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On the stability under persistent disturbances for systems with impulse effect (**)

1 - Introduction

The present paper considers systems of equations with impulse effect of the form

$$\frac{dx}{dt} = f(t, x) \quad t \neq \tau_i(x), \qquad \Delta X|_{t=\tau_i(x)} = B_i(x).$$

Similar systems can be obtained in many problems of physics, technology and biology. Their study was initiated by the papers of Millman and Mishkis $[\mathbf{1}]_{1,2}$ and Mishkis and Samoilenko [**2**]. The stability of the solutions is the concern of the contributions of Samoilenko and Perestjuk $[\mathbf{3}]_{1,2}$, where linear and quasilinear systems are considered in detail.

This paper deals with the problems of stability of the solutions of systems with impulse effect under persistent disturbances. The definitions of stability are in accord with those in $[\mathbf{3}]_1$ and are given in the form used in [**4**].

2 - Preliminary remarks

Let R^n be an n-dimensional Euclidean space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, $I = [0, \infty)$ and Ω is an open connected set in R^n.

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Consider the system of n differential equations with impulse effect

$$\frac{dx}{dt} = f(t, x) \quad t \neq \tau_i(x), \qquad \Delta x|_{t=\tau_i(x)} = B_i(x) \tag{1}$$

and its corresponding perturbed system with impulse effect

$$\frac{dx}{dt} = f(t, x) + g(t, x) \quad t \neq \tau_i(x), \qquad \Delta x|_{t=\tau_i(x)} = B_i(x) + P_i(x)\,, \tag{2}$$

where

$x: I \to R^n$; $f, g: I \times \Omega \to R^n$; $B_i, P_i: \Omega \to R^n$; $\tau_i: R^n \to R$, $i \in N = \{1, 2, \ldots\}$.

Such systems describe processes whose state at definite moments changes jumpwise. Concerning the systems (1) and (2) this happens when the mapping point (t, x) of the extended phase space meets one of the hypersurfaces σ_i, given by the equations $t = \tau_i(x)$, $0 < \tau_1(x) < \ldots < \tau_i(x) < \ldots$.

At these moments, under the action of a momentary effect (hit, impulse) the mapping point «instantly» jumps from the position (t, x) into the position $(t, x + B_i(x))$. The solutions of the systems (1) and (2) will be considered to be left continuous, i.e. the following conditions hold when the integral curve $(t, x(t))$ meets the hypersurfaces σ_i at the moments t_i

$$x(t_i - 0) = x(t_i)\,, \qquad \Delta x|_{t=t_i} = x(t_i + 0) - x(t_i - 0)\,.$$

We say that conditions (A) hold if the following conditions are fulfilled:

A1. The function $f(t, x)$ is continuous in the domain $I \times \Omega$ and constants $a \geq 0$ and $M \geq 0$ exist, such that

$$\|f(t, x) - f(t, y)\| \leq a\|x\| \qquad \text{for} \quad t \in I\,, \quad x \in \Omega\,, \quad y \in \Omega\,,$$
$$\|f(t, x)\| \leq M \qquad \text{for} \quad t \in I\,, \quad x \in \Omega\,.$$

A2. The function $g(t, x)$ is continuous in the domain $I \times \Omega$ and a constant $b \geq 0$ exists, such that

$$\|g(t, x) - g(t, y)\| \leq b\|x - y\| \qquad \text{for} \quad t \in I\,, \quad x \in \Omega\,, \quad y \in \Omega\,.$$

A3. A constant $c \geq 0$ exists so that

$$\|U(x) - U(y)\| \leq c\|x - y\| \qquad \text{for } x \in \Omega\,, \quad y \in \Omega\,,$$

where the function U is any of the functions B_i, P_i, $i \in N$.

A4. The functions $\tau_i(x)$, $i \in \boldsymbol{N}$ are continuously differentiable in the domain Ω and constants $N > 0$ and $\theta > 0$ exist, such that

$$\sup_{x\in\Omega} \left\| \frac{\partial \tau_i(x)}{\partial x} \right\| \leqslant N , \qquad \inf_{x\in\Omega} \tau_{i+1}(x) - \sup_{x\in\Omega} \tau_i(x) \geqslant \theta .$$

A5. A number $h > 0$ exists, such that

$$\left\langle \frac{\partial \tau_i(x + s(B_i(x) + z))}{\partial x} , \ B_i(x) + z \right\rangle \leqslant 0$$

for $x \in \Omega$, $s \in [0, 1]$, $z \in R^n$, $\|z\| \leqslant h$, $i \in \boldsymbol{N}$.
Let $t_0 \in I$, $x_0 \in \Omega$ and by $x(t; t_0, x_0)$ denote the solution of system (1) (or (2)) for which $x(t_0 + 0; t_0, x_0) = x_0$. By $J^+(t_0, x_0)$ denote the maximal interval of the form $(t_0, \bar{t})$ where the solution is right continuable.

The definitions for stability of the solutions of systems with impulse effect needed to our further considerations are supplied next.

Let $x = \varphi(t)$ $t \in I$ be a solution of system (1) and let the integral curve $(t, \varphi(t))$ meet the hypersurfaces σ_i at the moments $t_1 < t_2 < \ldots < t_i < \ldots$.

Def. 1. The solution $x = \varphi(t)$ of system (1) is called:

1.1. *uniformly stable* if $(\forall \varepsilon > 0)(\forall \eta > 0)(\exists \delta > 0)(\forall t_0 \in I)$

$$(\forall x_0 \in \Omega, \ \|x_0 - \varphi(t_0 + 0)\| < \delta) \, (\forall t \in J^+(t_0, x_0), \ |t - t_i| > \eta)$$

$$\|x(t; t_0, x_0) - \varphi(t)\| < \varepsilon;$$

1.2. *uniformly attractive* if $(\exists \Delta > 0)(\forall \varepsilon > 0)(\forall \eta > 0)(\exists \sigma > 0)$

$$(\forall t_0 \in I)(\forall x_0 \in \Omega, \ \|x_0 - \varphi(t_0 + 0)\| < \Delta) ,$$

$$t_0 + \sigma \in J^+(t_0, x_0) \qquad \text{and} \qquad (\forall t \geqslant t_0 + \sigma, \ t \in J^+(t_0, x_0), \ |t - t_i| > \eta)$$

$$\|x(t; t_0, x_0) - \varphi(t)\| < \varepsilon;$$

1.3. *quasiattractive* if $(\forall \varepsilon > 0)(\forall \eta > 0)(\exists \lambda, r\colon 0 < \lambda < r \leqslant \varepsilon)$

$$(\exists \{\theta_n\} \subset I, \ \lim_{n\to\infty} \theta_n = \infty, \ \theta_{n+1} > \theta_n, \ \sup_{n\in N} (\theta_{n+1} - \theta_n) < \infty, \ |\theta_n - t_i| > \eta \quad \text{for } i, n \in \boldsymbol{N})$$

$$(\forall n \in \boldsymbol{N})(\forall z \in \Omega, \ \|z - \varphi(\theta_n)\| < r)$$

$$\theta_{n+1} \in J^+(\theta_n, z) \quad \text{and} \quad \|x(\theta_{n+1}; \theta_n, z) - \varphi(\theta_{n+1})\| < \lambda;$$

1.4. *uniformly asymptotically stable* if it is uniformly stable and uniformly attractive;

1.5. *uniformly stable under persistent disturbances* if

$$(\forall \varepsilon > 0)(\forall \eta > 0)(\exists r > 0)(\exists \varrho > 0) , \quad (\forall x_0 \in \Omega, \ \|x_0 - \varphi(t_0 + 0)\| < r) ,$$

the solution $x(t; t_0, x_0)$ of system (2) satisfies $(\forall g(t, x), \ \|g(t, x)\| < \varrho$ for

$$(t, x) \in I \times \Omega) \ (\forall \{P_i\}, \ \|P_i(x)\| < \varrho \text{ for } x \in \Omega) \ (\forall t \in J^+(t_0, x_0), \ |t - t_i| > \eta)$$
$$\|x(t; t_0, x_0) - \varphi(t)\| < \varepsilon .$$

We say that *the solution* $\varphi(t)$ *of system* (1) *satisfies condition* B if the following condition is fulfilled

$$\{x \in R^n : \ \|x - \varphi(t)\| \leqslant d\} \subset \Omega \text{ for all } t \in I \qquad (d > 0) .$$

3 - Main results

The following lemmas will be required for proving the main results.

Lemma 1 [**3**]$_1$. *Let conditions* A1, A3, A4, A5 *be fulfilled and let* $\varphi(t)$ *be a solution of system* (1) *lying in* Ω *for* $t \in [t_0, t_0 + T] \subset I$.

Then, if $MN < 1$ *the integral curve* $(t, \varphi(t))$ *for* $t \in [t_0, t_0 + T]$ *meets every hypersurface* σ_i *once most.*

Lemma 1 supplies sufficient conditions excluding the «beating» of the solution on the hypersurfaces σ_i, i.e. the event when the integral curve $(t, \varphi(t))$ meets the hypersurface σ_i several or infinitely many times.

Lemma 2. *Let conditions* A1, A3, A4, A5 *hold and* $MN < 1$. *Let* $\varphi(t)$ *be a solution of system* (1) *satisfying condition* B, *while the integral curve* $(t, \varphi(t))$ *meets the hypersurfaces* σ_i *at the moments* $t_1 < t_2 < \ldots < t_i < \ldots$.

Let $x(t) = x(t; t_0, x_0)$ *be a solution of system* (1) *and the numbers* $\varepsilon > 0$ *and* $\eta > 0$ *be such that* $\eta < \theta/2$ *and* $(1 + c)(\varepsilon + 5M\eta) < d$.

If the condition

$$\|x(t) - \varphi(t)\| < \varepsilon \quad \text{holds for } t \in J^+(0, x_0) , \quad |t - t_i| > \eta , \tag{3}$$

then $J^+(0, x_0) = [0, \infty)$.

Proof. In view of Lemma 1 the integral curve $(t, \varphi(t))$ meets each of the hypersurfaces σ_i only once at the moments $t = t_i$, for which, according to condition A4, we have

$$\inf_{x\in\Omega} \tau_i(x) \leqslant t_i \leqslant \sup_{x\in\Omega} \tau_i(x) \qquad i \in \boldsymbol{N}.$$

Let $\alpha_i = \frac{1}{2}\big(\sup_{x\in\Omega} \tau_i(x) + \inf_{x\in\Omega} \tau_{i+1}(x)\big)$, $i \in \boldsymbol{N}$.

We will first show that the solution $x(t)$ is continuable in the interval $[0, \alpha_1]$. Condition A4 implies that for $t \in [0, \alpha_1]$ the integral curve $(t, x(t))$ may meet only the hypersurface σ_1 and let t_1' be the first moment when this happens.

Let $t_1' < t_1 - \eta$. Then, in view of inequality (3) and conditions A1 and B, the solution $x(t)$ is continuable to $t = t_1'$ and $\|x(t_1' + 0) - \varphi(t_1')\| < \varepsilon < d$, i.e. $x(t_1' + 0) \in \Omega$ and $x(t)$ is continuable to $t = t_1 - \eta$, and moreover, $\|x(t_1 - \eta) - \varphi(t_1 - \eta)\| \leqslant \varepsilon$.

Inequality (3) and Lemma 1 yield that for $t \in [0, t_1 - \eta]$ $x(t)$ does not leave Ω and the integral curve $(t, x(t))$ does not meet the hypersurface σ_1 any more.

For $t \in J^+(0, x_0) \cap [t_1 - \eta, \alpha_1]$ the following estimates hold

$$\begin{aligned} &\|x(t) - \varphi(t)\| < \varepsilon \qquad \text{for } t \in (t_1 + \eta, \alpha_1], \\ (4)\qquad &\|x(t) - \varphi(t_1 - \eta)\| \leqslant \|x(t) - x(t_1 - \eta)\| + \|x(t_1 - \eta) - \varphi(t_1 - \eta)\| \\ &\qquad \leqslant \varepsilon + 2M\eta \qquad \text{for } |t - t_1| \leqslant \eta. \end{aligned}$$

Hence, for $t \in J^+(0, x_0) \cap [0, \alpha_1]$, the solution $x(t)$ does not leave Ω, it is continuable to $t = \alpha_1$ and the integral curve $(t, x(t))$ meets the hypersurface σ_1 only once. Besides, for $|t - t_1| \leqslant \eta$ the estimate (4) holds.

If $t_1' > t_1 + \eta$, we come to the same conclusions by analogous arguments.

Let $t_1' \in [t_1 - \eta, t_1 + \eta]$. Then the solution $x(t)$ is continuable to t_1' and the following estimates hold

$$\|x(t_1 - \eta) - \varphi(t_1 - \eta)\| \leqslant \varepsilon, \qquad \|x(t) - \varphi(t_1 - \eta)\| \leqslant \varepsilon + 2M\eta,$$

$$\|x(t_1' + 0) - \varphi(t_1 + 0)\| \leqslant (1 + c)(\varepsilon + 3M\eta) < d \qquad \text{for } t \in [t_1 - \eta, t_1'].$$

Hence, $x(t_1' + 0) \in \Omega$ and the solution $x(t)$ is continuable on the right of t_1'.

For $t \in J^+(0, x_0) \cap [t_1', \alpha_1]$ the estimates, as follows, hold

$$\begin{aligned} &\|x(t) - \varphi(t)\| < \varepsilon && \text{for } t > t_1 + \eta; \\ &\|x(t) - \varphi(t_1 + 0)\| \leqslant (1 + c)(\varepsilon + 5M\eta) < d && \text{for } t \leqslant t_1 + \eta. \end{aligned}$$

Hence, for $t \in J^+(0, x_0) \cap [0, \alpha_1]$ the solution $x(t)$ does not leave Ω, it is continuable to $t = \alpha_1$ and the integral curve $(t, x(t))$ meets the hypersurface σ_1 only once.

By the method of mathematical induction and by analogous arguments, it is easily verified that the solution $x(t)$ is continuable to any $t = \alpha_i$, i.e. $J^+(0, x_0) = [0, \infty)$.

Lemma 3. *Let the following conditions be fulfilled:*

1. *For $t \geqslant t_0$, the function $u(t)$ is nonnegative and piecewise continuous with discontinuities of the first order at which $u(t)$ is left continuous.*

2. *For $t \geqslant t_0$ the function $a(t)$ is nonnegative and continuous.*

3. *The sequence $\{t_i\}$ satisfies the condition $t_0 < t_1 < \ldots < t_i < \ldots \lim_{i\to\infty} t_i = \infty$.*

4. *For $t \geqslant t_0$, the inequality*

$$u(t) \leqslant u_0 + \int_{t_0}^{t} a(s)u(s)\,\mathrm{d}s + \sum_{t_0<t_i<t} \beta_i u(t_i)$$

holds, where $u_0 \geqslant 0$, $\beta_i \geqslant 0$ are constants.

Then for $t \geqslant t_0$

$$u(t) \leqslant u_0 \prod_{t_0<t_i<t} (1 + \beta_i) \exp\left(\int_{t_0}^{t} a(s)\,\mathrm{d}s\right).$$

The proof of Lemma 3 is completed by the method of mathematical induction and is based on the lemma of Gronwall-Bellman.

Theorem 1. *Suppose the following conditions hold:*

1. *Conditions* A *hold and $MN < 1$.*

2. *The solution $\varphi(t)$ of the system* (1) *is uniformly stable and qnasiattractive and fulfills condition* B.

Then the solution $\varphi(t)$ is uniformly stable under persistent disturbances.

Proof. In view of Lemma 1, the integral curve $(t, \varphi(t))$ meets each hypersurface exactly once and let this happen at the moments $\{t_i\}$.

Let $\varepsilon > 0$ and $\eta > 0$ be given and satisfy the inequality $(1+c)(\varepsilon + 5M\eta) < d$. Put $\bar{\varepsilon} = \min\left(\varepsilon, ((1-MN)/3N)\eta\right)$, $\bar{\eta} = \min\left(\eta, ((1-MN)/8MN)\eta\right)$.

By $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; t_0, x_0)$ denote the solutions of system (1) and (2), respectively, and let the integral curves $(t, x(t))$ and $(t, y(t))$ meet the hypersurfaces σ_i at the moments $\{t_i'\}$ and $\{t_i''\}$ respectively. Let $J^+(x)$ and $J^+(y)$

be the maximal intervals where the solutions $x(t)$ and $y(t)$ are right continuable.

The uniform stability of $\varphi(t)$ implies that $\bar{\delta} > 0$, $\bar{\delta} < \bar{\varepsilon}/2$ exists, such that for every $t_0 \in I$, for every $x_0 \in \Omega$, $\|x_0 - \varphi(t_0 + 0)\| < \bar{\delta}$ and for every $t \in J^+(x)$, $|t - t_i| > \bar{\eta}$

$$\|x(t;\, t_0, x_0) - \varphi(t)\| < \frac{\bar{\varepsilon}}{2}\,. \tag{5}$$

Then $(1 + c)(\bar{\varepsilon}/2 + 5M\bar{\eta}) < (1 + c)(\varepsilon + 5M\eta) < d$ and in view of Lemma 2 $J^+(x) = (t_0, \infty)$. Besides, $|t_i - t_i'| \leqslant MN|t_i - t_i'| + \bar{\varepsilon}N/2 + 2MN\bar{\eta}$. Hence, the estimate $|t_i - t_i'| \leqslant (N\bar{\varepsilon} + 4M\bar{\eta})/[2(1 - MN)]$ holds.

Since the solution $\varphi(t)$ of system (1) is quasiattractive, it follows that there exist constants λ, r: $0 < \lambda < r < \bar{\delta}$ and a sequence $\{\theta_n\} \subset I$, $\lim_{n\to\infty} \theta_n = \infty$ $\theta_{n+1} > \theta_n$, $\sup_{n\in N} (\theta_{n+1} - \theta_n) < \infty$, $|\theta_n - t_i| > \eta$ $\forall n$, $i \in \boldsymbol{N}$ such that for all $n \in \boldsymbol{N}$ and $x_0 \in \Omega$, $\|x_0 - \varphi(\theta_n)\| < r$

$$\|x(\theta_{n+1};\, \theta_n, x_0) - \varphi(\theta_{n+1})\| < \lambda\,. \tag{6}$$

We set $\theta_0 = 0$, $T = \sup_{n\geqslant 0} (\theta_{n+1} - \theta_n)$ and choose $\delta > 0$, $\varrho > 0$ so that $(\forall t_0 \in I)\big(\forall x_0 \in \Omega,\ \|x_0 - \varphi(t_0 + 0)\| < \delta\big)(\forall t > t_0,\ |t - t_i| > \eta)$

$$\|x(t;\, t_0, x_0) - \varphi(t)\| < \frac{r}{2}\,, \tag{7}$$

$$\varrho < \min\Big(h, \frac{1 - MN}{2N}\Big)\,, \tag{8}$$

$$\varrho T\Big(\frac{\theta + 1}{\theta}\Big)\big(1 + c(1 - MN - \varrho N)^{-1}\big)^{T/\theta} \exp[aT] < \min\Big(\frac{r}{2}, r - \lambda\Big)\,. \tag{9}$$

Suppose that the perturbations g and $\{P_i\}$ are such that $\|g(t, x)\| < \varrho$, $\|P_i(x)\| < \varrho$ for $i \in \boldsymbol{N}$ and $(t, x) \in I \times \Omega$.

We must show that for this choice of δ and ϱ, the solution $y(t)$ of system (2) satisfies the inequality

$$\|y(t) - \varphi(t)\| < \varepsilon \quad \text{for all } t \in J^+(y),\ |t - t_i| > \eta. \tag{10}$$

Let $(t_0, \omega] \subset J^+(y)$. We distinguish two cases:

I. $t_0 - \omega \leqslant T$. Let for $\tau \in (t_0, t]$ the integral curves $(\tau, \varphi(\tau))$, $(\tau, x(\tau))$

and $(\tau, y(\tau))$ meet the hypersurfaces σ_i, $n(t)$, $n_1(t)$ and $n_2(t)$ times, respectively. We prove that for $t \in (t_0, \omega]$, $|t - t_i| > \eta$, the following relations hold

(11) $$\|x(t) - y(t)\| < \frac{\delta}{2},$$ (12) $$|t_i' - t_i| \leqslant \eta, \quad |t_i'' - t_i| \leqslant \eta,$$

(13) $$n(t) = n_1(t) = n_2(t).$$

For $t \in (t_0, \omega]$, the solutions $x(t)$ and $y(t)$ satisfy the relations

(14) $$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds + \sum_{t_0 < t_i' < t} B_i(x(t_i')),$$

(15) $$y(t) = x_0 + \int_{t_0}^{t} \big(f(s, y(s)) + g(s, y(s))\big)\, ds + \sum_{t_0 < t_i'' < t} \big(B_i(y(t_i'')) + P_i(y(t_i''))\big).$$

Then for $t_0 < t \leqslant \min(t_1', t_1'')$ the estimate

$$\|y(t) - x(t)\| \leqslant \int_{t_0}^{t} a\|y(s) - x(s)\|\, ds + \varrho(t - t_0)$$

holds. Hence in view of Gronwall-Bellman's lemma and (9), we obtain

$$\|y(t) - x(t)\| \leqslant \varrho T \exp[aT] < \frac{r}{2} < \frac{\bar{\varepsilon}}{4}.$$

Then, taking into account A1, A4 and (8), we obtain consequently the estimates

$$|t_1'' - t_1| \leqslant \frac{N\bar{\varepsilon}}{4(1 - MN - N\varrho)}, \quad |t_1' - t_1| \leqslant \frac{N\bar{\varepsilon} + 4MN\bar{\eta}}{2(1 - MN)} < \eta,$$

$$|t_1'' - t_1| \leqslant \frac{3N\bar{\varepsilon} + 8MN\bar{\eta}}{2(1 - MN)} < \eta.$$

Hence, for $t \in (t_0, t_1 - \eta]$ inequalities (11), (12) hold and (13) is fulfilled.

In the case when $t \in [t_i + \eta, t_{i+1} - \eta] \cap (t_0, \omega]$ the inequalities (11) and (12) and equality (13) are proved by induction with respect to i. For completeness we will only prove that the condition (13) implies estimate (11).

Indeed, (14) and (15) yield for the function $u(t) = \|y(t) - x(t)\|$

(16) $$u(t) \leqslant \int_{t_0}^{t} a u(s)\, ds + \varrho(t - t_0) + \varrho n(t) + \sum_{t_0 < t_i < t} c\|y(t_i'') - x(t_i')\|.$$

If we denote $\bar{t}_i = \min(t_i, t_i'')$, then, taking into account A1 and A4, we get

$$\|y(t_i'') - x(t_i')\| \leqslant (1 - MN - N\varrho)^{-1}\|y(\bar{t}_i) - x(\bar{t}_i)\| . \tag{17}$$

(16) and (17), having in mind that $t - t_0 \leqslant T$ and $n(t) \leqslant (t - t_0)/\theta \leqslant T/\theta$, imply that

$$u(t) \leqslant \varrho T\left(\frac{\theta + 1}{\theta}\right) + \int_{t_0}^{t} a u(s)\, ds + \sum_{t_0 < \bar{t}_i < t} c(1 - MN - N\varrho)^{-1} u(\bar{t}_i) . \tag{18}$$

Applying Lemma 3 to inequality (18), we obtain the estimate

$$\|y(t; t_0, x_0) - x(t; t_0, x_0)\| \leqslant \varrho T \left(\frac{\theta + 1}{\theta}\right) (1 + c(1 - MN - N\varrho)^{-1})^{T/\theta} \exp[aT]. \tag{19}$$

Then, (7), (19) and (9) imply that for $t \in (t_0, \omega]$, $|t - t_i| > \eta$

$$\|y(t) - \varphi(t)\| \leqslant \|y(t) - x(t)\| + \|x(t) - \varphi(t)\| < \frac{r}{2} + \frac{r}{2} = r < \varepsilon .$$

Besides, since $(1 + c)(\varepsilon + 5M\eta) < d$, Lemma 2 implies that for $t \in (t_0, \omega]$ the solution $y(t)$ of system (2) does not leave Ω and the integral curve $(t, y(t))$ meets the hypersurfaces σ_i only once.

II. $t_0 - \omega > T$. Then some points of the sequence $\{\theta_n\}$ are included in (t_0, ω). Let $\theta_m, \theta_{m+1}, \ldots, \theta_p$ be all these points. It is sufficient to show that:

(a) For $t \in (t_0, \theta_m]$, $|t - t_i| > \eta$, $\|y(t) - \varphi(t)\| < r$.

(b) $k \in \{m, m + 1, \ldots, p - 1\}$ and $\|y(\theta_k) - \varphi(\theta_k)\| < r$ imply $\|y(t) - \varphi(t)\| < \varepsilon$ for $t \in [\theta_k, \theta_{k+1}]$, $|t - t_i| > \eta$ and $\|y(\theta_{k+1}) - \varphi(\theta_{k+1})\| < r$.

(c) For $t \in [\theta_p, \omega]$, $|t - t_i| > \eta$, $\|y(t) - \varphi(t)\| < \varepsilon$.

Since $\theta_m - t_0 \leqslant T$, then (a) is proved as in case I.

Let $\|y(\theta_k) - \varphi(\theta_k)\| < r$. Accounting that $t - \theta_k \leqslant \theta_{k+1} - \theta_k \leqslant T$ and the choice of r and ϱ, for $t \in [\theta_k, \theta_{k+1}]$, $|t - t_i| > \eta$ we get the estimates

$$\|y(t; \theta_k, y(\theta_k)) - x(t; \theta_k, y(\theta_k))\| < \min\left(\frac{r}{2}, r - \lambda\right) , \tag{20}$$

$$\|x(t; \theta_k, y(\theta_k)) - \varphi(t)\| < \frac{\varepsilon}{2} , \tag{21}$$

$$\|x(\theta_{k+1}; \theta_k, y(\theta_k)) - \varphi(\theta_{k+1})\| < \lambda . \tag{22}$$

Then, (20) and (21) imply that $\|y(t)-\varphi(t)\|<\varepsilon$ for $t\in[\theta_k,\theta_{k+1}]$, $|t-t_i|>\eta$, while (20) and (22) yield that $\|y(\theta_{k+1})-\varphi(\theta_{k+1})\|<r$. So, (b) is fulfilled.

The proof of (c) is completed as in case I.

Thus, Theorem 1 is proved.

As a corollary of Theorem 1 we get the following

Theorem 2. *Let the following conditions hold:*

1. *Conditions* A *are fulfilled and* $MN<1$.

2. *The function* $\varphi(t)$ *is a uniform asymptotically stable solution of system* (1) *satisfying condition* B.

Then the solution $\varphi(t)$ *is uniformly stable under persistent disturbances.*

Proof. In view of Theorem 1, it is sufficient to prove that the solution $\varphi(t)$ is quasiattractive.

Since $\varphi(t)$ is a uniformly attractive solution of system (1), then $\Delta>0$ exists, such that

$$(\forall\varepsilon>0)(\forall\eta>0)(\exists\sigma>0)(\forall t_0\in I)(\forall z\in\Omega,\ \|z-\varphi(t_0+0)\|<\Delta)\ t_0+\sigma\in J^+(t_0,z)$$
$$\text{and } (\forall t\geqslant t_0+\sigma,\ t\in J^+(t_0,z),\ |t-t_i|>\eta)$$

$$\|x(t;t_0,z)-\varphi(t)\|<\frac{\varepsilon}{2}. \tag{23}$$

Then choose the numbers r, λ and the sequence $\{\theta_n\}$, satisfying the conditions of Def. 1, as follows

$$r=\varepsilon\,,\quad \lambda=\frac{\varepsilon}{2}\,,\quad \alpha_n=n(\sigma+2\eta)\,,$$

$$\theta_n=\begin{cases}\alpha_n & \text{if } |\alpha_n-t_i|>\eta \quad \text{for every } i\in\boldsymbol{N}\\ \alpha_n+2\eta & \text{if } |\alpha_n-t_i|\leqslant\eta \quad \text{for some } i\in\boldsymbol{N}.\end{cases}$$

Thus Theorem 2 is proved.

References

[1] V. D. Mill'man and A. D. Myshkis: $[\bullet]_1$ *On the stability of motion in the presence of impulses*, Sibarski Math. Z. (1) **2** (1960), 233-237; $[\bullet]_2$ *Random impulses in linear dynamic systems. Asymptotic methods for solving differential equations*, Ed. AN UkSSR, Kiev (1963), 64-81.

[2] A. D. Myshkis and A. M. Samoilenko, *Systems with impulses in prescribed moments of time*, Math. Sb. **74** (1967), 2.

[3] A. M. Samoilenko and N. A. Perestjuk: $[\cdot]_1$ *Stability of the solutions of differential equations with impulse effect*, Differencial'nye Uravnenija **11** (1977), 1981-1992; $[\cdot]_2$ *On the stability of the solutions of systems with impulse effect*, Differencial'nye Uravnenija **11** (1981), 1995-2001.

[4] N. Rouche, P. Habets and M. Laloy, *Stability theory by Liapunov's direct method*, Springer-Verlag, New York 1977.

Summary

In this paper problems concerning the stability of solution of systems of ordinary differential equations with impulse effect under persistent disturbances are considered.

Definitions for stability of the system considered are introduced. In the proof of the theorem, a new analogue of the Gronwall-Bellman's inequality for piecewise continuous functions is applied.

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