Riv. Mat. Univ. Parma (4) 9 (1983), 197-205

MIGUEL A. CANELA (*)

Sequential barrelledness and other sequential properties related with it (**)

1 - Spaces which are complete in the sense of Mackey

Let (E, τ) be a locally convex space (we suppose that all the spaces considered are separated), and E^* its topological dual. We denote by $\sigma(E, E^*)$, $\mu(E, E^*)$ and $\beta(E, E^*)$ the weak, Mackey and strong topologies, respectively on E. $\gamma(E, E^*)$ stands for the topology on E of uniform convergence on the strongly bounded subsets of E^*.

We say that E is *complete in the sense of Mackey* (locally complete for some authors) when one of the following equivalent conditions holds: (a) The Von Neumann bornology of (E, τ) is complete. (b) Every Cauchy-Mackey sequence in E is τ-convergent. (c) The closed absolutely convex hull of every weakly convergent sequence is weakly compact (see [**5**] for the first two and [**2**] for the third).

We say that E is *sequentially barrelled* when every $\sigma(E^*, E)$-convergent sequence is equicontinuous. The property (c) above allows us to state the following characterization, which extends Proposition 4.2 of [**14**]$_1$ (see also Chapter VI, Section 4 of [**7**]).

1.1 – Proposition. *Let* (E, τ) *be a locally convex space. The following are equivalent*:

(i) (E, τ) *is complete in the sense of Mackey.* (ii) $\mu(E^*, E)$ *is sequentially barrelled.*

(*) Indirizzo: Dept. de Teoria de Funcions, Facultat de Matemàtiques, Universitat, Gran Via 585, Barcelona 7, Spain.

(**) Ricevuto: 3-IX-1981.

Also

1.2 – Corollary. *Let* (E, τ) *be a Mackey space. The following are equivalent:*

(i) (E, τ) *is sequentially barrelled.* (ii) $(E^*, \sigma(E^*, E))$ *is complete in the sense of Mackey.*

N. J. Kalton has shown in [**9**] that a Mackey space (E, τ) in which every sequentially continuous linear functional is continuous (*Mazur space*) is sequentially barrelled if and only if every bounded set is strongly bounded (we call such a space a *Banach-Mackey space*). The proceding corollary provides us with an analogous setting, in which the hypothesis of E being Mazur is omitted and the Banach-Mackey property is replaced by the completeness in the sense of Mackey, which is stronger. Both properties are not equivalent, as follows from the existence of non-complete barrelled normed spaces. The Corollary **1.2** together with the following, gives us a trivial proof of the result of Kalton mentioned above.

1.3 – Corollary. *Let* (E, τ) *be a Mazur space satisfying the Banach-Mackey property. Then* $(E^*, \sigma(E^*, E))$ *is complete in the sense of Mackey.*

Proof. Consider on E the topology τ^n of uniform convergence on τ-convergent sequences of E, and E^* is complete for this topology (see [**1**]$_1$). Then E^* is strongly complete. The Banach-Mackey property follows from the coincidence of $\beta(E^*, E)$- and $\sigma(E^*, E)$-bounded subsets, and thus $(E^*, \sigma(E^*, E))$ has complete Von Neumann bornology.

For a locally convex space (E, τ), we denote by τ^s (respectively by τ^+) the finest topology (resp. locally convex topology) on E which has the same convergent sequences as τ. If α is a topology on E^*, compatible with the dual pair $\langle E^*, E\rangle$, we denote by α^f (resp. α^{lf}) the finest topology (resp. locally convex topology) on E which coincides with $\sigma(E, E^*)$ on every α-equicontinuous subset of E. Now, we can state

1.4 – Proposition *Let* (E, τ) *be a locally convex space, complete in the sense of Mackey. Then*

$$\text{(i)}\ (\tau^n)^f = \tau^s. \qquad \text{(ii)}\ (\tau^n)^{lf} = \tau^+.$$

Proof. Remark first that the completeness hypothesis is equivalent to the fact that τ^n is compatible with respect to the duality $\langle E^*, E\rangle$ (see [**2**]). Actually, $(\tau^n)^f$ is the finest topology on E equal to $\sigma(E, E^*)$ on the closed aqsolutely convex hulls of τ-null sequences of E, and τ^s is the finest one which coincides with τ (and thus with $\sigma(E, E^*)$) on every τ-metrizable compact subset of E. According to Theorem 1.4 of [**9**], these hulls are metrizable, and $(\tau^n)^f$ is finer than τ^s. Conversely, $(\tau^n)^f$ has the same convergent sequences as τ, and τ^s is finer than $(\tau^n)^f$.

To obtain (ii), it suffices to remark that $(\tau^n)^{lf}$ is the finest locally convex topology which is coarser than $(\tau^n)^f$, and analogously for τ^+ and τ^s.

A locally convex space (E, τ) is Mazur if and only (E^*, τ^n) is complete, as we have shown in $[\mathbf{1}]_1$. We see now that for E complete in the sense of Mackey, this result can be made more precise.

1.5 – Corollary. *If (E, τ) is complete in the sense of Mackey, E^* is τ^n-dense in the space E^+ of sequentially continuous linear functionals on (E, τ).*

Proof. It suffices to calculate the dual in the identity **1.4** (ii).

The following corollary extends Proposition 3.5 of $[\mathbf{14}]_1$.

1.6 – Corollary. *If (E, τ) is complete in the sense of Mackey, the following holds:*

(i) *E^* is τ^n-complete if and only if every sequentially closed hyperplane of E is closed.*

(ii) *(E^*, τ^n) is a Pták space if and only if every sequentially closed subspace of E is closed.*

(iii) *(E^*, τ) is an infra-Pták space if and only if every dense sequentially closed subspace of E is closed.*

(iv) *(E^*, τ^n) is hypercomplete if and only if every sequentially closed convex subset of E is closed.*

2 - Spaces with Schauder basis

We are going to see now some conditions, introduced by O. T. Jones in [**8**], which grant some sequential properties for spaces with Schauder basis. Recall that a locally convex space (E, τ) is said to be a *C-sequential space* when $\tau = \tau^+$, or, equivalently, when for each locally convex space F and each sequentially continuous linear mapping f of E into F, f is continuous (see [**12**] for details).

Suppose that (E, τ) is a locally convex space with a Schauder basis $\{e_n\}_{n=1}^{\infty}$, and denote by $\{e_n^*\}_{n=1}^{\infty}$ the sequence of coefficient functionals orthogonal to this basis, and by S_n the n^{th} partial sum operator defined by

$$S_n(x) = \sum_{k=1}^{n} \langle e_k^*, x \rangle e_k , \qquad x \in E .$$

We consider now the following conditions for E.

(A) If F is a locally convex space and $f\colon E \to F$ is linear and satisfies

$$\lim_n f(S_n(x)) = f(x) \quad \text{for all } x \in E , \tag{1}$$

then f is continuous.

(A)′ If u is a linear functional on E satisfying

$$\lim_n u(S_n(x)) = u(x) \quad \text{for all } x \in E , \tag{2}$$

then u is continuous.

Actually, we can state

2.1 – Proposition. *If (E, τ) has a Schauder basis satisfying condition* (A), *it is C-sequential. If it has a Schauder basis satisfying* (A)′, *it is Mazur.*

Proof. If F is a locally convex space and $f\colon E \to F$ is sequentially continuous and linear, f satisfies relation (1). Thus (A) implies C-sequentiality. The same for the remaining.

Using the completeness results proved in [**1**]$_1$, we obtain directly the Theorems 5 and 6 of [**8**].

2.2 – Corollary. *Let (E, τ) be a space with a Schauder basis. Then:*

(i) *If the basis satisfies condition* (A) *and F is complete* (*resp. quasi-complete, sequentially complete, etc.*) *the space $L(E, F)$ of continuous linear operators of E into F is complete* (*resp. quasi-complete, sequentially complete, etc.*) *for the topology of uniform convergence on bounded subsets of E.*

(ii) *If the basis satisfies condition* (A)′, *the strong dual of E is complete.*

We are going to see now how certain degrees of barrelledness imply conditions (A) and (A)′.

2.3 – Proposition. *Let (E, τ) be a space with a Schauder basis $\{e_n\}_{n=1}^{\infty}$. Then:*

(i) *If (E, τ) is barrelled, the basis satisfies* (A).

(ii) *If E^* is sequentially complete for $\sigma(E^*, E)$, the basis satisfies* (A)′.

Proof. The first assertion has been proved in [**8**]. For the second, take a linear functional u and E satisfying (2). Consider the operator

$$S_n : E^{\#} \to E^*$$
$$v \to \sum_{k=1}^{n} \langle v, e_k \rangle e_k^* ,$$

defined on the algebraic dual $E^{\#}$ of E. Actually

$$\lim_n \langle S_n^*(u), x \rangle = \lim_n u(S_n(x)) = u(x) ,$$

for all $x \in E$, and hence $\{S_n^*(u)\}_{n=1}^{\infty}$ is a Cauchy sequence in $(E^*, \sigma(E^*, E))$. Its limit coincides with u.

2.4 – Corollary. *Let (E, τ) be a locally convex space with a Schauder basis. Then:*

(i) *If E is barrelled, it is C-sequential.*

(ii) *If E^* is $\sigma(E^*, E)$-sequentially complete, E is Mazur.*

Part (i) of the preceding corollary has been proved directly by Hampson and Wilansky [**3**] and part (ii) by Webb [**14**]$_2$.

For $\sigma(E^*, E)$-sequential completeness, we can state a result analogous to **1.1**. We say that (E, τ) is a *C-barrelled space* when every $\sigma(E^*, E)$-Cauchy sequence in E^* is τ-equicontinuous. Actually, $(E^*, \sigma(E^*, E))$ is sequentially complete if and only if $(E, \sigma(E, E^*))$ is C-barrelled, as has been shown by Kalton in [**9**].

2.5 – Example. According to the preceding, every C-barrelled space with Schauder basis is Mazur. Nevertheless, C-barrelledness cannot be replaced in this setting by sequential barrelledness, as the following example shows. Take $E = l^1$, and $\tau = \mu(l^1, c_0)$. c_0 is not $\sigma(c_0, l^1)$-sequentially complete, and thus (E, τ) is not Mazur, because, according to Theorem 3.3 of [**9**], a Mackey Mazur space with the Banach-Mackey property must be C-barrelled.

2.6 – Example. A barrelled space can fail to be Mazur. To see this, take a completely regular topological space X, X being a Q-space but not a Q_1-space in the sense of Shirota [**11**]. Then, the space $C_c(X)$ of continuous real functions on X is barrelled for the compact-open topology, but not Mazur, according to Theorem 1 of [**10**]. An example of a separable barrelled space which is not Mazur can be obtained from the classical example of a reflexive space which is separable and non-complete.

3 - Spaces with weak* angelic dual

We recall first the definition of *angelic space*. A Hausdorff topological space X is said to be *angelic* when every relatively countably compact subset A of X satisfies: (a) A is relatively compact in X; (b) for every $x \in \overline{A}$, there is a sequence in A with limit x. Every (relatively) compact subset of an angelic space is (relatively) sequentially compact, and conversely. [**3**] is a good reference on this field. We are interested in locally convex spaces whose weak * dual is angelic. A sufficient condition is that the space is the union of a sequence of weakly relatively countably compact subsets ([**3**], 3.10).

3.1 – Proposition. *Let (E, τ) be a locally convex space. Then:*

(i) *If (E, τ) is sequentially barrelled, $\sigma(E^*, E)^s$ is finer than τ^f (resp. $\sigma(E^*, E)^+$ finer than τ^{lf}).*

(ii) *If every equicontinuous subset $H \subset E^*$ is weak* angelic, $\sigma(E^*, E)^s$ is coarser than τ^f (resp. $\sigma(E^*, E)^+$ coarser than τ^{lf}).*

Proof. For the first assertion, it suffices to remark that τ^f and τ^{lf} have the same convergent sequences as $\sigma(E^*, E)$, since every weak* convergent sequence is equicontinuous. For the second, note that $\sigma(E^*, E)$ restricted to H is a sequential topology, and thus $\sigma(E^*, E)$ and $\sigma(E^*, E)^s$ coincide on H. Moreover, $\sigma(E^*, E)^+$ is a locally convex topology on E^* which is finer than $\sigma(E^*, E)$ and coarser than τ^f, and, hence, it is coarser than τ^{lf}.

The condition of (ii) is fulfilled by a large class of spaces: the sub*WCG* of Hunter and Lloyd [**6**], the weakly K-analytic Banach spaces of Talagrand [**13**], and, more generally, every space for which the weak* dual is angelic. The coincidence $\tau^f = \tau^{lf} = \sigma(E^*, E)^+$ has been stated for a Banach space with weak* angelic dual in a previous work [**1**]$_2$.

3.2 – Corollary. *Let (E, τ) be a sequentially barrelled space for which* *every equicontinuous subset of* E^* *is weak* angelic. Then the following holds:*

(i) $\sigma(E^*, E)^s = \tau^f$. (ii) $\sigma(E^*, E)^+ = \tau^{lf}$.

(iii) $(E^*, \sigma(E^*, E))$ *is Mazur if and only if* E *is complete.*

(iv) (E, τ) *is Pták (resp, infra-Pták, hypercomplete) if and only if every subspace (resp. dense subspace, convex subset) which is weak* sequentially closed is closed.*

The assertion **3.2** (iv) implies in particular Proposition 3.14 of [**6**], because of the remarks made after **3.1**. Note that the hypothesis on the equicontinuous subsets is not necessary, in general, for the completeness of E, since there are Banach spaces E for which $(E^*, \sigma(E^*, E))$ is Mazur, and whose closed unit dual ball is not sequentially compact (see [**1**]$_2$), and since we have

3.3 – Proposition. *Let* (E, τ) *be a sequentially barrelled space such that* E^* *is weak* Mazur. Then* (E, τ) *is complete.*

Proof. E must be complete with respect to the topology of uniform convergence on weak* null sequences, namely $\sigma(E^*, E)^n$. But $\sigma(E^*, E)^n$ is coarser than τ because of the sequential barrelledness of (E, τ).

Note also that, in general, the hypothesis mentioned above is not superfluous for the converse implication, because Banach spaces whose weak* dual is not Mazur are known, e.g. l^∞, or, more generally, every Grothendieck space.

We give now two results which extend previous results of Webb.

3.4 – Proposition. *Let* (E, τ) *be a sequentially barrelled space, and* τ_1 *a locally convex topology on* E, *polar with respect to the dual pair* $\langle E, E^*\rangle$, *and finer than* τ. *If* (E, τ_1) *has weak* angelic dual, then the completions of* (E, τ) *and* (E, τ_1) *coincide.*

Proof. By a well-known theorem of Bourbaki and Robertson, the completion $\hat{E}_1$ of (E, τ_1) can be considered as a subspace of the completion $\hat{E}$ of (E, τ), and we must show that every z in $\hat{E}$ belongs to $\hat{E}_1$. Both spaces can be considered as subspaces of the algebraic dual $(E^*)^\#$. Because of the sequential barrelledness of (E, τ), z is weak* continuous on E^*. On the other hand, τ_1 is the topology of uniform convergence on the sets of a certain saturated family $\mathscr{A}$ of $\sigma(E^*, E)$-bounded subsets, and to show that z is $\sigma(E^*, E)$-continuous on every $A \in \mathscr{A}$ is to show that z belongs to E (Grothendieck's completeness theorem). Putting $F = (E, \tau_1)^*$, every $A \in \mathscr{A}$ is $\sigma(F, E)$-relatively compact in F. The restriction of $\sigma(F, E)$ to A is an angelic topology, and, since z is $\sigma(E^*, E)$-sequentially continuous, z is $\sigma(F, E)$-continuous on A.

The preceding Proposition has been stated by Webb for the case in which $\tau_1 = \gamma(E, E^*)$ is separable (see [7]). The following corollary appears also in [7] for $\beta(E^*, E)$ complete and $\gamma(E, E^*)$ separable.

3.5 – Corollary. *Let (E, τ) be a locally convex space satisfying*:

(i) *$\gamma(E, E^*)$ is complete and has weak* angelic dual.*

(ii) *$\beta(E^*, E)$ is sequentially complete.*

Then, $\mu(E, E^*)$ is complete.

Proof. If $\beta(E, E^*)$ is complete, the dual pair $\langle E, E^*\rangle$ has the Banach-Mackey property, and the $\sigma(E^*, E)$- and the $\beta(E^*, E)$-bounded subsets of E^* coincide ([**14**]$_1$, 4.1). Thus, any degree of completeness on $\beta(E^*, E)$ implies that $(E^*, \sigma(E^*, E))$ is complete in the sense of Mackey, and, through **1.2**, $\mu(E, E^*)$ is sequentially barrelled. We can apply now **3.4** to the situation $\tau = \mu(E, E^*)$ and $\tau_1 = \gamma(E, E^*)$.

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Abstract

Sequential barrelledness for locally convex spaces is studied, specially for the Mackey topology. In this case, the sequential barrelledness is shown to be equivalent to some degree of completeness for the dual space. These conditions are considered in connection with other sequential properties, like the condition of Mazur or angelicity. Special attention is paid to the case of a space with a Schauder basis.

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