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Spirallike and Robertson functions with fixed second coefficient (**)

1 - Introduction

Let S denote the class of functions

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots , \tag{1.1}$$

which are analytic and univalent in the unit disc $E \equiv \{z \setminus |z| < 1\}$.

A function $f(z)$ is said to be in $S^\lambda(\alpha, \beta)$ if the inequality

$$\left| \frac{zf'(z)/f(z) - 1}{2\beta(zf'(z)/f(z) - 1 + (1-\alpha)\ e^{-i\lambda} \cos \lambda) - (zf'(z)/f(z) - 1)} \right| < 1 \tag{1.2}$$

holds for some α, β, λ $(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant 1,\ -\pi/2 < \lambda < \pi/2)$ and for all $z \in E$. The class $S^\lambda(\alpha, \beta)$ of λ-spirallike functions of order α and type β was introduced in [**10**]. For $\beta = 1$, we get the class, $S^\lambda(\alpha)$, of λ-spirallike functions of order α introduced by Libera [**7**], while $(\alpha, \beta) = (0, 1)$ gives the class S^λ of so-called spirallike functions, defined and shown to be in S by Spacek [**14**]. In recent years, such functions have been the source of useful and important counter examples in geometric function theory (see [**3**], [**6**], ...).

An associated class consisting of those functions $f(z)$ for which $zf'(z)$ is in

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$S^{\lambda}(\alpha, \beta)$ was introduced in [**9**]. Thus, a function $f(z)$ is said to be in $M^{\lambda}(\alpha, \beta)$ if the inequality

$$(1.3) \qquad \left| \frac{zf''(z)/f'(z)}{2\beta(zf''(z)/f'(z) + (1-\alpha)e^{-i\lambda}\cos\lambda) - zf''(z)/f'(z)} \right| < 1$$

holds for some α, β, λ $(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant 1,\ -\pi/2 < \lambda < \pi/2)$ and for all $z \in E$. The functions of the class $M^{\lambda}(\alpha, \beta)$ we call « Robertson functions of order α and type β ». The class $M^{\lambda}(\alpha)$ introduced by Chichra [**2**] is obtained by fixing $\beta = 1$ in this class, while taking $(\alpha, \beta) = (0, 1)$, we get the class M^{λ} of so-called Robertson functions introduced and studied in [**11**]$_2$.

Def. 1. A function $f(z) = z + a_2 z^2 + \ldots$ analytic in E is said to be in $S_p^{\lambda}(\alpha, \beta)$ $(|\lambda| < \pi/2,\ |a_2| = 2p,\ 0 \leqslant p \leqslant \beta(1-\alpha)\cos\lambda)$ if it satisfies (1.2).

Def. 2. A function $f(z)$ is in $M_p^{\lambda}(\alpha, \beta)$ if $zf'(z)$ is in $S_p^{\lambda}(\alpha, \beta)$.

Note that functions in $M_p^{\lambda}(\alpha, \beta)$ must satisfy (1.3). Although $S_p^{\lambda}(\alpha, \beta) \subset S$, the functions in $M_p^{\lambda}(\alpha, \beta)$ need not be univalent, as shown in [**11**]$_2$.

The concept of « radius of convexity » and « radius of starlikeness » for classes of univalent and related analytic functions are quite useful and have attracted several investigators. Recently, Libera in [**7**] introduced the concept of « γ-spiral radius » for the classes of univalent functions. Following Libera [**7**], if $f \in S$ and $|\gamma| < \pi/2$, then γ-spiral radius of f is

$$(1.4) \qquad \gamma - \text{s.r.}\ \{f\} = \sup\left\{r \colon \operatorname{Re}\left(e^{i\gamma} z \frac{f'(z)}{f(z)}\right) > 0,\ z \in E\right\},$$

and if $F \subset S$, then γ-spiral radius of F is

$$(1.5) \qquad \gamma - \text{s.r.}\ F = \inf_{f \in F}\left[\gamma - \text{s.r.}\ \{f\}\right].$$

Motivated by Libera [**7**], we introduce γ-convex radius as follows.

Def. 3. If $f \in N$, the class of analytic functions in E and $|\gamma| < \pi/2$, then the *γ-convex radius of f* is

$$(1.6) \qquad \gamma - \text{c.r.}\ \{f\} = \sup\left\{r \colon \operatorname{Re}\left\{e^{i\gamma}\left(1 + z\frac{f''(z)}{f'(z)}\right)\right\} > 0,\ z \in E\right\}.$$

Def. 4. If $G \subset N$ and $|\gamma| < \pi/2$, then *γ-convex radius of G* is

$$\gamma - \text{c.r. } G = \inf_{f \in G} [\gamma - \text{c.r. } \{f\}] . \tag{1.7}$$

Results in terms of a fixed second coefficient have been obtained for various subclasses of S. Finkelstein [4] investigated the classes $S_p^0(0, 1)$ and $M_p^0(0, 1)$, the starlike and convex functions with pre-assigned second coefficient. Extensions of these results can be found in [**1**], [**5**], [**12**], ... In the present paper, we obtain lower bounds for the classes introduced above and apply them to determine γ-spiral radius for functions of the class $S_p^\lambda(\alpha, \beta)$ and γ-convex radius for functions of the class $M_p^\lambda(\alpha, \beta)$.

2 - Growth estimates

We need the following lemma.

Lemma. *Let $w(z) = b_1 z + \ldots$ be an analytic map of the unit disc into itself. Then $|b_1| \leqslant 1$ and*

$$|w(z)| \leqslant \frac{r(r + |b_1|)}{(1 + |b_1| r)} , \qquad z = r .$$

Equality holds at some point z ($\neq 0$) if and only if

$$w(z) = \frac{e^{-it} z(z + b_1 e^{it})}{1 + \bar{b}_1 e^{-it} z} , \qquad t \geqslant 0 .$$

This lemma is an iterated form of Schwarz's lemma and is due to Lowner [**8**].

To obtain growth estimates for the classes $S_p^\lambda(\alpha, \beta)$ and $M_p^\lambda(\alpha, \beta)$ it is useful to consider the following class.

Def. 5. A function $h(z) = 1 + 2a_2 z + \ldots$, analytic in the unit disc E, is in $H_p^\lambda(\alpha, \beta)$ $(|\lambda| < \pi/2,\ |a_2| = p,\ 0 \leqslant p \leqslant \beta(1-\alpha)\cos\lambda)$ if the inequality

$$\left| \frac{h(z) - 1}{2\beta(h(z) - 1 + (1-\alpha) e^{-i\lambda} \cos\lambda) - (h(z) - 1)} \right| < 1 \tag{2.1}$$

holds for some α, β, λ $(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant 1, -\pi/2 < \lambda < \pi/2)$ and for all $z \in E$.

Observe that $f(z) \in S_p^\lambda(\alpha, \beta)$ if and only if $zf'(z)/f(z) \in H_p^\lambda(\alpha, \beta)$ and $f(z) \in M_p^\lambda(\alpha, \beta)$ if and only if $1 + zf''(z)/f'(z) \in H_p^\lambda(\alpha, \beta)$.

Theorem 1. *Suppose* $h(z) \in H_p^\lambda(\alpha, \beta)$, $|\gamma| < \pi/2$ *and*

$$u = pr + \beta(1-\alpha)\cos\lambda\,, \qquad v = \beta(1-\alpha)\cos\lambda \cdot r + p\,.$$

Then, for $|z| = r < 1$ *and for all* $\alpha, \beta, \lambda, p$ $(0 \leqslant \alpha < 1,\ 0 < \beta \leqslant 1,\ -\pi/2 < \lambda < \pi/2,\ 0 \leqslant p \leqslant \beta(1-\alpha)\cos\lambda)$,

$$\operatorname{Re}\{e^{i\gamma} h(z)\} \tag{2.2}$$

$$\geqslant \frac{u^2\cos\gamma - 2uv\beta(1-\alpha)\cos\lambda\cdot r + v^2(2\beta-1)(2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda)-(2\beta-1)\cos\gamma)r^2}{u^2-(2\beta-1)^2v^2r^2}\,.$$

The result is sharp.

Proof. Since $h(z) \in H_p^\lambda(\alpha, \beta)$, an application of Schwarz's lemma gives

$$h(z) = \frac{1 + \big(2\beta - 1 - 2\beta(1-\alpha)\,e^{-i\lambda}\cos\lambda\big)\,w(z)}{1 + (2\beta-1)w(z)}\,, \tag{2.3}$$

where $w(z) = b_1 z + \ldots$ $(b_1 = p/(\beta(1-\alpha)\cos\lambda))$ satisfies the hypothesis of the lemma. Thus

$$|w(z)| \leqslant \frac{vr}{u}\,, \tag{2.4}$$

where u and v are as given in the statement of the theorem and $|z| = r$. Let $B(z) = e^{i\gamma}h(z)$ and $|\gamma| < \pi/2$. Then (2.3) may be written as

$$w(z) = \frac{e^{i\gamma} - B(z)}{(2\beta-1)B(z) - e^{i\gamma}(2\beta - 1 - 2\beta(1-\alpha)\,e^{-i\lambda}\cos\lambda)}\,. \tag{2.5}$$

From (2.4) and (2.5), we have

$$\left|\frac{e^{i\gamma} - B(z)}{(2\beta-1)B(z) - e^{i\gamma}(2\beta - 1 - 2\beta(1-\alpha)\,e^{-i\lambda}\cos\lambda)}\right| \leqslant \frac{vr}{u}\,. \tag{2.6}$$

Setting $B(z) = \xi + i\eta$ and simplifying, (2.6) gives

$$\left|\xi + i\eta - e^{i\gamma}\Big(1 + \frac{(2\beta(1-\alpha)(2\beta-1)\,e^{-i\lambda}\cos\lambda)\,v^2r^2}{u^2-(2\beta-1)^2v^2r^2}\Big)\right| \leqslant \frac{2uv\beta(1-\alpha)\cos\lambda\cdot r}{u^2-(2\beta-1)^2v^2r^2}\,. \tag{2.7}$$

From (2.7), it follows that

$$\text{(2.8)} \qquad \operatorname{Re}(B(z)) \geqslant \operatorname{Re}\{e^{i\gamma}(1+\frac{(2\beta(1-\alpha)(2\beta-1)\,e^{-i\lambda}\cos\lambda)\,v^2r^2}{u^2-(2\beta-1)^2v^2r^2})\} - \frac{2uv\beta(1-\alpha)\cos\lambda\cdot r}{u^2-(2\beta-1)^2v^2r^2}\,.$$

Now the result follows immediately from (2.8).

The bound in (2.2) is sharp for the function

$$h(z) = \frac{u+((2\beta-1)\,e^{i\theta}-2\beta(1-\alpha)\cos\lambda\cdot e^{i(\theta-\lambda)})\,vz}{u+(2\beta-1)\,e^{i\theta}vz} \qquad (\beta \neq \tfrac{1}{2}),$$

where θ is given by the relation

$$\text{(2.9)} \qquad \theta = 2\arctan[(\frac{u-(2\beta-1)\,vr}{u+(2\beta-1)\,vr})\cot\frac{\lambda}{2}]\,,$$

while for $\beta = \frac{1}{2}$, it is sharp for the function

$$h(z) = z\exp[-(1-\alpha)\cos\lambda\cdot e^{i(\theta-\lambda)}\frac{v}{u}\,z]\,,$$

where $\theta = \lambda$.

3 - The γ-spiral and γ-convex radius

In the sequel set $S_p^\lambda(\alpha, 1) = S_p^\lambda(\alpha)$, $S_1^\lambda(\alpha, \beta) = S^\lambda(\alpha, \beta)$, $S_p(0, 1) = S_p^\lambda$ and $S_p^0(0, 1) = S_p^*$.

Theorem 2. γ-s.r. $S_p^\lambda(\alpha, \beta)$ *is the smallest positive root* r_0 *of the equation*

$$\text{(3.1)} \qquad u^2\cos\gamma - 2uv\beta(1-\alpha)\cos\lambda\cdot r + v^2(2\beta-1)(2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda)-(2\beta-1)\cos\gamma)\,r^2 = 0\,,$$

where $u = pr + \beta(1-\alpha)\cos\lambda$ *and* $v = \beta(1-\alpha)\cos\lambda\cdot r + p$. *The result is sharp for all admissible values of* α, β, γ, λ *and* p.

Proof. Setting $h(z) = zf'(z)/f(z)$, in Theorem 1, we get

$$\operatorname{Re}\left\{e^{i\gamma} z \frac{f'(z)}{f(z)}\right\} \tag{3.2}$$

$$\geqslant \frac{u^2 \cos\gamma - 2uv\beta(1-\alpha)\cos\lambda\cdot r + v^2(2\beta-1)\,(2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda)}{u^2-(2\beta-1)^2v^2r^2}$$

$$\frac{-(2\beta-1)\cos\gamma)\,r^2}{u^2-(2\beta-1)^2v^2r^2}\,.$$

Thus, from (1.5) and the inequality (3.2), f is γ-spiral in $|z|<r_0$, where r_0 is the smallest positive root of the equation (3.1). Hence the theorem.

The result is sharp for the function $f(z)$ given by

$$z\frac{f'(z)}{f(z)} = \frac{u + \left((2\beta-1)\,e^{i\theta} - 2\beta(1-\alpha)\cos\lambda\cdot e^{i(\theta-\lambda)}\right) vz}{u + (2\beta-1)\,e^{i\theta} vz} \qquad (\beta \neq \tfrac{1}{2})\,, \tag{3.3}$$

where θ satisfy the relation (2.9); while for $\beta = \frac{1}{2}$, it is sharp for the function $f(z)$ given by

$$z\frac{f'(z)}{f(z)} = z\exp\left[-(1-\alpha)\cos\lambda\cdot e^{i(\theta-\lambda)}\frac{v}{u}z\right], \tag{3.4}$$

where $\theta = \lambda$.

Corollary 1. *γ-s.r. $S_p^\lambda(\alpha)$ is the smallest positive root r_0 of the equation*

$$u^2\cos\gamma - 2uv(1-\alpha)\cos\lambda\cdot r + v^2(2(1-\alpha)\cos\lambda\cos(\gamma-\lambda) - \cos\gamma)r^2 = 0\,,$$

where $u = pr + (1-\alpha)\cos\lambda$ and $v = (1-\alpha)\cos\lambda\cdot r + p$. The result is sharp.

The above result is obtained by fixing $\beta = 1$ in Theorem 2. Further, taking $p = \beta(1-\alpha)\cos\lambda$ in Theorem 2, we get the following result found in [**10**].

Corollary 2. *γ-s.r. $S^\lambda(\alpha, \beta)$ is the smallest positive root r_0 of the equation*

$$\cos\gamma - 2\beta(1-\alpha)\cos\lambda\cdot r$$
$$+ (2\beta-1)(2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda) - (2\beta-1)\cos\gamma)r^2 = 0\,.$$

The result is sharp.

The following result relate λ-spirallike functions to starlike functions; can be determined from Theorem 2 by fixing $\gamma = 0$ and $(\alpha, \beta) = (0, 1)$. This result was also obtained by Silverman and Telage [**13**] by using a different technique.

Corollary 3. *Let $f(z)$ is in S_p^λ. If we set $c = \cos\lambda + |\sin\lambda|$, then $f(z)$ is starlike in $|z| < r_0(p)$, where*

$$r_0(p) = \frac{p(1-c) + \sqrt{p^2(1-c)^2 + 4c\cos^2\lambda}}{2c\cos\lambda}.$$

Further $r_0(p)$ is decreasing $(0 \leq p \leq \cos\lambda)$ with

$$r_0(0) = \frac{1}{\sqrt{c}} \qquad \textit{and} \qquad r_0(\cos\lambda) = \frac{1}{c}.$$

Remark. Since c is maximized at $|\lambda| = \pi/4$, for all p, we have

$$(3.5) \qquad \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)^{-1} = \frac{1}{\sqrt{2}} \leq r_0(p) \leq \frac{1}{\sqrt[4]{2}} = \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)^{-1}.$$

The lower bound in (3.5) is the radius of *starlikeness* of λ-spirallike functions, found by Robertson [**11**]$_1$. The upper bound in (3.5) shows that odd λ-spirallike functions are starlike for $|z| < 1/\sqrt[4]{2}$.

In the result that follows we relate starlike to spirallike functions.

Corollary 4. *Let $f(z)$ is in S_p^*. If we set $D = \sec\gamma + |\tan\gamma|$, then $f(z)$ is γ-spirallike function in $|z| < \bar{r}_0(p)$, where*

$$\bar{r}_0(p) = \frac{p(1-D) + \sqrt{p^2(1-D)^2 + 4D}}{2D}.$$

Furthermore $\bar{r}_0(p)$ is decreasing $(0 \leq p \leq 1)$ with

$$\bar{r}_0(0) = \frac{1}{\sqrt{D}} \qquad \textit{and} \qquad \bar{r}_0(1) = \frac{1}{D}.$$

The above result is obtained by fixing $\lambda = 0$, $(\alpha, \beta) = (0, 1)$ in Theorem 2, which was also determined in [**14**] by using a different technique.

Setting $h(z) = 1 + zf''(z)/f'(z)$ in Theorem 1 and using Def. 4, we get the following result for the class $M_p^\lambda(\alpha, \beta)$.

Theorem 3. *γ-c.r. $M_p^\lambda(\alpha, \beta)$ is the smallest positive root r_0 of the equation*

$$u^2 \cos\gamma - 2uv\beta(1-\alpha)\cos\lambda \cdot r + v^2(2\beta-1)(2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda) - (2\beta-1)\cos\gamma)r^2 = 0\,,$$

where $u = pr + \beta(1-\alpha)\cos\lambda$ and $v = \beta(1-\alpha)\cos\lambda \cdot r + p$. The result is sharp for the function $f(z)$ given by

$$1 + z\frac{f''(z)}{f'(z)} = \frac{u + ((2\beta-1)e^{i\theta} - 2\beta(1-\alpha)\cos\lambda \cdot e^{i(\theta-\lambda)})vz}{u + (2\beta-1)e^{i\theta}vz} \quad (\beta \neq \tfrac{1}{2})\,,$$

where θ satisfy the relation (2.9); *while for $\beta = \frac{1}{2}$, it is sharp for the function $f(z)$ given by*

$$1 + z\frac{f''(z)}{f'(z)} = z\exp[-(1-\alpha)\cos\lambda \cdot e^{i(\theta-\lambda)}\frac{v}{u}z]\,,$$

where $\theta = \lambda$.

Remark. For different values of the parameters α, β, λ, γ $(0 \leqslant \alpha < 1,$ $0 < \beta \leqslant 1,$ $|\lambda| < \pi/2,$ $|\gamma| < \pi/2)$, Theorem 3 gives γ-c.r. for the classes introduced by Chichra [**2**], Robertson [**11**]$_2$ etc. with fixed second coefficient.

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Summary

Soit $H_p^\lambda(\alpha,\beta)(|\lambda|<\pi/2,\ |a_2|=p,\ 0\leqslant p\leqslant\beta(1-\alpha)\cos\lambda)$ *la classe des fonctions* $h(z)=1+2a_2z+\ldots$ *qui sont analytiques et satisfont à l'inégualité*

$$|(h(z)-1)/\{2\beta(h(z)-\alpha)-(h(z)-1)\}|<1$$

pour les valeurs α,β,λ $(0\leqslant\alpha<1,\ 0<\beta\leqslant 1,\ -\pi/2<\lambda<\pi/2)$ *et pour tout* $z\in E\equiv\{z: |z|<1\}$. *Par ailleurs, une fonction* $f(z)$ *appartient à* $S_p^\lambda(\alpha,\beta)$ *si et seulement si* $zf'(z)/f(z)\in H_p^\lambda(\alpha,\beta)$ *et* $f(z)$ *appartient à* $M_p^\lambda(\alpha,\beta)$ *si et seulement si* $1+zf''(z)/f'(z)\in H_p^\lambda(\alpha,\beta)$. *Dans le papier, nous obtenons une limite inferieure exacte pour* $\mathrm{Re}\{e^{i\nu}h(z)\}$, $|\nu|<\pi/2$ *et nous utilisons ce résultats pour trouver le rayon des ν-spirales pour les fonctions de la classe* $S_p^\lambda(\alpha,\beta)$ *et le rayon des ν-convexes pour les fonctions de la classe* $M_p^\lambda(\alpha,\beta)$.

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