

At the beginning of their research, Fermi, Pasta and Ulam presumed that the process of transfer of energy to higher modes would continue perhaps until the energy became uniformly distributed throughout all the modes (only a finite number was allowed by the discretisation adopted for the numerical method chosen). Instead the approximate numerical examination lead to a completely different result: the energy, after flowing in some of the higher modes, seemed to concentrate again periodically in the first mode.

Deeper studies showed that this result had nothing to do with the properties of the solution of (1.1), but was only due to the numerical approximation adopted. In fact, Zabusky and Kruskal and later, in a concise way, Lax [**5**]$_1$ proved that the problem (1.1) cannot have smooth solutions beyond a critical instant t_c. For $t > t_c$, the discretisation adopted by Fermi, Pasta and Ulam (who tacitly assume the regularity of the solution) is not a suitable technique for the approximate solution of (1.1).

In order to study the solution for $t > t_c$, a weak formulation of the problem is necessary, within which it is meaningful to discuss solutions with discontinuous first derivatives; in fact *S.* Antman [**1**] has proposed recently a weak form of the problem of non-linear vibrations of strings which is also suitable for the description of the F.P.U. problem.

In this paper we present a numerical method appropriate for the approximate solution of the problem in weak form. The method uses finite space-time elements and has been applied already with some success on the numerical solution of hyperbolic linear problems [2], [3]; when applied to the F.P.U. problem the method leads to very convincing results; it has allowed us also to explore the response of strings with different tension laws.

2 - Weak formulation of the problem

Let us take a quasi-linear class of propagation problems to which the problem of F.P.U. belongs

$$(2.1)_1 \qquad u_{tt} - k^2(u_x)u_{xx} = 0 \qquad \forall (x, t) \in \mathscr{D} = [0, 1] \times [0, T] ,$$

$$(2.1)_2 \qquad u(x, 0) = u_0(x) ,$$

$$(2.1)_3 \qquad u_t(x, 0) = \dot{u}_0(x) ,$$

$$(2.1)_4 \qquad u(0, t) = u(1, t) = 0 ,.$$

In (2.1) $k^2(y)$ is a function of class C^2 with positive values for all y.
The problem (2.1) is called *genuinely non-linear* if

$$(2.2) \qquad \left| \frac{\mathrm{d}k}{\mathrm{d}y} \right| \geqslant m > 0 .$$

We notice that in the F.P.U. problem when, as we shall assume, $\varepsilon > 0$, to insure the correct sign of the coefficient of u_{xx}, u_x must be bounded from below

$$(2.3) \qquad u_x > -1/\varepsilon .$$

If the problem is genuinely non-linear, no regular solution exists for t greater than a critical time t_c, which depends on k and on the choice of the initial conditions.

Rather it is convenient to put the problem in a weak form which is formally obtained by multiplying the left hand side of $(2.1)_1$ by a sufficient smooth function v vanishing for $t = T$ and $x = 0, 1$ and integrating by parts over $(0, 1) \times (0, T)$ (see Antmann [1]). Then one is required to find $u(x, t)$ such that:

(i) the equality

$$(2.4)_1 \quad -\int_0^1 \dot{u}_0(x)\cdot v(x,0)\,dx - \int_0^1\int_0^T u_t\cdot v_t\,dx\,dt + \int_0^1\int_0^T F(u_x)\cdot v_x\,dx\,dt = 0$$

is satisfied for every function $v(x,t) = \varphi(x)\cdot\psi(t)$, where $\varphi(x)$ and $\psi(t)$ are continuous piecewise linear functions with compact support on $(0,1)$ and $[0,T]$ respectively;

(ii) conditions $(2.1)_2$-$(2.1)_4$ are satisfied in the sense of the trace $[\mathbf{5}]_2$, i.e.,

$$(2.4)_2 \quad \lim_{x\to 0}\int_{t_1}^{t_2} u(x,t)\,dt = \lim_{x\to 1}\int_{t_1}^{t_2} u(x,t)\,dt = 0 \qquad \text{for every } (t_1,t_2)\in[0,T];$$

$$(2.4)_3 \quad \lim_{t\to 0}\int_a^b \big(u(x,t)-u_0(x)\big)\,dx = \lim_{t\to 0}\int_a^b \big(\dot{\boldsymbol{u}}(x,t)-\dot{\boldsymbol{u}}_0(x)\big)\,dx = 0$$

for every $(a,b)\in(0,1)$.

In this new formulation we can relax considerably the smoothness conditions on u and although there is no strict equivalence between the weak and strong formulations, where there are discontinuities in the first derivatives, the former has the predominance from a physical point of view.

3 - Numerical solution of the weak problem

We can proceed now to a discretisation of problem (2.4) by « space-time elements » [2] leading to an efficient numerical process of solution.

Starting from the assumption that the solution $u(x,t)$ fullfills the « semigroup property », the process is based on the division of the domain $\mathscr{D}$ into N rectangles $\mathscr{D}_n = (0,1)\times(\tau_{n-1},\tau_n)$, $n = 1,\ldots,N$; $\tau_0 = 0$, $\tau_N = T$. Knowing the values of u and u_t at instant $t = 0$, we can find u and u_t at instant $t = \tau_1$. Knowing u and u_t for $t = \tau_1$ the values for $t = \tau_2$ are determined and so on until the pre-established time T is reached.

Every rectangle $(0,1)\times(\tau_{n-1},\tau_n)$ is divided into « rectangular elements » R as shown in Fig. 1.

The approximation of the function $u(x,t)$ within the element is obtained in this way

$$(3.1) \qquad u(x,t) = \sum_1^4 {}_i\, N_i(x,t)\,u(P_i) = \boldsymbol{N}^T\boldsymbol{u}\,,$$

where the shape functions $N_i(x, t)$ have the following properties

$$N_i(x, t) = \varphi_i(x) \cdot \psi_i(t) , \tag{3.2}$$

with φ_i and ψ_i linear functions,

$$N_i(P_j) = \delta_{ij} \quad (i, j = 1, 2, 3, 4), \tag{3.3}$$

$$\sum_1^4{}_i N_i(x, t) = 1 \qquad \forall (x, t) \in R_k . \tag{3.4}$$

For the derivatives, we obtain from (3.1)

$$u_x(x, t) = \sum_1^4{}_i \frac{\partial N_i}{\partial x}(x, t) u(P_i) = \boldsymbol{N}_x^T \boldsymbol{u} , \tag{3.5}_1$$

$$u_t(x, t) = \sum_1^4{}_i \frac{\partial N_i}{\partial t}(x, t) u(P_i) = \boldsymbol{N}_t^T \boldsymbol{u} , \tag{3.5}_2$$

$$\dot{u}_0(x) = \sum_1^2{}_i N_i(x, 0) \dot{u}_0(P_i) = \boldsymbol{N}_0^T \dot{\boldsymbol{u}}_0 . \tag{3.5}_3$$

Similar expressions are used for the approximation of function v and its derivatives.

Accepting the approximation (3.1) and (3.5), the problem (2.4) can be formulated for the single element as follows

$$-\boldsymbol{v}_0^T \Big(\int_{a_k}^{a_{k+1}} \boldsymbol{N}_0 \boldsymbol{N}_0^T \, \mathrm{d}x \Big) \dot{\boldsymbol{u}}_0 - \boldsymbol{v}^T \Big(\int_{a_k}^{a_{k+1}} \int_0^{\tau_1} (\boldsymbol{N}_t \boldsymbol{N}_t^T) \, \mathrm{d}x \, \mathrm{d}t \Big) \boldsymbol{u} + \boldsymbol{v}^T \Big(\int_{a_k}^{a_{k+1}} \int_0^{\tau_1} (\boldsymbol{N}_x F(\boldsymbol{N}_x^T \boldsymbol{u})) \, \mathrm{d}x \, \mathrm{d}t \Big) = 0 , \tag{3.6}$$

or

$$\boldsymbol{v}_0^T Q \boldsymbol{u}_0 + \boldsymbol{v}^T M \boldsymbol{u} + \boldsymbol{v}^T W = 0, \tag{3.7}$$

$$Q = -\int_{a_k}^{a_{k+1}} (\boldsymbol{N}_0 \boldsymbol{N}_0^T) \, \mathrm{d}x , \quad M = -\int_{a_k}^{a_{k+1}} \int_0^{\tau_1} (\boldsymbol{N}_t \boldsymbol{N}_t^T) \, \mathrm{d}x \, \mathrm{d}t ,$$

$$W = \int_{a_k}^{a_{k+1}} \int_0^{\tau_1} (\boldsymbol{N}_x F(\boldsymbol{N}_x^T \boldsymbol{u})) \, \mathrm{d}x \, \mathrm{d}t . \tag{3.8}$$

Indicating with $\boldsymbol{u}_0$ the known vector of the nodal values of u for $t = 0$ and with $\boldsymbol{u}_1$ the unknown vector of the nodal values of u for $t = \tau_1$, (3.7) may be written in the following way

$$\boldsymbol{v}_0^T Q \dot{\boldsymbol{u}}_0 + (\boldsymbol{v}_0^T; \boldsymbol{v}_1^T) \begin{bmatrix} M_{11}; & M_{12} \\ M_{21}; & M_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \end{bmatrix} + (\boldsymbol{v}_0^T; \boldsymbol{v}_1^T) \begin{bmatrix} \boldsymbol{W}_0 \\ \boldsymbol{W}_1 \end{bmatrix} = 0 \,, \tag{3.9}$$

where, keeping in mind the hypothesis $\boldsymbol{v}_1 = 0$, we obtain

$$\boldsymbol{v}_0^T Q \dot{\boldsymbol{u}}_0 + \boldsymbol{v}_0^T M_{11} \boldsymbol{u}_0 + \boldsymbol{v}_0^T M_{12} \boldsymbol{u}_1 + \boldsymbol{v}_0^T \boldsymbol{W}_0 = 0 \,. \tag{3.10}$$

As a result of the arbitrary choice of v, the following condition is obtained for the single element

$$Q \dot{\boldsymbol{u}}_0 + M_{11} \boldsymbol{u}_0 + M_{12} \boldsymbol{u}_1 + \boldsymbol{W}_0 = 0 \,. \tag{3.11}$$

(3.11) has to be written for all the n elements in which the strip $\mathscr{D}_1$ has been divided. The K systems of the two equations so obtained shall be combined one with the other, adding up the equations that refer to the same unknown value and taking into consideration the boundary conditions. This process, known as « assembly », leads us to a system with $K - 1$ equations for the same number of unknown values.

We may therefore write, for the whole domain $\mathscr{D}_1$, a system of equations formally analogous to (3.11)

$$\boldsymbol{r}(\boldsymbol{u}_1) = Q \dot{\boldsymbol{u}}_0 + M_{11} \boldsymbol{u}_0 + M_{12} \boldsymbol{u}_1 + \boldsymbol{W}_0 = 0 \,. \tag{3.12}$$

This system (3.12) is non linear because of the presence of the term $\boldsymbol{W}_0$. For its solution, Newton-Raphson's method is adopted.

Once the vector $\boldsymbol{u}_1$ is determined, even the initial conditions $\boldsymbol{u}_1$ for the subsequent step in time will have to be calculated. The following method is considered the best for the calculation of $\boldsymbol{u}_1$, even though it is expensive because of the amount of computation involved.

Let us determine the value of function u for $t = \tau_1$, $t_1 = (1 + \beta)\tau_1$ and $t_2 = (1 - \beta)\tau_1$ for an appropriate β

$$\dot{\boldsymbol{u}}_1 = \frac{\boldsymbol{u}(t_1) - \boldsymbol{u}(t_2)}{2\beta\tau} \,, \tag{3.13}$$

assuming in the calculation that $\beta = 0.1$.

Finally a recurrent process is obtained which brings us to a unique value of u for any value of t even if t is bigger than t_c. We will go back to this point in the next paragraph.

4 - Numerical results of the problem of F.P.U.

In the numerical example quoted below we have taken $\varepsilon = 0.2$. On the basis of the results obtained by Lax [5]$_1$ we find the critical instant is $t_c = 4/\pi^2\varepsilon \simeq 2.026$.

The interval $[0, 1]$ in x is divided into 100 equal parts and the width of every step in time is taken as 0.01. The solution $u(x, t)$ is calculated in the domain

$$\mathscr{D} = (0, 1) \times (0, 5) . \tag{4.1}$$

The reduced Gaussian integration method is adopted: one single point for every element. In figures 2, 3 the graph of the solution $u(x, t)$ is shown as a function of x at various instants, with a step in time equal to 0.05 up to $T = 5$.

For $t > t_c \simeq 2$ (Fig. 3), the graph of the function u shows a sharp change of slope. See also Fig. 4, where u_t is plotted as function of x for $3.5 < t < 4$.

As it has already been remarked, more than one solution to the problem (2.4) is possible when shock waves occur and uniqueness can be recovered imposing, for example, Lax's condition on the characteristic paths [5]$_2$, [7].

Lax's condition for a system of the 2nd order demands that if λ_1 and λ_2 are two eigenvalues in ascending order (in our case $-k$ and k) the following results obtain

$$\begin{aligned} &\lambda_1(w_s) < s\,, \quad \lambda_2(w_s) > s > \lambda_2(w_r) \qquad \text{for } s > 0\,, \\ &\lambda_1(w_s) > s > \lambda_1(w_r)\,, \quad s < \lambda_2(w_r) \qquad \text{for } s < 0\,, \end{aligned} \tag{4.2}$$

where w_s, w_r indicate respectively the values of $w = u_x$ to the left and to the right of the discontinuity and $s = \mathrm{d}x/\mathrm{d}t$ is the slope of the path of discontinuity.

Fig. 5 shows the behaviour of the characteristic path in the domain $\mathscr{D} = (0, 1) \times (2, 3)$ of the (x, t) plane and the discontinuity path (which is recognized in the figure by its irregular behaviour).

Observing Fig. 5, and keeping in mind that λ_1 and λ_2 are respectively the slope $(\mathrm{d}x/\mathrm{d}t)$ of the two families of characteristic paths, it can be checked that the condition is satisfied. In fact, for $t < 2.6$.

(a) the characteristics of the family corresponding to the eigenvalue $+k$ converge, for an increasing value of t, from the left and from the right of the discontinuity path;

(b) the characteristics of the family corresponding to the eigenvalue $-k$ intersects the discontinuity path and gives $\lambda_1(w_s) = k(w_r) < s$.

At the instant when the discontinuity path meets the edge of $\mathcal{D}(x = 1)$ and is reflected, the two families of characteristics change their roles. This verification may be done for the whole period in which the numeric solution is calculated: hence the validity of Lax's condition is verified.

For $t > t_c$ (1.2) does not hold. Let $F(y)$ be a primitive of $k^2(y)$. We have from (2.1)

$$\int_0^1 (u_{tt}u_t)\,dx - \int_0^1 u_t\Big(\frac{d}{dx}F(u_x)\Big)\,dx = 0\,, \qquad \text{whence}$$

$$\int_0^1 \frac{d}{dt}\Big(\frac{1}{2}u_t^2\Big)\,dx - \int_0^1 \Big(\frac{d}{dx}\big(F(u_x)u_t\big) - F(u_x)u_{tx}\Big)\,dx = 0.$$

It follows that

$$\frac{d}{dt}\Big(\int_0^{s(t)} \frac{1}{2}u_t^2\,dx + \int_{s(t)}^1 \frac{1}{2}u_t^2\,dx\Big) = \int_0^1 \frac{d}{dt}\Big(\frac{1}{2}u_t^2\Big)\,dx - s'(t)\Big[\frac{1}{2}u_t^2\Big],$$

$$\frac{d}{dt}\Big(\int_0^{s(t)} G(u_x)\,dx + \int_{s(t)}^1 G(u_x)\,dx\Big) = -s'(t)[G(u_x)] + \int_0^1 F(u_x)u_{xt}\,dx\,,$$

$$\int_0^1 \frac{d}{dx}\big(F(u_x)u_t\big)\,dx = F(u_x)u\Big|_0^1 - [F(u_x)\,u_t]\,,$$

where $s = dx/dt$ is the slope of the discontinuity path, $G(y)$ is a primitive of $F(y)$ and $[\cdot]$ represents the value of the jump along the discontinuity path.

Because $u_t(0) = u_t(1) = 0$, we have

$$\frac{d}{dt}\Big(\int_0^1 \Big(\frac{1}{2}u_t^2 + G(u_x)\Big)\,dx = -s'(t)\Big[\frac{1}{2}u_t^2 + G(u_x)\Big] - [F(u_x)u_t]\,. \tag{4.3}$$

The left hand side of (4.3) is the time derivative of the mechanical energy. It can be proved, [5]$_2$, that if a solution $u(x, t)$ of (2.4) satisfies the condition (4.2), the right hand side of (4.3) is negative. Thus the equation (4.3) shows that the mechanical energy decreases when shocks occur.

The expression $\int_0^1 \big(\frac{1}{2}u_t^2 + G(u_x)\big)\,dx$ has been numerically evaluated at every time step whose values are plotted versus time as shown in Fig. 6.

5 - Other numerical results obtained

The numerical method was used for the solution of the problem (2.1) also for other expressions of $k(u_x)$.

Let us take for example the problem (2.1) for $k(y) = 1/(1 + y^2)$. Because $(dk/dy)(0) = 0$; the problem is not genuinely non-linear and the hypotheses of Lax are no longer valid. Moreover because $k(y)$ satisfies the following conditions

$$k(y) > 0 \qquad \text{and} \quad k(0) = 1\,,$$

$$k'(y) \begin{cases} < 0 & \text{for } y > 0\,, \\ > 0 & \text{for } y < 0\,, \end{cases}$$

$$\int_{-\infty}^{+\infty} k(y)\,dy < +\infty\,,$$

it can be proved [6] that the problem (2.1) admits a unique smooth solution for every $t > 0$.

The numerical solution is determined for $0 < t < 3$. Figure 7 shows the behaviour of the solution $u(x, t)$ for $2 < t < 3$.

From an analysis of the graphs the solution appears to be always regular and symmetric with respect to $x = 0.5$.

The behaviour of the characteristic lines (Fig. 8) confirms the absence of shock fronts.

We wish to thank G. Capriz for many critical remarks.

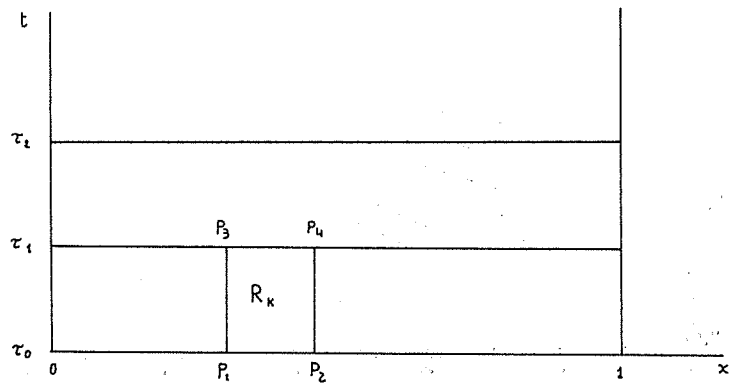

Fig. 1

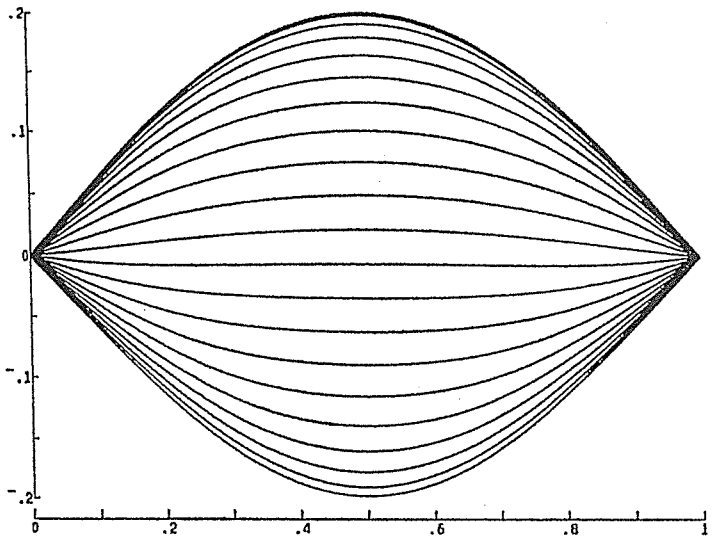

Fig. 2

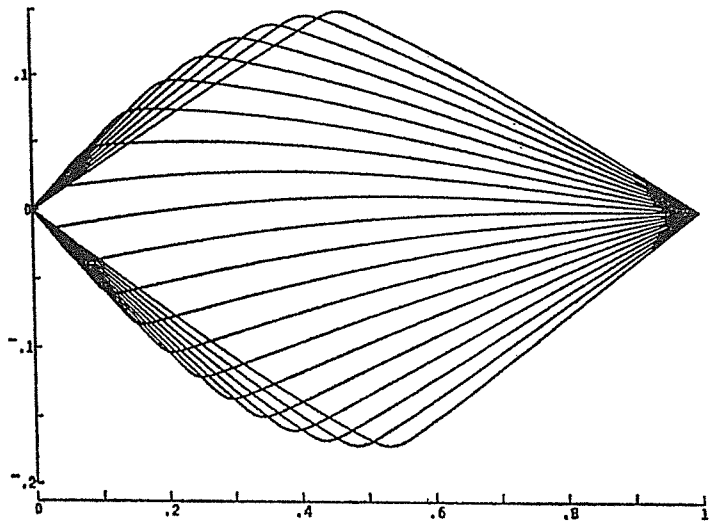

Fig. 3

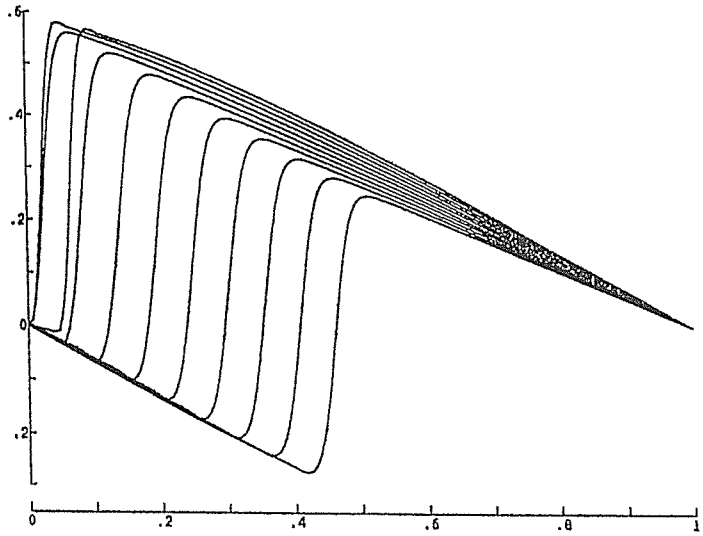

Fig. 4

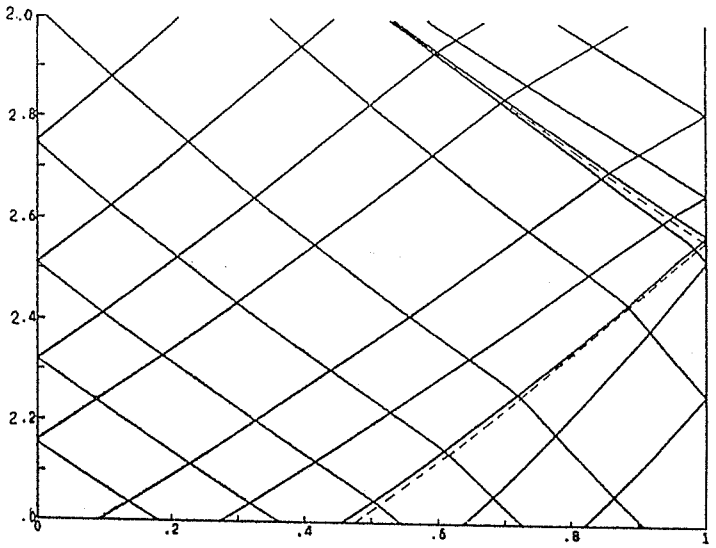

Fig. 5

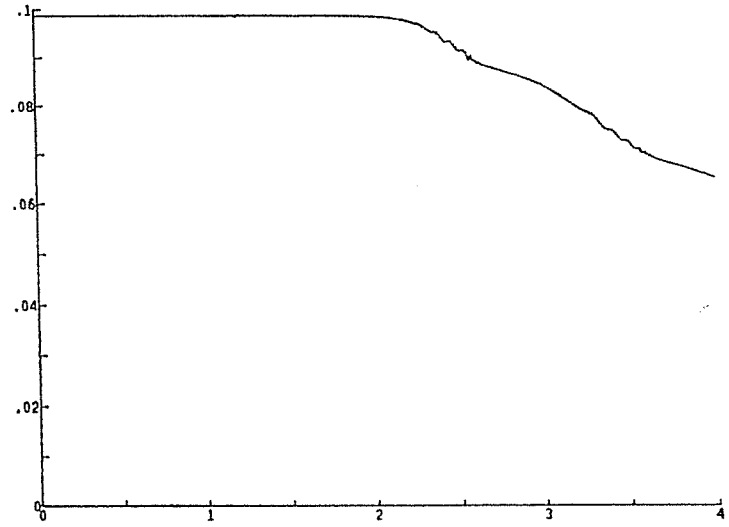

Fig. 6

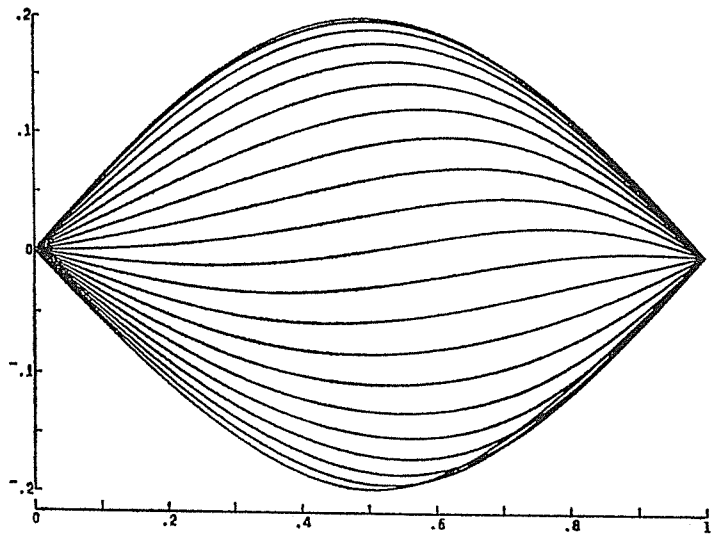

Fig. 7

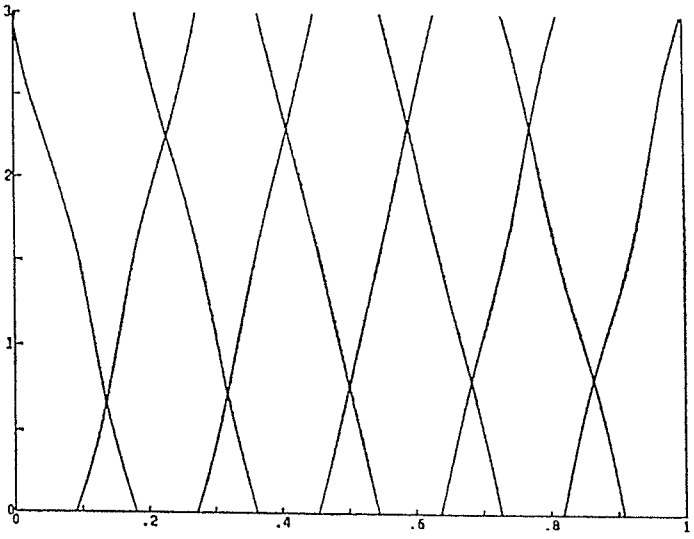

Fig. 8

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Riassunto

Si descrive un metodo numerico per lo studio di problemi di vibrazione non lineare dei fili. Il metodo si basa su una formulazione debole del problema proposta da Antman e sull'uso di elementi finiti spazio-tempo che consentono di approssimare la soluzione anche in presenza di discontinuità nelle derivate prime. Il metodo viene applicato allo studio del problema di Fermi-Pasta-Ulam la cui soluzione, dopo un certo istante critico, non è liscia. Viene calcolata la diminuzione di energia meccanica nel tempo dopo l'istante critico.

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