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Convex Clarkson inequalities with applications to nonlinear operators (**)

The following Hilbert space identity is well-known

$$(*)\qquad |(1-t)x+ty|^2+t(1-t)|x-y|^2+(1-t)|x|^2+t|y|^2 \qquad (0\leqslant t\leqslant 1)\,.$$

This identity, which can be regarded as a convex parallelogram law, has proven to be very useful in the study of certain problems arising in the geometric study of nonlinear operators acting on L_2 (see, for example [**11**]). In this note we use Clarkson's inequalities to derive a version of $(*)$ for the Lp spaces, and give an application of the resulting inequality in the study of nonlinear semigroups.

Throughout, we assume that (X, M, μ) is a σ-finite measure space and $Lp(\mu)$ $(p>1)$ denotes the Banach space of all functions f satisfying $\int_X |f|^p\, d\mu<\infty$; we also assume that $1/p+1/q=1$.

Theorem 1. *Suppose* $2\leqslant p<\infty$ *and* $t=2^{-n}$ *for some integer* n. *Then for* $x, y\in Lp(\mu)$

$$(1)\qquad |(1-t)x+ty|_p^p+(2^{p-1}-1)^{-1}t(1-t^{p-1})|x-y|_p^p\leqslant(1-t)|x|_p^p+t|y|_p^p\,.$$

Notice that this reduces to $(*)$ in case $p=2$, and reduces to Clarkson's inequality [**3**]

$$\left|\frac{x+y}{2}\right|^p+\left|\frac{x-y}{2}\right|^p\leqslant\frac{1}{2}\,|x|^p+\frac{1}{2}\,|y|^p$$

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in case $t = \frac{1}{2}$. For $1 < p \leqslant 2$, the inequality is similar.

Theorem 2. *Suppose* $1 < p \leqslant 2$ *and* $t = 2^{-n}$ *for some integer* n. *Then for* $x, y \in Lp(\mu)$

$$(2) \qquad |(1-t)x + ty|_p^q + (2^{q-1}-1)^{-1}t(1-t^{q-1})|x-y|_p^q \leqslant (1-t)|x|_p^q + t|y|_p^q .$$

An almost immediate consequence of these results is the following

Theorem 3. *Let* K *be a convex and bounded subset of* Lp. *If* $d = \sup\{|x-y| : x, y \in K\}$ *and* $r = \inf\{\varrho > 0 : \bigcap_{x \in K} B(x; \varrho) \neq \emptyset\}$, *then*

$$(3) \qquad (1-2^{1-p})^{-1} r^p \leqslant d^p \quad \textit{if } 2 \leqslant p < \infty ,$$

$$(4) \qquad (1-2^{1-q})^{-1} r^q \leqslant d^q \quad \textit{if } 1 < p \leqslant 2 .$$

The number d defined above is the *diameter* of K, while the number r is the *Cebysev radius* of K. It follows routinely from weak-compactness of K and weak lower semicontinuity of the norm that there is an $x \in \mathrm{cl}(K)$ (called the *Cebysev Center* of K) such that $K \subseteq B(x; r)$. In case $p = 2$, the above estimates are known to be sharp [**1**]; we do not know if they are sharp for $p \neq 2$.

Before stating our final result, recall that a (nonlinear) semigroup acting on a set K is a function σ: $[0, \infty) \times K \to K$ satisfying

$$(5) \qquad \sigma(0, x) = x \qquad \text{for all } x ,$$

$$(6) \qquad \sigma(s+t, x) = \sigma(s, \sigma(t, x)) \qquad \text{for all } x, s, t .$$

In addition, σ is said to be *uniformly γ-Lipschitzian* if, for all t and all $x, y \in K$

$$(7) \qquad |\sigma(t, x) - \sigma(t, y)| \leqslant \gamma |x - y| .$$

The uniformly Lipschitizian semigroups were introduced in [**6**], and studied in greater detail in [**4**], [**7**] and [**8**].

Theorem 4. *Let* K *be a closed, bounded and convex subset of* Lp $(1 < p < \infty)$ *and let* σ *be a uniformly* γ*-Lipschitizian semigroup acting on* K. *Then* *there is an $\overline{x}$ in K such that $\sigma(t, \overline{x}) = \overline{x}$ for all t if*

$$1 < p < 2 \quad \text{and } \gamma < (1 - 2^{1-q})^{-1/q}, \quad \text{or} \tag{8}$$

$$2 < p < \infty \quad \text{and } \gamma < (1 - 2^{1-p})^{-1/p}. \tag{9}$$

In case $p = 2$, the above estimates were derived in [8] using substantially different techniques than those we apply below; it is not known, even in the case $p = 2$, whether the above estimates are sharp.

We now turn to the proofs of the above results; except for Theorem 2, we prove only the case $2 < p < \infty$, the proof for $1 < p \leqslant 2$ being identical. We begin with the simplest result, Theorem 3, then prove Theorem 4 which is a refinement of the approach in Theorem 3. Finally, we derive the inequalities (1) and (2).

Proof of Theorem 3. Let z denote the Cebysev center of K; fix $\lambda = 2^{-n}$ and choose sequences $\{x_m\}$ and $\{y_m\}$ in K so that

$$|x_m - z| \to r \quad \text{as } m \to \infty, \qquad |(1-\lambda)z + \lambda x_m - y_m| \geqslant r - \frac{1}{m} \quad \text{for all } m.$$

Applying (1)

$$r^p + (2^{p-1}-1)^{-1}\lambda(1-\lambda^{p-1})r^p$$
$$\leqslant \varliminf_{m\to\infty} |(1-\lambda)(z-y_m) + \lambda(x_m - y_m)|^p + (2^{p-1}-1)^{-1}\lambda(1-\lambda)^{p-1}|x_m - z|^p$$
$$\leqslant \varlimsup_{m\to\infty} (1-\lambda)|z-y_m|^p + \lambda|x_m - y_m|^p \leqslant (1-\lambda)r^p + \lambda d^p.$$

From this, recalling $\lambda = 2^{-n}$,

$$\big(1 + (2^{p-1}-1)^{-1}(1 - 2^{-n(p-1)})\big)r^p \leqslant d^p.$$

Letting $n \to \infty$ gives (3).

Proof of Theorem 4. We follow the general outline of [6], and so omit some of the details. For $x, y \in K$ define

$$r(x, y) = \lim_{t\to\infty} \sup_{s \geqslant t} |\sigma(s, x) - y|, \qquad d(x) = r(x, x).$$

It is readily checked that, for each x, the map $r(x, \cdot)$ is nonexpansive and convex (cf. [7]) and thus weakly lower semicontinuous. From this, there is a $z(x) \in K$ for which $r(x, z(x)) = \inf\{r(x, y): y \in K\}$.

Now fix $\varepsilon > 0$ and select $t_1 > 0$ so that $|\sigma(t_1, z(x)) - z(x)| \geqslant d(z(x)) - \varepsilon$; select $t_2 > 0$ so that $s \geqslant t_2$ implies $|\sigma(s, x) - z(x)| \leqslant r(x, z(x)) + \varepsilon$; thus $s - t_1 \geqslant t_2$ implies

$$|\sigma(t_1, z(x)) - \sigma(s, x)| \leqslant \gamma |z(x) - \sigma(s - t_1, x)|$$
$$\leqslant \gamma[r(x, z(x)) + \varepsilon] .$$

Now fix $\lambda = 2^{-n}$ and set $m = (1 - \lambda)z(x) + \lambda\sigma(t_1, z(x))$. Then, applying (1) and the above estimates,

$$|m - \sigma(s, x)|^p + (2^{p-1} - 1)^{-1}\lambda(1 - \lambda^{p-1})[d(z(x)) - \varepsilon]^p$$
$$\leqslant |m - \sigma(s, x)|^p + (2^{p-1} - 1)^{-1}\lambda(1 - \lambda^{p-1})|\sigma(t_1, z(x)) - z(x)|^p$$
$$\leqslant (1 - \lambda)|z(x) - \sigma(s, x)|^p + \lambda|\sigma(t_1, z(x)) - \sigma(s, x)|^p$$
$$\leqslant (1 - \lambda)|z(x) - \sigma(s, x)|^p + \lambda\gamma^p[r(x, z(x)) + \varepsilon]^p .$$

Taking the limit superior as $s \to \infty$ gives

$$r(x, z(x))^p + (2^{p-1} - 1)^{-1}\lambda(1 - \lambda^{p-1})[d(z(x)) - \varepsilon]^p$$
$$\leqslant (1 - \lambda)r(x, z(x))^p + \lambda\gamma^p[r(x, z(x)) + \varepsilon]^p .$$

Now we may let $\varepsilon \to 0$ and, upon gathering terms, obtain,

$$(2^{p-1} - 1)^{-1}\lambda(1 - \lambda^{p-1})\, d(z(x)) \leqslant \lambda(\gamma^p - 1)r(x, z(x))^p .$$

Dividing by $\lambda = 2^{-n}$ and letting $n \to \infty$ gives

$$d(z(x))^p \leqslant (\gamma^p - 1)(2^{p-1} - 1)r^p(x, z(x)) . \tag{10}$$

Now set $\alpha = [(\gamma^p - 1)(2^{p-1} - 1)]^{1/p}$; then $\alpha < 1$ by our choice of γ; also

$$|z(x) - x| \leqslant d(x) + r(x, z(x)) \leqslant 2d(x) = \frac{2}{1 - \alpha}\,(d(x) - \alpha d(x))$$
$$\leqslant \frac{2}{1 - \alpha}\, d(x) - d(z(x)) .$$

Since d is continuous (indeed, $|d(x) - d(y)| \leqslant (1 + \gamma)|x - y|$), Carisit's theorem [2] now implies that $z(\bar{x}) = \bar{x}$ for some $x \in K$. Then, in light of (10), $d(\bar{x}) = 0$, so $\sigma(t, x) \to \bar{x}$ as $t \to \infty$. From this $\sigma(s, \bar{x}) = \lim_{t\to\infty} \sigma(s, \sigma(t, \bar{x})) = \lim_{t\to\infty} \sigma(s + t, \bar{x})$, so $\sigma(s, \bar{x}) = \bar{x}$ for all s, completing the proof.

We remark that, in [6], an estimate for γ is derived using the modulus of convexity; for $p \geqslant 2$, this estimate is $(1 + 2^{-p})^{1/p} (< (1 - 2^{1-p})^{-1/p})$.

Proof of Theorem 1. As we have already remarked, for $n = 1$ this reduces to Clarkson's inequality for $p \geqslant 2$; for $n > 1$, we proceed by induction; setting $\lambda_n = 2^{-n}$

$$|(1 - \lambda_{n+1})x + \lambda_{n+1} y|^p = |(1 - \lambda_n)x + \lambda_n (\frac{x+y}{2})|^p$$

$$\leqslant (1 - \lambda_n)|x|^p + \lambda_n |\frac{x+y}{2}|^p - (2^{p-1} - 1)^{-1} \lambda_n (1 - \lambda_n^{1-p}) |\frac{x-y}{2}|^p$$

$$\leqslant (1 - \lambda_n)|x|^p + \lambda_{n+1}|x|^p + \lambda_{n+1}|y|^p - (\lambda_n 2^{-p} + 2^{-p}(2^{p-1} - 1)^{-1}\lambda_n(1 - \lambda_n^{1-p}))|x - y|^p$$

$$\leqslant (1 - \lambda_{n+1})|x|^p + \lambda_{n+1}|y|^p - (2^{p-1} - 1)^{-1}\lambda_{n+1}(1 - \lambda_{n+1}^{1-p})|x - y|^p,$$

completing the proof.

Proof of Theorem 2. First observe that for any numbers $a, b \geqslant 0$,

$$(\tfrac{1}{2}a^p + \tfrac{1}{2}b^p)^{q-1} \leqslant \tfrac{1}{2}a^q + \tfrac{1}{2}b^q . \tag{11}$$

To see this, suppose without loss of generality that $0 < b \leqslant a$, and divide by a^q; (11) is then equivalent to

$$2^{p-2}(1 + x^p) \leqslant (1 + x^q)^{p-1}, \qquad 0 \leqslant x \leqslant 1 .$$

Set $f(x) = 2^{(p-2)/p}(1 + x^p)^{1/p}(1 + x^q)^{-1/q}$; an elementary computation shows that

$$f'(x) = \frac{2^{(p-2)/p}(1 + x^p)^{1/p}}{x(1 + x^q)^{1/q}} \left(\frac{x^p}{1 + x^p} - \frac{x^q}{1 + x^q}\right) .$$

Since $p \leqslant q$, $f'(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$, and thus $f(x) \leqslant f(1) = 1$; thus

$$2^{(p-2)/p}(1 + x^p)^{1/p} \leqslant (1 + x^q)^{1/q} .$$

Raising both sides to the power p establishes (11). From this and Clarkson's inequality for $1 < p \leqslant 2$, we see that

$$\left|\frac{x+y}{2}\right|^q + \left|\frac{x-y}{2}\right|^q \leqslant \frac{1}{2}\,|x|^q + \frac{1}{2}\,|y|^q .$$

The proof of Theorem 2 can now be completed in exactly the same fashion as Theorem 1.

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Abstract

A version of Clarkson's inequalities involving convex combinations is derived. Applications to Cebysev centers and nonlinear semigroups are considered.

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