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On the compatibility and uniqueness of the equilibrium solution for the problem of pure traction of an elastic dielectric (**)

Introduction

In two previous papers $[\mathbf{1}]_{1,2}$ we studied the equilibrium problem for non linear dielectrics; in $[\mathbf{1}]_1$ a theorem of existence and uniqueness for the elastostatics of a dielectric was proved, whereas in $[\mathbf{1}]_2$ Signorini's method was generalized to the case of a non linear dielectric when a portion of the boundary of the system is constrained (mixed boundary problem).

In this paper we shall prove a theorem of compatibility and uniqueness of the equilibrium solution for traction boundary condition problems when the whole boundary of the dielectric is free. We will obtain results which are similar to the corresponding ones that Signorini obtained in the elastostatics [4] (p. 223 and foll).

More precisely, if we develop the displacement $\boldsymbol{u}$ in power series of the two parameters λ and μ, the first due to the mechanical actions and the second to the electric ones and if we impose that the total momentum of all the loads is equal to zero, then we will be able to determine the conditions that have to be satisfied by all terms in the power series of $\boldsymbol{u}$. Such conditions with *double indexes*, which represent a double infinity of compatibility conditions to be satisfied, will turn out to be Signorini's compatibility condition for the elastostatics when one of the two indexes is supposed equal to zero.

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1 - Statement of the problem

Let $\mathscr{B}$ be a continuous dielectric. If $\mathscr{C}^*$ and $\mathscr{C}$ represent a reference configuration and equilibrium one of $\mathscr{B}$ respectively, the deformation that $\mathscr{B}$ will experience on passing from $\mathscr{C}^*$ to $\mathscr{C}$ is expressed by the set of scalar functions

$$x^i = x^i(X^L) \qquad (i, L = 1, 2, 3) , \tag{1.1}$$

where X^L are the Lagrangian coordinates of the point $X \in \mathscr{C}^*$ and x^i are the Eulerian coordinates of $x \in \mathscr{C}$.

The boundary problem with pure traction condition for the equilibrium of a dielectric $\mathscr{B}$ is the following (1)

$$\begin{aligned} &\operatorname{Div} \boldsymbol{A}(u_{i,L}, \varphi_{,L}) + \varrho^* \boldsymbol{b} = 0 , \qquad \operatorname{Div} \boldsymbol{B}(u_{i,L}, \varphi_{,L}) = 0 , \\ &\varphi = \bar{\varphi} \quad \text{on } \partial\mathscr{C}^* , \qquad \boldsymbol{T}_* \cdot \boldsymbol{n}_* = \boldsymbol{t}_* \quad \text{on } \partial\mathscr{C}^* , \end{aligned} \tag{1.2}$$

where: $u^i = x^i - X^i =$ displacement field, $H_{i,L} = u_{i,L} =$ displacement gradient field, $\varrho_* =$ reference mass density, $\varphi =$ electrostatic potential, $\mathscr{E}_L = -\varphi_{,L}$ = Lagrangian electric field, $\boldsymbol{b} =$ specific force density, $\boldsymbol{T}_* = \boldsymbol{A}(H_{i,L}, \mathscr{E}_L)$ = Piola-Kirchhoff stress tensor, $\mathscr{D} = \boldsymbol{B}(H_{i,L}, \mathscr{E}_L) =$ Lagrangian induction fields, $\boldsymbol{n}_* =$ unit exterior normal vector to $\partial\mathscr{C}^*$, and $\bar{\varphi}$ is a sectionally constant function on $\partial\mathscr{C}^*$.

Let us assume that the following representations of $\boldsymbol{b}$ and $\boldsymbol{t}_*$ hold

$$\boldsymbol{b} = \sum_{n=1}^{\infty} \lambda^n \boldsymbol{b}_n , \qquad \varphi = \sum_{n=1}^{\infty} \mu^n \bar{\varphi}_n , \qquad \boldsymbol{t} = \sum_{n=1}^{\infty} \lambda^n \boldsymbol{t}_{*n} , \tag{1.3}$$

where λ and μ are characteristic parameters of the problem, and $\bar{\varphi}_n$ are sectionally constant on $\partial\mathscr{C}^*$.

Moreover, let us assume that $\mathscr{C}^*$ is a natural state of the system, that is

$$\boldsymbol{T} = \boldsymbol{\omega}(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0} , \qquad \mathscr{D} = \boldsymbol{B}(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0} ,$$

and let $\mathscr{D}$ and $\boldsymbol{T}_*$ be analytic functions with respect to $\boldsymbol{H}$ and $\mathscr{E}$, which are obtainable from the entropy function $z(\boldsymbol{H}, \mathscr{E})$ according to the formulas (2)

$$\boldsymbol{T}_* = \varrho_* \frac{\partial z}{\partial \boldsymbol{F}} , \qquad \mathscr{D} = -\varrho_* \frac{\partial z}{\partial \mathscr{E}} .$$

(1) See [2].
(2) See [3].

Upon these hypotheses the problem here to be dealt represents a particular case of the one which we introduced in $[1]_{1,2}$ therefore all the results which we obtained there can be immediately applied here and a solution of problem (1.2) can then be written in the form

$$u = \sum_{n,m=0}^{\infty} \lambda^n \mu^m u_{nm} , \quad \varphi = \sum_{n,m=0}^{\infty} \lambda^n \mu^m \varphi_{nm} , \tag{1.4}$$

where the terms u_{nm} and φ_{nm} satisfy the system of the elastostatics of a linear dielectric, cfr. $[1]_2$,

$$(1.5)_1 \qquad \operatorname{Div}(\boldsymbol{A}_{10}\cdot\boldsymbol{H}_{nm} + \boldsymbol{A}_{01}\cdot\mathscr{E}_{nm} + \mathscr{A}_{nm}) + \varrho_* \boldsymbol{b}_n \delta_{0n} = \boldsymbol{0} ,$$

$$(1.5)_2 \qquad \operatorname{Div}(\boldsymbol{B}_{10}\cdot\boldsymbol{H}_{nm} + \boldsymbol{B}_{01}\cdot\mathscr{E}_{nm} + \mathscr{B}_{nm}) = 0 ,$$

$$(1.5)_3 \qquad \varphi_{nm} = \delta_{nn} \forall_m \qquad \text{on } \partial\mathscr{C} ,$$

$$(1.5)_4 \qquad (\boldsymbol{A}_{10}\cdot\boldsymbol{H}_{nm} + \boldsymbol{A}_{01}\cdot\mathscr{E}_{nm} + \mathscr{A}_{nm})\cdot\boldsymbol{n}_* = \boldsymbol{t}_{*n}\delta_{0n} \qquad \text{on } \partial\mathscr{C}^* .$$

In (1.5) $(\boldsymbol{H}_{nm})_{ij} = (\boldsymbol{u}_{nm})_{ij}$ and moreover $\boldsymbol{A}_{10}$ and $\boldsymbol{A}_{01}$ are the elasticity tensor and the linear polarization tensor, $\boldsymbol{B}_{01}$ is the dielectric tensor and $\boldsymbol{B}_{10}$ represents the linear interaction between the displacement and the electric induction.

Furthermore $\mathscr{A}_{nm}$ and $\mathscr{B}_{nm}$ are polinomial functions of the displacement gradients $\boldsymbol{H}_{ij}$ and of the fields $\mathscr{E}_{ij}$, where the indexes i and j are smaller than n and m respectively.

We recall that in $[1]_2$ we proved that the determination of u_{nm}, φ_{nm} brings us back to the integration of the successive linear boundary problems; nevertheless, in the present case, not every solution of system (1.5) represents an acceptable term of the power series (1.4), since the solution $\boldsymbol{u}$, φ, as it is well known, must be such that the total momentum of the system has to be equal to zero cfr. [4] (p. 127 and p. 128). That is, for $\boldsymbol{u}$, φ to be a solution of (1.5), the following compatibility condition must hold

$$\int_{\partial\mathscr{C}^*} \boldsymbol{u}\times\boldsymbol{t}_*\, d\sigma_* + \int_{\mathscr{C}^*} \boldsymbol{u}\times\varrho_*\boldsymbol{b}\, d\mathscr{C}^* = \boldsymbol{0} . \tag{1.6}$$

2 - An analysis of the compatibility condition

By inserting $(1.4)_1$ into (1.6), and taking into account (1.3), we obtain the condition

$$\int_{\partial\mathscr{C}^*} \Big(\sum_{n,m=0}^{\infty} \lambda^n\mu^m \boldsymbol{u}_{nm}\Big)\times\Big(\sum_{p=0}^{\infty} \lambda^p \boldsymbol{t}_{*p}\Big) d\sigma^* + \int_{\mathscr{C}^*} \Big(\sum_{n,m=0}^{\infty} \lambda^n\mu^m \boldsymbol{u}_{nm}\Big)\times\varrho_*\Big(\sum_{p=0}^{\infty} \lambda^p \boldsymbol{b}_p\Big) d\mathscr{C}^* = 0 ,$$

which can be written in the following form

$$\sum_{n,m=0}^{\infty} \lambda^n \mu^m \sum_{p=1}^{n-1} \Big(\int_{\partial\mathscr{C}^*} \boldsymbol{u}_{n-p,m} \times \boldsymbol{t}_{*p} \, d\sigma^* + \int_{\mathscr{C}^*} \boldsymbol{u}_{n-p,m} \times \varrho_* \boldsymbol{b} \, d\mathscr{C}^* \Big) = 0 \, .$$

Therefore (1.6) requires that the terms $\boldsymbol{u}_{nm}$ of the series $(1.4)_1$ satisfy the following other compatibility conditions

$$(2.1) \qquad \sum_{p=1}^{n=1} \Big(\int_{\partial\mathscr{C}^*} \boldsymbol{u}_{n-p,m} \times \boldsymbol{t}_{*p} \, d\sigma^* + \int_{\mathscr{C}^*} \boldsymbol{u}_{n-p,m} \times \varrho_* \boldsymbol{b}_p \, d\mathscr{C}^* \Big) = 0 \qquad (n, m \; 1, 2 \, \ldots) \, ,$$

and (2.1) provides us with Signori's compatibility condition, when we set $m = 0$ (cfr. [4], p. 224).

On the other hand, it is well known, that system (1.5) exhibits a solution $\boldsymbol{u}_{nm}$, φ_{nm} such that $\boldsymbol{u}_{nm}$ is determined to within an infinitesimal rotation of $\mathscr{B}$. We shall prove that (2.1), upon suitable conditions, defines completely such a infinitesimal rotation for each n and m, so that, $\boldsymbol{u}_{nm}$ and consequently $\boldsymbol{u}$, is determined in one and only one way.

To this extent, let us assume that $(\boldsymbol{u}_{1m}, \varphi_{1m})$, $(\boldsymbol{u}_{2m}, \varphi_{2m}) \ldots (\boldsymbol{u}_{p-1m}, \varphi_{p-1m})$ are solutions previously found for system (1.5), such that they satisfy condition (2.1), when $n = 2, 3 \ldots p$ and for a set value of m.

Let $(\boldsymbol{u}_{pm}, \varphi_{pm})$ be a solution of (1.5) with $n = p$ and let us set

$$(2.2) \qquad \boldsymbol{r}_p = \sum_{n=1}^{p} \Big(\int_{\partial\mathscr{C}^*} \boldsymbol{u}_{p+1-n,m} \times \boldsymbol{t}_{*n} \, d\sigma^* + \int_{\mathscr{C}^*} \boldsymbol{u}_{p+1-n,m} \times \varrho_* \boldsymbol{b}_n \, d\mathscr{C}^* \Big) \, .$$

If $\boldsymbol{\omega}_{pm}$ is some constant vector which is a solution of (1.5), then also $\boldsymbol{u}_{pm} + \boldsymbol{\omega}_{pm} \times \boldsymbol{u}_*$ is a solution of the same system when $n = p$ and for a set m; here the quantity $\boldsymbol{u}_*$ is the position vector of some point $X \in \mathscr{C}^*$ with respect to a fixed origin.

Condition (2.1) will be satisfied by such solution if

$$\int_{\partial\mathscr{C}^*} (\boldsymbol{\omega}_{pm} \times \boldsymbol{u}_*) \times \boldsymbol{t}_{*1} \, d\sigma_* + \int_{\mathscr{C}^*} (\boldsymbol{\omega}_{pm} \times \boldsymbol{u}_*) \times \varrho_* \boldsymbol{b}_1 \, d\mathscr{C}^* = - \boldsymbol{r}_{pm} \, .$$

By the use of the following vectorial identity $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c}) \boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c}) \boldsymbol{a}$, the last condition takes the form

$$(2.3) \qquad (\boldsymbol{A}_1 - \boldsymbol{I} \operatorname{tr} \boldsymbol{A}_1) \boldsymbol{\omega}_{pm} = - \boldsymbol{r}_{pm} \, ,$$

where $\boldsymbol{A}_1$ the «astatic load», corresponding to $\boldsymbol{t}_{*1}$ and $\boldsymbol{b}_1$, is given by the relation

$$\boldsymbol{A}_1 \equiv \int_{\partial\mathscr{C}^*} \boldsymbol{u}_* \otimes \boldsymbol{t}_{*1} \, d\sigma_* + \int_{\mathscr{C}^*} \boldsymbol{u}_* \otimes \varrho_* \boldsymbol{b}_1 \, d\mathscr{C}^* \, .$$

If the load $\boldsymbol{t}_{*1}, \boldsymbol{b}_1$ does not have equilibrium axes that is $\det(\boldsymbol{A}_1 - \boldsymbol{I}\,\mathrm{tr}\,\boldsymbol{A}_1) \neq 0$, system (2.3) exhibits one and only one solution $\boldsymbol{\omega}_{pm}$ so that $\boldsymbol{u}_{nm}$ will be univocally determined from the compatibility condition (2.1).

On the other hand for system (1.5) to have a solution $(\boldsymbol{u}_{nm}, \varphi_{nm})$ it is necessary that all the loads acting on the system be balanced and so we are led to prove that condition (2.1) is equivalent to the requirement that the moments of such loads are equal to zero for each n and m.

In fact, system (1.5) shows that the loads which actually determine the dispacement $\boldsymbol{u}_{nm}$ are given by

$$(2.4) \qquad \boldsymbol{t}^*_{nm} = \boldsymbol{t}_{*nm}\delta_{0m} - \mathscr{A}_{nm}\cdot\boldsymbol{n}_*, \qquad \boldsymbol{b}_{nm} = \boldsymbol{b}_n\delta_{0m} + \mathrm{Div}\mathscr{A}_{nm},$$

whereas the condition for the momentum of the loads to be equal to zero is

$$(2.5) \qquad \int_{\partial\mathscr{C}^*}\boldsymbol{u}_*\times\boldsymbol{t}_{*nm}\,\mathrm{d}\sigma_* + \int_{\mathscr{C}^*}\boldsymbol{u}_*\times\varrho_*\boldsymbol{b}_{nm}\,\mathrm{d}\mathscr{C}^* = \boldsymbol{0} \qquad (n, m\ 1, 2, \ldots).$$

If we sobstitute the expansions (1.3) into (1.6) we see that $\boldsymbol{t}_{*n}$ and $\boldsymbol{b}_n$ are equilibrated.

Therefore (2.5) can be written as follows

$$\int_{\partial\mathscr{C}^*}\boldsymbol{u}_*\times(\boldsymbol{t}_{*nm} - \boldsymbol{t}_{*n}\delta_{0m})\,\mathrm{d}\sigma_* + \int_{\mathscr{C}^*}\boldsymbol{u}_*\times\varrho_*(\boldsymbol{b}_{nm} - \boldsymbol{b}_n\delta_{0m})\,\mathrm{d}\mathscr{C}^* = \boldsymbol{0} \qquad (n, m\ 1, 2, \ldots),$$

which, taking into account (2.4), becomes

$$(2.6) \qquad -\int_{\partial\mathscr{C}^*}\boldsymbol{u}_*\times(\mathscr{A}_{nm}\cdot\boldsymbol{n}_*)\,\mathrm{d}\sigma_* + \int_{\mathscr{C}^*}\boldsymbol{u}_*\times\varrho_*\,\mathrm{Div}\,\boldsymbol{A}_{nm}\,\mathrm{d}\mathscr{C}^* = 0 \qquad (n, m = 1, 2, 3, \ldots).$$

If we introduce Green's identity

$$\int_{\mathscr{C}^*}(\mathrm{Grad}\,\boldsymbol{s})\boldsymbol{S}^T\,\mathrm{d}\mathscr{C}^* = \int_{\partial\mathscr{C}^*}\boldsymbol{s}\otimes\boldsymbol{S}\cdot\boldsymbol{n}_*\,\mathrm{d}\sigma_* - \int_{\mathscr{C}^*}\boldsymbol{s}\otimes\mathrm{Div}\,\boldsymbol{S}\,\mathrm{d}\mathscr{C}^*,$$

where $\boldsymbol{s} = \boldsymbol{u}_*$ and $\boldsymbol{S} = \mathscr{A}_{nm}$, then from (2.6) the following condition is obtained

$$(2.7) \qquad \int_{\mathscr{C}^*}\mathscr{A}_{nm}\,\mathrm{d}\mathscr{C}^* = \text{symmetric tensor}.$$

On the other hand, as we proved in $[\mathbf{1}]_1$ (formulas (3.7) and (3.4)), and it is showed by (1.5), the following relation holds

$$\boldsymbol{T}_{*nm} = \boldsymbol{A}_{10}\cdot\boldsymbol{u}_{nm}\boldsymbol{A}_{01}\cdot\mathscr{E}_{nm} + \mathscr{A}_{nm},$$

so that, if we recall that $\boldsymbol{A}_{10}\cdot\boldsymbol{u}_{nm} = \boldsymbol{L}\cdot\boldsymbol{E}_{nm}$, where $\boldsymbol{L}$ is the infinitesimal elasticity tensor and $\boldsymbol{E}_{nm} = \frac{1}{2}(\boldsymbol{u}_{nm} + \boldsymbol{u}^T_{mn})$, and observe that the hypothesis of existence of the specific enthalpy implies that $A_{01}\cdot\mathscr{E}_{nm}$ is symmetric (see [**3**]), condition (2.7) implies that

$$\int_{\mathscr{C}^*} \boldsymbol{T}_{*nm}\,\mathrm{d}\mathscr{C}^* = \text{symmetric tensor}\,. \tag{2.8}$$

Furthermore, because of the symmetry of the stress tensor $\boldsymbol{T}$ and recalling the definition of the Piola-Kirchhoff tensor, it follows that $\boldsymbol{T}_*F^T = \boldsymbol{T}_*(1 + H^T) = $ symmetric tensor, where $\boldsymbol{F}$ is the deformation gradient.

Since we have $\boldsymbol{T}_* = \sum_{n,m=0}^{\infty} \lambda^n\mu^m \boldsymbol{T}_{*nm}$, $\boldsymbol{H} = \sum_{n,m=0}^{\infty} \lambda^n\mu^m \boldsymbol{H}_{nm}$, the last condition becomes

$$\boldsymbol{T}_{*nm} + \sum_{p=1}^{n-1} \boldsymbol{T}_{*pm} H^T_{n-p,m} = \text{symmetric tensor}\,. \tag{2.9}$$

By integrating (2.9), taking into account (2.8), we obtain

$$\sum_{p=1}^{n-1} \int_{\mathscr{C}^*} \boldsymbol{T}_{*pm} H^T_{n-p,m}\,\mathrm{d}\mathscr{C}^* = \text{symmetric tensor}\,. \tag{2.10}$$

If we apply Green's identity to (2.10) with $s = \boldsymbol{u}_{n-p,m}$, $\boldsymbol{S} = \boldsymbol{T}_{n-p,m}$, the following relation is obtained

$$\sum_{p=1}^{n-1} \Big(\int_{\partial\mathscr{C}^*} \boldsymbol{u}_{n-p,m} \times \boldsymbol{T}_{*p,m}\cdot \boldsymbol{n}_*\,\mathrm{d}\sigma_* - \int_{\mathscr{C}^*} \boldsymbol{u}_{n-p,m} \times \mathrm{Div}\,\boldsymbol{T}_{*p,m}\,\mathrm{d}\mathscr{C}^*\Big) = \boldsymbol{0}\,,$$

which, taking into account system (1.5), brings back conditions (2.1).

Therefore we have proved that the compatibility condition (2.1) coincides with condition (2.5) which represents the necessary conditions to be satisfied in order that system (1.5) admit a solution for the displacement and the electric field $\mathscr{E}$.

References

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Abstract

We prove a compatibility theorem and one of uniqueness for the problem of equilibrium of an elastic dielectric with pure traction boundary conditions, upon the hypotheses that the solution can be expanded in power series of two suitable parameters. We prove that such theorems reduce themselves to the ones proposed by Signorini in elasticity, when the parameter connected with the electric field is equal to zero.

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