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A weak formulation of the 2-point boundary value problem for hyperbolic equations (**)

0 - Introduction

While studing [**11**]$_1$ the question of existence of solutions for abstract hyperbolic boundary value problems of the form

(0.1) $$\mathrm{d}^2u/\mathrm{d}t^2 + A(t)u = f(t, u) \qquad 0 < t < T$$

(0.2) $$\begin{aligned} B_1u &= a_{11}u(0) + a_{12}u'(0) + b_{11}u(T) + b_{12}u'(T) = 0\,, \\ B_2u &= a_{21}u(0) + a_{22}u'(0) + b_{21}u(T) + b_{22}u'(T) = 0\,, \end{aligned}$$

the question of a weak formulation arose. Here we assume that $u\colon (0, T) \to V$ is a Hilbert space valued function, $A(t)\colon V \to V^*$ (the dual) is a linear map, and f is not necessarily a linear map. The Cauchy problem ($B_1u = u(0)$, $B_2u = u'(0)$) had already been given a weak formulation by Lions [**6**] and Lions-Magenes [7]. In this paper we give a weak formulation for the general two-point boundary value problem above, which agrees with the formulation of Lions-Magenes in the case of the Cauchy problem. Nonhomogeneous conditions are also considered for some special cases of B_1 and B_2.

Our work can be applied for instance to problems of the form

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} \partial/\partial x_i\big(a_{ij}(t, x)\frac{\partial u}{\partial x_j}\big) = f(t, x, u, u_t, u_x) \qquad (t, x) \in (0, T)\times\Omega\,,$$

$$u(t, x) = 0 \qquad (t, x) \in (0, T)\times\partial\Omega\,,$$

$$B_1u = a_{11}u(0, x) + a_{12}u_t(0, x) + b_{11}u(T, x) + b_{12}u_t(T, x) = 0 \qquad x\,\delta\,\Omega\,,$$

$$B_2u = a_{21}u(0, x) + a_{22}u_t(0, x) + b_{21}u(T, x) + b_{22}u_t(T, x) = 0 \qquad x \in \Omega\,,$$

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where $0 < T < +\infty$, Ω is an open bounded subset of $\boldsymbol{R}^n$, and the 4-tuples of real numbers, $(a_{11}, a_{12}, b_{11}, b_{12})$ and $(a_{21}, a_{22}, b_{21}, b_{22})$, are assumed to be linearly independent in $\boldsymbol{R}^4$. Observe that we are prescribing boundary conditions on the entire surface of the cylinder $(0, T) \times \Omega$. Above we have indicated a Dirichlet boundary condition on the lateral surface $(0, T) \times \partial\Omega$ although this is not essential. A Neumann condition or boundary conditions of mixed type could also be prescribed.

The results of this paper concern only the formulation of the weak problem. In subsequent papers $[11]_{2,3}$ we shall prove existence theorems, under suitable hypothesis, for problems of the above type.

1 - Preliminaries

Let H be a real separable Hilbert space with norm $\|\cdot\|_H$ and let $u: \boldsymbol{R} \to H$ be a function of the real variable t taking values in H. We say that $u \in C(0, T; H)$ if u is continuous from $(0, T)$ into H (with the norm topology). Similarly we say that $u \in C([0, T]; H)$ if u is continuous from $[0, T]$ into H. We employ the following notations.

$$C^m(0, T; H) = \{u \in C(0, T; H): \frac{\mathrm{d}^k u}{\mathrm{d}t^k} \in C(0, T; H) \text{ for } k = 1, \ldots, m\},$$

$$C_0^m(0, T; H) = \{u \in C^m(0, T; H): \text{support}\,(u) \subset (0, T)\},$$

$$C_{\text{per}}^m(0, T; H) = \{u \in C^m(0, T; H): \frac{\mathrm{d}^k u}{\mathrm{d}t^k} \text{ is } T\text{-periodic for } k = 0, 1, \ldots, m\}.$$

All the derivatives above are assumed to be strong derivatives. In the case $H = \boldsymbol{R}$ we will simplify the above notations to $C^m(0, T)$, $C_0^m(0, T)$, and $C_{\text{per}}^m(0, T)$ respectively. If u is infinitely differentiable we will set $m = \infty$. If u is m-times continuously differentiable on some open interval containing $[0, T]$ then we say $u \in C^m([0, T]; H)$ or $u \in C^m[0, T]$ in the case $H = \boldsymbol{R}$.

Let V be another real separable Hilbert space with $V \subset H$. We assume that the inclusion map is a continuous injection from V into H and that V is dense in H. Let $c_0 > 0$ be the constant such that $\|u\|_H \leqslant c_0 \|u\|_V$ for $u \in V$. For convenience we will denote the norms and inner products of V, H by $\|\cdot\|$, $|\cdot|$ and $((\cdot,\cdot))$, $(\cdot,\cdot)$ respectively.

Let V^* denote the dual of V and identify H with its dual H^*. Thus $V \subset H \subset V^*$. We consider a family, $A(t)$, $0 \leqslant t \leqslant T$, of continuous linear mappings from V to its dual V^*. Thus for each t we have $A(t) \in \mathscr{L}(V, V^*)$, where $\mathscr{L}(V, V^*)$

denotes the space of bounded linear mappings from V into V^*. If $u, v \in V$ $A(t)u \in V^*$, $0 \leqslant t \leqslant T$, and $(A(t)u, v) \in \mathbf{R}$, $0 \leqslant t \leqslant T$, where the bracket denotes the dual action of V^* on V. If $A(t)u \in H$ this bracket coincides with the inner product on H. We associate a family of bilinear forms on $V \times V$ to $A(t)$ by setting $a(t; u, v) = (A(t)u, v)$. We make the following assumptions:

(1.1) $a(t; u, v) = a(t; v, u)$ $\forall u, v \in V$,

(1.2) $\exists c_1 > 0$ such that $|a(t; u, v)| \leqslant c_1 \|u\| \|v\|$ $\forall u, v \in V$,

(1.3) $a(t; u(t), v)$ is measurable whenever $u(t)$ is measurable and is continuous on $(0, T)$ whenever $u(t)$ is continuous on $(0, T)$.

Let $L^2(0, T; V)$ and $L^2(0, T; H)$ denote the spaces of (equivalence classes of) measurable functions from $(0, T)$ into V and H respectively which are norm-square integrable. We point out that $L^2(0, T; V)$ and $L^2(0, T; H)$ are Hilbert spaces (cfr. Dunford-Schwartz [**4**]). We define another Hilbert space (cfr. Lions-Magenes [**7**]) by

$$(1.4) \qquad W(0, T) = \{u \in L^2(0, T; V) : u' = \frac{du}{dt} \in L^2(0, T; H)\} .$$

The derivative in (1.4) is assumed to be a weak derivative in the sense that (cfr. Schwartz [**10**]).

$$(1.5) \qquad \int_0^T u'(t)\varphi(t)\, dt = -\int_0^T u(t)\varphi'(t)\, dt \qquad \forall \varphi \in C_0^\infty(0, T) .$$

Since $V \subset H$ we observe that the integrals above have values in H. Equipped with the norm $\|u\|_W = (\|u\|^2_{L^2(0,T;V)} + \|u'\|^2_{L^2(0,T;H)})^{1/2}$, $W(0, T)$ is a Hilbert space.

2 - Basic lemmas

We state some needed results concerning the spaces introduced in the previous section.

Lemma 2.1. *$C_0^\infty(0, T; H)$ is dense in $L^2(0, T; H)$.*

We briefly outline the proof (cfr. Smiley [**11**]$_1$) which is well-known in the real-valued case. Define a smoothing kernel $k \in C_0^\infty(-\infty, \infty)$ by

$$k(t) = \begin{cases} K \exp[(t^2-1)^{-1}] & -1<t<1 \\ 0 & \text{otherwise} \end{cases} \qquad \Big(K = \big(\int_{-\infty}^{\infty} \exp[(t^2-1)^{-1}]\,dt\big)^{-1}\Big).$$

If $u \in L^2(0, T; H)$ and n is an integer we set

$$u_n(t) = \int_{-\infty}^{\infty} \bar{u}(t - s/n)\,k(s)\,ds\,, \tag{2.1}$$

where $\bar{u}$ denotes the zero extension of u. One then shows $\|u_n\|_{L^2(0,T;H)} \leq \|u\|_{L^2(0,T;H)}$ for $n = 1, 2, 3, \ldots$. Next for $\delta > 0$ set $u_\delta(t) = u(t)$ on $(\delta, T-\delta)$ and $u_\delta(t) = 0$ otherwise. Then $(u_\delta)_n \in C_0^\infty(0, T; H)$ for n large and by the triangle inequality $(u_\delta)_n \to u$ as $\delta \to 0^+$, $n \to \infty$.

Def. A function $u\colon [0, T] \to H$ is *absolutely continuous* if, given $\varepsilon > 0$, there is a $\delta > 0$ such that for any finite set of disjoint intervals $((a_1, b_1), \ldots, (a_n, b_n))$ contained in $[0, T]$ with $\sum_{i=1}^{n} |b_i - a_i| < \delta$, we have $\sum_{i=1}^{n} |u(b_i) - u(a_i)| < \varepsilon$.

Lemma 2.2. *If $u \in W(0, T)$ then $u\colon [0, T] \to H$ is in the equivalence class of a unique absolutely continuous function. Moreover* $u'(t) = \lim_{\Delta t \to 0} [u(t + \Delta t) - u(t)]/\Delta t$ (a.e.) $t \in (0, T)$ *and for almost all* $\alpha, \beta \in (0, T)$

$$u(\alpha) - u(\beta) = \int_{\alpha}^{\beta} u'(s)\,ds\,. \tag{2.2}$$

Note that (2.2) is an equality between elements of H.

Here again the proof parallels the case for real-valued functions (cfr. Smiley [11]$_1$). We set $\tilde{u}(t) = \int_0^t u'(s)\,ds$. From the theory of functional analysis (cfr. Yosida [12], p. 134]) we know that $\tilde{u}'(t) = u'(s)$ (a.e.) $t \in (0, T)$. Clearly $\tilde{u}$ is absolutely continuous; for $0 < \alpha < \beta < T$ we define a sequence of real-valued functions $\{\varphi_n\}$ where $\varphi_n(t) = 0$ if $t \in [0, \alpha - 1/n] \cup [\beta + 1/n, T]$, and $\varphi_n(t) = 1$ if $t \in [\alpha, \beta]$. Elsewhere we define φ_n so that it is linear and continuous on $[0, T]$. From (1.5) it follows that (for n large)

$$-\int_0^T u(t)\varphi_n'(t)\,dt = \int_0^T u'(t)\varphi_n(t)\,dt\,.$$

Passing to the limit as $n \to \infty$ we obtain (2.2) except possibly for $\alpha, \beta \in Z$, a set of measure zero. We now observe that $\tilde{u}(\beta) - u(\beta)$, for $\beta \in [0, T] - Z$, is independent of β and conclude that u is equal (a.e.) to an absolutely continuous function.

Lemma 2.3. *$W(0, T) \subset C([0, T]; H)$ continuously, where*

$$\|u\|_{C([0,T];H)} = \max\{|u(t)| : 0 \leqslant t \leqslant T\}.$$

This follows at once from Lemma 2.2 and the existence of $t_0 \in [0, T]$ such that $|u(t_0)|$ is less than or equal to the mean value of $|u(t)|$.

Lemma 2.4. *Let δ be any real number, $0 < \delta < T/2$. If $u \in W(0, T)$ then u_n, defined by* (2.1), *converges to u in $W(\delta, T - \delta)$.*

This is a technical result used in Theorem 2.5.

We already know that $u_n \to u$ in $L^2(0, T; V)$ and hence in $L^2(\delta, T - \delta; V)$. If $\varphi \in C_0^\infty(\delta, T - \delta)$ and $n > \delta^{-1}$ then one can show $(u_n)' = (u')_n$, where the primes denote weak derivatives with respect to φ. From this and the proof of Lemma 2.1 the result follows. Note that for n large $u_n \in C^\infty(0, T; V)$.

Theorem 2.5. *The set $C^\infty(0, T; V) \cap W(0, T)$ is dense in $W(0, T)$.*

Proof. We set $\delta_n = T/4n$ and define a sequence of open intervals by $I_n = (\delta_n, T - \delta_n)$, $n = 1, 2, 3, \ldots$. Using these sets we define an open cover for $(0, T)$ by setting $\mathcal{O}_1 = I_1$, $\mathcal{O}_2 = I_2$, and $\mathcal{O}_n = I_n - I_{n-2}$ for $n \geqslant 3$. We know (cfr. Adams [1]) that there exists a C^∞-partition of unity subordinate to the cover $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \ldots\}$. Let $\{\varphi_1, \varphi_2, \varphi_3, \ldots\}$ denote such a partition of unity. Since $\varphi_n \in C_0^\infty(\mathcal{O}_n)$ it follows that $\varphi_n u \in W(0, T)$ for each $u \in W(0, T)$. Moreover, we have just seen that

$$(\varphi_n u)_m(t) = \int_{-\infty}^{\infty} (\varphi_n u)(t - s/m)\, k(s)\, \mathrm{d}s\,,$$

converges to $\varphi_n u$ in $W(\delta_{n+1}, T - \delta_{n+1})$. Notice that if $m > 4n(n+1)/T$ then support $[(\varphi_n u)_m] \subset I_{n+1} - I_{n-3}$ for $n \geqslant 4$.

Let $\varepsilon > 0$. Since $(\varphi_n u)_m$ converges to $\varphi_n u$ in $W(\delta_{n+1}, T - \delta_{n+1})$ we may select a sequence of numbers $m_n > 4n(n+1)/T$, $n = 1, 2, \ldots$, such that

$$\|(\varphi_n u)_{m_n} - (\varphi_n u)\|_{W(0,T)} = \|(\varphi_n u)_{m_n} - (\varphi_n u)\|_{W(\delta_{n+1},\, T - \delta_{n+1})} < \varepsilon/2^n\,.$$

We now set $\psi_n = (\varphi_n u)_{m_n}$ $(n = 1, 2, 3, \ldots)$ and define $\psi = \sum_{n=1}^{\infty} \psi_n$. Clearly

$\psi \in C^\infty(0, T; V)$ since each $\psi_n \in C^\infty(0, T; V)$ and $\operatorname{supp}(\psi_n) \subset I_{n+1} - I_{n-3}$. We let m be any positive integer and observe that

$$\|u - \psi\|_{W(\delta_m,\, T-\delta_m)} = \Big\|\sum_{n=1}^{m+2} \varphi_n u - \sum_{n=1}^{m+2} \psi_n\Big\|_{W(\delta_m,\, T-\delta_m)} \leqslant \sum_{n=1}^{m+2} \|\varphi_n u - \psi_n\|_{W(0,T)} < \varepsilon .$$

By using this fact in conjuntion with the dominated convergence theorem we conclude first that $\psi \in W(0, T)$ and second that $\|u - \psi\|_{W(0,T)} < \varepsilon$. This concludes the proof.

3 - Boundary condition subspaces of $W(0, T)$.

We consider the following homogeneous scalar version of boundary value problem (0.1), (0.2). Let $\varphi \in C^2(0, T)$, $a \in C^0(0, T)$, and consider

$$\varphi'' + a(t)\varphi = 0 \qquad 0 < t < T , \tag{3.1}$$

$$\begin{aligned} B_1\varphi &= a_{11}\varphi(0) + a_{12}\varphi'(0) + b_{11}\varphi(T) + b_{12}\varphi'(T) = 0 ,\\ B_2\varphi &= a_{21}\varphi(0) + a_{22}\varphi'(0) + b_{21}\varphi(T) + b_{22}\varphi'(T) = 0 , \end{aligned} \tag{3.2}$$

where $(a_{11}, a_{12}, b_{11}, b_{12})$ and $(a_{21}, a_{22}, b_{21}, b_{22})$ are linearly independent in $\boldsymbol{R}^4$. The coefficients a_{ij}, $b_{ij} (i, j = 1, 2)$ appearing in (3.2) are assumed to be the same as in (0.2). We make no notational distinction between the above linear boundary operators and those appearing in (0.2).

The adjoint boundary value problem (in parametric form) to (3.1), (3.2) is (cfr. Cole [**3**])

$$\varphi'' + a(t)\varphi = 0 \qquad 0 < t < T , \tag{3.3}$$

$$\begin{bmatrix} \varphi(0) \\ \varphi'(0) \end{bmatrix} = \begin{bmatrix} a_{12} & a_{22} \\ -a_{11} & -a_{21} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad \begin{bmatrix} \varphi(T) \\ \varphi'(T) \end{bmatrix} = \begin{bmatrix} -b_{12} & -b_{22} \\ b_{11} & b_{21} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \tag{3.4}$$

where α, β are arbitrary real numbers. At least theoretically, we can always write the parametric boundary conditions above in the form (cfr. Cole [**3**], p. 141)

$$\begin{aligned} B_1^*\varphi &= a_{11}^*\varphi(0) + a_{12}^*\varphi'(0) + b_{11}^*\varphi(T) + b_{12}^*\varphi'(T) = 0 ,\\ B_2^*\varphi &= a_{21}^*\varphi(0) + a_{22}^*\varphi'(0) + b_{21}^*\varphi(T) + b_{22}^*\varphi'(T) = 0 . \end{aligned} \tag{3.5}$$

As a notational convenience we shall write $B_1^*\varphi = B_2^*\varphi = 0$ to denote boundary conditions (3.4).

We define subsets Φ_B, Φ_{B^*} of $C^\infty[0, T]$ as follows. We set

$$\Phi_B = \{\varphi \in C^\infty[0, T]: B_1\varphi = B_2\varphi = 0\} , \quad \Phi_{B^*} = \{\varphi \in C^\infty[0, T]: B_1^*\varphi = B_2^*\varphi = 0\} .$$

Note that $C_0^\infty(0, T)$ is a subset of both Φ_B and Φ_{B^*}. Let $A_0 = a_{11}a_{22} - a_{12}a_{21}$ and $B_0 = b_{11}b_{22} - b_{12}b_{21}$. We show that a necessary and sufficient condition for problem (3.1)-(3.2) to be self-adjoint is that $A_0 = B_0$.

Lemma 3.1. *$\Phi_B = \Phi_{B^*}$ if and only if $A_0 = B_0$.*

Proof. Let M be the (4×4) matrix having $(a_{11}, a_{12}, b_{11}, b_{12})$ for its first row, $(a_{21}, a_{22}, b_{21}, b_{22})$ for its second row, and $(0, 0, 0, 0)$ for its third and fourth rows. Note that the null space of M is 2-dimensional. Let $X \subset \boldsymbol{R}^4$ denote the 2-dimensional space spanned by the vectors $v_1 = (a_{12}, -a_{11}, -b_{12}, b_{11})$ and $v_2 = (a_{22}, -a_{21}, -b_{22}, b_{21})$. If $P: C^1[0, T] \to \boldsymbol{R}^4$ denotes the map $P\varphi = (\varphi(0), \varphi'(0), \varphi(T), \varphi'(T))$ we have $MP\varphi = o$ if and only if $\varphi \in \Phi_B$, and $P\varphi \in X$ if and only if $\varphi \in \Phi_{B^*}$.

Let $x \in X \subset \boldsymbol{R}^4$ so that $x = \alpha v_1 + \beta v_2$. Since

$$Mx = \operatorname{col}\left(\beta(A_0 - B_0), \alpha(A_0 - B_0), 0, 0\right)$$

we see that X is the null space of M if and only if $A_0 = B_0$.

We next define subsets Ψ_B, Ψ_{B^*} of $C^\infty([0, T]; V)$ as follows. We set

$$\Psi_B = \{\psi \in C^\infty([0, T]; V): \psi = \sum_{\text{finite}} \varphi_i v_i, \varphi_i \in \Phi_B, v_i \in V\} ,$$

$$\Psi_{B^*} = \{\psi \in C^\infty([0, T]; V): \psi = \sum_{\text{finite}} \varphi_i v_i, \varphi_i \in \Phi_{B^*}, v_i \in V\} .$$

Corollary 3.2. *$\Psi_B = \Psi_{B^*}$ if and only if $A_0 = B_0$.*

Finally we introduce our closed boundary condition subspaces of $W(0, T)$. We set

$$W_B(0, T) = \text{closure}\,\{u \in C^\infty([0, T]; V): B_1 u = B_2 u = 0\} ,$$

$$W_{B^*}(0, T) = \text{closure}\,\{u \in C^\infty([0, T]; V): B_1^* u = B_2^* u = 0\} ,$$

where the closure is taken with respect to the norm on $W(0, T)$. The notation needs some explanation. First by $B_1 u = B_2 u = 0$ we are referring to boundary conditions (0.2). Second, by $B_1^* u = B_2^* u = 0$ we are referring to boundary conditions in the form of (0.2) with the coefficients a_{ij}, b_{ij} $(i, j = 1, 2)$ replaced by the adjoint coefficients a_{ij}^*, b_{ij}^* $(i, j = 1, 2)$ appearing in (3.5). As we now show, an alternate definition in terms of the sets Ψ_B and Ψ_{B^*}, is possible.

Lemma 3.2. *$W_B(0, T)$ and $W_{B^*}(0, T)$ are the closures in $W(0, T)$ of Ψ_B and Ψ_{B^*} respectively.*

Proof. Let $\{w_1, w_2, w_3, \ldots\}$ be a complete orthonormal basis for V. For $u \in C^\infty([0, T]; V)$ we define $P_n u = \sum_{i=1}^{n} g_i(t) w_i$, where $g_i(t) = ((u(t), w_i))$. Notice that

$$B_j(P_n u) = \sum_{i=1}^{n} (B_j g_i) w_i = \sum_{i=1}^{n} ((B_j u, w_i)) w_i \qquad (j = 1, 2) .$$

Thus, if $B_1 u = B_2 u = 0$, then $B_1(P_n u) = B_2(P_n u) = 0$ for $n = 1, 2, 3, \ldots$.

Let $u \in W_B(0, T)$. We may assume that $u \in C^\infty([0, T]; V)$ and $B_1 u = B_2 u = 0$. In this case we see from the above remarks that $P_n u \in \Psi_B$ for $n \to 1, 2, 3, \ldots$. We show that $P_n u \to u$ in $W(0, T)$. From basic Hilbert space theory we know that both $\|u(t) - P_n u(t)\| \to 0$ and $\|u'(t) - (P_n u)'(t)\| \to 0$ in V as $n \to \infty$, for each $t \in [0, T]$. Since $\|u(t) - (P_n u)(t)\|^2 \leqslant 4\|u(t)\|^2$ and $\|u'(t) - (P_n u)'(t)\|^2 \leqslant 4\|u'(t)\|^2$, we may apply the dominated convergence theorem to conclude that $P_n u \to u$ in $W(0, T)$. The argument for $W_{B^*}(0, T)$ is similar.

Corollary 3.3. *$W_B(0, T) = W_{B^*}(0, T)$ if and only if $A_0 = B_0$.*

4 - The weak problem

We are now in a position to give a precise meaning to the concept of a weak solution of problem (0.1), (0.2). In this section we will assume that the boundary conditions (0.2) are homogeneous. In the next section we will consider some nonhomogeneous problems. We will assume henceforth that the right-hand member of equation (0.1) is a function of t alone. If f depends on u the definitions are the same.

Def. 4.1. Let $f \in L^2(0, T; H)$. We say that $u \in W(0, T)$ is *a weak* *solution* of the boundary value problem

$$\frac{d^2u}{dt^2} + A(t)u = f, \quad 0 < t < T, \tag{4.1}$$

$$B_1u = B_2u = 0, \tag{4.2}$$

if $u \in W_B(0, T)$ and

$$\int_0^T [(u, \varphi'' v) + a(t; u, \varphi v)]\,dt = \int_0^T (f, \varphi v)\,dt \qquad \forall \varphi \in \Phi_{B^*}, \quad v \in V. \tag{4.3}$$

Due to the linearity involved we may equivalently write (4.3) as

$$\int_0^T [(u, \psi'') + a(t; u, \psi)]\,dt = \int_0^T (f, \psi)\,dt \qquad \forall \psi \in \Psi_{B^*}. \tag{4.4}$$

We will show that the weak problem formulated in Def. 4.1 is a proper generalization of the classical problem (4.1), (4.2). But first we prove a needed result.

Lemma 4.1. *If $u \in C^1([0, T]; V)$ and $B_1u = B_2u = 0$, then $u \in W_B(0, T)$.*

Proof. Let $\delta > 0$ and define a function $u_\delta \in C([0, T]; V)$ by

$$u_\delta(t) = \begin{cases} u(0) + tu'(0) & 0 \leq t \leq \delta, \\ \delta^{-1}(t-\delta)[u(2\delta) - u(0) - \delta u'(0)] + u(0) + \delta u'(0) & \delta \leq t \leq 2\delta, \\ u(t) & 2\delta \leq t \leq T - 2\delta, \\ \delta^{-1}(T - \delta - t)[u(T-2\delta) - u(T) + \delta u'(T)] + u(T) - \delta u'(T) & T - 2\delta \leq t \leq T - \delta, \\ u(T) + (t - T)u'(T) & T - \delta \leq t \leq T. \end{cases}$$

It is easy to show that $u_\delta \to u$ uniformly on $[0, T]$ as $\delta \to 0^+$. Also we find $u'_\delta \to u'$ uniformly on $[0, T] - \{\delta, 2\delta, T - 2\delta, T - \delta\}$ as $\delta \to 0^+$. Hence $u_\delta \to u$ in $W(0, T)$ as $\delta \to 0^+$. We now set

$$u_n(t) = \int_{-\infty}^{\infty} \overline{u}_\delta(t - s/n)\,k(s)\,ds,$$

where $\overline{u}_\delta$ denotes the linear extension of u_δ to all of $(-\infty, \infty)$ and $k(s)$ is the smoothing kernel of section **2**. We already know that $u_n \to u_\delta$ in $W(0, T)$ and thus $u_n \to u$ in $W(0, T)$ as $n \to \infty$ (and $\delta \to 0^+$). It remains to show that u_n satisfies the boundary conditions (for n large). To this end we observe that $sk(s)$ is an odd function and hence

$$\int_{-\infty}^{\infty} (sv_1 + v_2) k(s)\, ds = v_2 \qquad \forall v_1, v_2 \in V .$$

Thus if $n > \delta^{-1}$ we have

$$u_n(0) = \int_{-\infty}^{\infty} \bar{u}_\delta(-s/n) k(s)\, ds = \int_{-\infty}^{\infty} \big(u(0) - (s/n)\;\; u'(0)\big) k(s)\, ds = u(0),$$

$$u_n'(0) = \int_{-\infty}^{\infty} u_\delta'(-s/n) k(s)\, ds = \int_{-\infty}^{\infty} u'(0) k(s)\, ds = u'(0) .$$

Similarly $u_n(T) = u(T)$ and $u_n'(T) = u'(T)$. Thus $B_i u_n = B_i u = 0$ $(i = 1, 2)$. This completes the proof.

Theorem 4.2. *Let* $u \in C^2(0, T; V)$ *with* $u'' \in L^2(0, T; V)$ *and let*

$$f \in C(0, T; H) \cap L^2(0, T; H) .$$

Then u satisfies (4.1), (4.2) *in the strong sense if and only if u is a weak solution.*

Proof. Under the assumption that $u'' \in L^2(0, T; V)$ we integrate by parts twice in (4.3) to obtain

$$\text{(4.5)} \quad \int_0^T [(u'', \varphi v) + a(t; u, \varphi v)]\, dt + (u, \varphi' v)\Big|_0^T - (u', \varphi v)\Big|_0^T = \int_0^T (f, \varphi v)\, dt \;\; \forall \varphi \in \Phi_{B^*},\; v \in V.$$

Let $\varphi \in \Phi_{B^*}$ so that $\varphi(0), \varphi'(0), \varphi(T), \varphi'(T)$ are given by (3.4) for some $\alpha, \beta \in \boldsymbol{R}$. An easy computation shows that for any $v \in V$

$$(u, \varphi' v)\Big|_0^T - (u', \varphi v)\Big|_0^T = (\alpha B_1 u + \beta B_2 u, v) .$$

Hence (4.5) can be written as

$$(4.6)\qquad \int_0^T [(u'', \varphi v) + a(t; u, \varphi v)]\,dt + (\alpha B_1 u + \beta B_2 u, v) = \int_0^T (f, \varphi v)\,dt \qquad \forall \varphi \in \Phi_{B^*}, \quad v \in V.$$

Since $C_0^\infty(0, T) \subset \Phi_{B^*}$ we see that (4.6) necessarily implies

$$(4.7)\qquad \int_0^T (u'' + A(t)u - f, v)\,\varphi\,dt = 0 \qquad \forall \varphi \in C_0^\infty(0, T), \quad v \in V.$$

Note that $A(t)u \in V^*$ for all $t \in [0, T]$ and $u \in V$.

Now if u is a classical solution of (4.1), (4.2) then it is clear from (4.6) that u must also satisfy (4.3). Moreover $u \in W_B(0, T)$ by Lemma 4.1. Thus u is a weak solution.

On the other hand, if u is a weak solution so that u satisfies (4.3), then it also must satisfy (4.7). Thus it follows that $(u'' + A(t)u - f, v) = 0$ for all $v \in V$ and $t \in (0, T)$. But then $u'' + A(t)u = f$ in V^* for all $t \in (0, T)$. This is the strong meaning of (4.1). Having established this we find that (4.6) reduces to $(\alpha B_1 u + \beta B_2 u, v) = 0$ for all $v \in V$ and arbitrary $\alpha, \beta \in \mathbf{R}$. But V is dense in H and thus we conclude $B_1 u = B_2 u = 0$.

Remarks. (1) The proof of the above result requires only that u satisfy $u, u', u'' \in L^2(0, T; V)$ and $f \in L^2(0, T; H)$. The former condition implies (by the same argument used in Lemma 2.2) that $u \in C^1([0, T]; V)$. Under these weaker assumptions on u and f we need only amend the statements above to hold almost everywhere.

(2) Taking $V = H = \mathbf{R}^n$, for $n \geqslant 1$, we see that the above results also apply to systems of 2nd order ordinary differential equations having the form (4.1), (4.2).

Let us now define a continuous symmetric bilinear form on $W(0, T)$ by

$$B(u, w) = \int_0^T [-(u', w') + a(t; u, w)]\,dt, \qquad u, w \in W(0, T).$$

Observe that after integrating by parts once we may write (4.4) in the equivalent form

$$(4.8)\qquad B(u, \psi) + (u, \psi')\Big|_0^T = (f, \psi)_{L^2(0,T;H)} \qquad \forall \psi \in \Psi_{B^*}.$$

Corollary 4.3. *If* $(u, \psi')\Big|_0^T = 0$ *for every* $u \in W_B(0, T)$ *and every* $\psi \in \Psi_{B^*}$, *then* u *is a weak solution of problem* (4.1), (4.2) *if and only if* $u \in W_B(0, T)$ *and*

$$B(u, w) = (f, w)_{L^2(0,T;H)} \qquad \forall w \in W_{B^*}(0, T) . \tag{4.9}$$

Proof. Clearly, under the assumption above, (4.8) is equivalent to

$$B(u, \psi) = (f, \psi)_{L^2(0,T;H)} \qquad \forall \psi \in \Psi_{B^*} . \tag{4.10}$$

Notice that, for fixed u and f, both sides of this equation are continuous linear functionals of $\psi \in W(0, T)$. According to Lemma 3.2, ψ_{B^*} is dense in $W_{B^*}(0, T)$. Hence u will satisfy (4.9) if and only if it satisfies (4.10).

Corollary 4.4. *If the boundary conditions* (4.2) *are either Dirichlet, Neumann, or periodic then* u *is a weak solution of problem* (4.1), (4.2) *if and only if* $u \in W_B(0, T)$ *and*

$$B(u, w) = (f, w)_{L^2(0,T;H)} \qquad \forall w \in W_B(0, T) . \tag{4.11}$$

Proof. In these cases we have $W_{B^*}(0, T) = W_B(0, T)$ and $(u, \psi')\Big|_0^T = 0$ for all $u \in W_B(0, T)$ and every $\psi \in \Psi_{B^*} = \Psi_B$.

To conclude this section we consider a special case of the boundary conditions (4.2). Let $B_1 u = u'(0)$, $B_2 u = u'(T)$ so that (4.2) becomes homogeneous Neumann conditions. We show that in this case the boundary condition subspace $W_B(0, T)$ is actually the whole space $W(0, T)$.

Theorem 4.5. *The set* $N = \{u \in C^\infty([0, T]; V) : u'(0) = u'(T) = 0\}$ *is dense in* $W(0, T)$.

Proof. According to Theorem 2.5 it suffices to show that for each $u \in C^\infty(0, T; V) \cap W(0, T)$ there is a sequence of functions in N converging to u in $W(0, T)$.

Let $u \in C^\infty(0, T; V) \cap W(0, T)$. We first observe that for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_0^s \|u(t)\|^2 \, dt < \varepsilon , \quad \int_{T-s}^T \|u(t)\|^2 \, dt < \varepsilon \qquad \forall s \in (0, \delta) .$$

Hence for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $s \in (0, \delta)$ we have

$$\frac{1}{s} \int_0^s t\|u(t)\|^2 \, dt < \varepsilon , \quad \frac{1}{s} \int_{T-s}^T (T - t)\|u(t)\|^2 \, dt < \varepsilon .$$

Thus there are sequences of numbers $\{t_{0n}\}$, $\{t_{1n}\}$ $(n = 1, 2, 3, \ldots)$ such that $t_{0n} \to 0^+$, $t_{1n} \to T^-$, and $t_{0n}\|u(t_{0n})\|^2 < 1/n$, $(T - t_{1n})\|u(t_{1n})\|^2 < 1/n$.
We use these numbers to define a sequence of functions $\{u_n\}$ in $W(0, T)$. Let $u_n(t) = u(t)$ for $t_{0n} \leqslant t \leqslant t_{1n}$ and $u_n(t) = u(t_{0n})$ or $u_n(t) = u(t_{1n})$ if $t \leqslant t_{0n}$ or $t \geqslant t_{1n}$ respectively. We have $u_n'(t) = u'(t)$ for $t_{0n} < t < t_{1n}$ and $u_n'(t) = 0$ otherwise (in the weak sense). Moreover we observe that

$$\|u - u_n\|_W^2 = \int_0^T \|u(t) - u_n(t)\|^2 \,dt + \int_0^T |u'(t) - u_n'(t)|^2 \,dt$$

$$= \int_0^{t_{0n}} (\|u(t) - u(t_{0n})\|^2 + |u'(t)|^2)\,dt + \int_{t_{1n}}^T (\|u(t) - u(t_{1n})\|^2 + |u'(t)|^2)\,dt$$

$$\leqslant 2\big(t_{0n}\|u(t_{0n})\|^2 + (T - t_{1n})\|u(t_{1n})\|^2\big) + \int_0^{t_{0n}} (2\|u(t)\|^2 + |u'(t)|^2)\,dt$$

$$+ \int_{t_{1n}}^T (2\|u(t)\|^2 + |u'(t)|^2)\,dt\,.$$

Hence $u_n \to u$ in $W(0, T)$. We may now proceed as in Lemma 4.1 to show that the regularizations of u_n,

$$(u_n)_m(t) = \int_{-\infty}^{\infty} u_n(t - s/m)\,k(s)\,ds \qquad (m > n)\,,$$

form a sequence of elements in N converging to u_n. We may then extract a sequence by a diagonal process which converges to u in $W(0, T)$. This proves the theorem.

5 - Nonhomogeneous problems

In this section we consider three important nonhomogeneous problems. To be precise we assume that B_1, B_2 denote the evaluation operators corresponding to either the Cauchy problem $(B_1u = u(0),\ B_2u = u'(0))$, the Dirichlet problem $(B_1u = u(0),\ B_2u = u(T))$, or the Neumann problem $(B_1u = u'(0),\ B_2u = u'(T))$. Let v_0, v_1 denote elements of V and let h_0, h_1 denote elements of H. We consider the nonhomogeneous problems

$$u'' + A(t)u = f \qquad 0 < t < T\,, \tag{5.1}$$

$$(5.2)_a \qquad u(0) = v_0\,, \quad u'(0) = h_0 \qquad \text{(Cauchy problem)}\,,$$

$$(5.2)_b \qquad u(0) = v_0\,, \quad u(T) = v_1 \qquad \text{(Dirichlet problem)}\,,$$

$$(5.2)_c \qquad u'(0) = h_0\,, \quad u'(T) = h_1 \qquad \text{(Neumann problem)}\,.$$

Note that all equations in $(5.2)_{a,b,c}$ are taken to have meaning as equations in H. However, if $u(0)$, $u(T) \in V$ then $u(0) = v_0$, $u(T) = v_1$ in V since the inclusion map is injective. It is our goal to define weak nonhomogeneous problems in these three cases. We do this for all three at once.

As before we use $W_B(0, T)$ to denote the homogeneous boundary condition subspace of $W(0, T)$. Let $u_B \in \{C([0, T]; V) : u' \in C([0, T]; H\}$ be any function satisfying the nonhomogeneous boundary conditions (5.2). That is we assume u_B satisfies $(5.2)_{a,b}$ or $(5.2)_c$ depending on whether the problem is Cauchy, Dirichlet, or Neumann respectively. We define $u \in W(0, T)$ to be a weak solution of problem (5.1), (5.2) if $u \in u_B + W_B(0, T)$ and

$$(5.3) \qquad \int_0^T [(u, \varphi'' v) + a(t; u, \varphi v)]\, dt - (u_B, \varphi' v)\Big|_0^T + (u_B', \varphi v)\Big|_0^T = \int_0^T (f, \varphi v)\, dt$$

$$\forall \varphi \in \Phi_{B^*}\,, \quad v \in V\,.$$

Observe that in two of the cases under consideration, the Dirichlet and Neumann problems, we have $\Phi_B = \Phi_{B^*}$ while in the third case $\Phi_B \neq \Phi_{B^*}$. As we now show, the above definition is independent of the choice of u_B.

Let $\varphi \in \Phi_{B^*}$ so that $\varphi(0), \varphi'(0), \varphi(T), \varphi'(T)$ are given by (3.4) for some $\alpha, \beta \in \boldsymbol{R}$. Then we see as before that

$$(5.4) \qquad -(u_B, \varphi' v)\Big|_0^T + (u_B', \varphi v)\Big|_0^T = -\langle \alpha B_1 u_B + \beta B_2 u_B, v\rangle\,.$$

Thus only the prescribed boundary values actual appear in (5.3). Also if u_{B^*} is another function satisfying all the required conditions then we should have $u_B - u_{B^*} \in W_B(0, T)$ and hence

$$u_B + W_B(0, T) = u_{B^*} + (u_B - u_{B^*}) + W_B(0, T) = u_{B^*} + W_B(0, T)\,.$$

That this is actually the case is shown in the following lemma.

Lemma 5.1. *If* $u \in \{C([0, T]; V)\,;\, u' \in C([0, T]; H\}$ *satisfies the homogeneous Cauchy, Dirichlet or Neumann boundary conditions, then* $u \in W_B(0, T)$, *where* $W_B(0, T)$ *is the corresponding boundary condition subspace.*

Proof. The case in which the boundary conditions are Neumann is an immediate consequence of Theorem 4.5.

Let us consider the Cauchy problem. We must show that for each $u \in \{u \in C([0, T]; V) : u' \in C([0, T]; H)$ and $u(0) = u'(0) = 0\}$ there is a sequence of functions $u_n \in \{u \in C^\infty([0, T]; V) : u(0) = u'(0) = 0\}$ which converges to u in $W(0, T)$. To this end we define for $\delta > 0$

$$u_\delta(t) = \begin{cases} 0 & -\infty < t \leqslant \delta\,, \\ \delta^{-1}(t-\delta)\,u(2\delta) & \delta \leqslant t \leqslant 2\delta\,, \\ u(t) & 2\delta \leqslant t \leqslant T-\delta\,, \\ u(T-\delta) & T-\delta \leqslant t < +\infty\,. \end{cases}$$

It is easy to see that $u_\delta \to u$ in $W(0, T)$ as $\delta \to 0^+$. If we define

$$(u_\delta)_n(t) = \int_{-\infty}^{\infty} u_\delta(t - s/n)\,k(s)\,ds\,, \tag{5.5}$$

where $k(s)$ is the smoothing kernel, then (for n sufficiently large) we have $(u_\delta)_n \in C^\infty([0, T]; V)$. Moreover, as in the proof of Lemma 4.1, we find that for $n > \delta^{-1}$ we have $(u_\delta)_n(0) = u_\delta(0) = 0$ and $(u_\delta)'_n(0) = u'_\delta(0) = 0$. Since $(u_\delta)_n \to u_\delta$ this establishes the result.

We use a similar argument for the Dirichlet problem. Let $\delta > 0$ and define $\Delta(t) = T(t-\delta)/(T-2\delta)$. Observe that $\Delta(t)$ is the real-valued linear function through the points $(\delta, 0)$ and $(T-\delta, T)$. For $u \in \{u \in C([0, T]: V) : u' \in C([0, T]; H)$ and $u(0) = u(T) = 0\}$ we define $u_\delta(t) = u(\Delta(t))$ for $\delta \leqslant t \leqslant T-\delta$ and $u_\delta(t) = 0$ otherwise. It is again easy to verify that $u_\delta \to u$ in $W(0, T)$ as $\delta \to 0^+$. Using (5.5) to define $(u_\delta)_n$ we obtain a sequence of functions in $C^\infty([0, T]; V)$ which also satisfy $(u_\delta)_n(0) = u_\delta(0) = 0$, $(u_\delta)_n(T) = u_\delta(T) = 0$. This completes the proof of the lemma.

We now show that our weak problem is a proper generalization of the classical problem.

Theorem 5.2. *Let $u \in C^2(0, T; V)$ with $u'' \in L^2(0, T; V)$ and let*

$$f \in C(0, T; H) \cap L^2(0, T; H)\,.$$

Then u is a classical solution of problem (5.1), (5.2) *if and only if u is a weak solution.*

Proof. For u as above we integrate by parts twice and use (5.4) to show that u satisfies (5.3) if and only if u satisfies

$$(5.5)\quad \int_0^T [(u'', \varphi v) + a(t; u, \varphi v)]\,dt + (\alpha B_1 u + \beta B_2 u, v) - (\alpha B_1 u_B + \beta B_2 u_B, v) = \int_0^T (f, \varphi v)\,dt \qquad \forall \varphi \in \Phi_{B^*}, \quad v \in V .$$

Note that α and β depend on the choice of $\varphi \in \Phi_{B^*}$ as dictated by (3.4). Taking $\varphi \in C_0^\infty(0, T)$ shows that $u'' + A(t)u = f$ in V^* for all $t \in (0, T)$. Hence (5.5) implies that for arbitrary $\alpha, \beta \in \boldsymbol{R}$ we have

$$\alpha(B_1(u - u_B), v) + \beta(B_2(u - u_B), v) = 0 \qquad \forall v \in V .$$

Since V is dense in H it follows that $B_1(u - u_B) = B_2(u - u_B) = 0$ as elements of H. From this we deduce immediately that u satisfies (5.2). Thus if u is a weak solution, it must also be a classical solution of (5.1), (5.2).

In showing the converse it is clear that u satisfies (5.5) and hence also (5.3). Since $B_1(u - u_B) = B_2(u - u_B) = 0$ we may apply Lemma 5.1 to conclude that $u - u_B \in W_B(0, T)$. Therefore u is a weak solution.

6 - Examples

I) We consider the following problem

$$(6.1)\qquad u_{tt} - u_{xx} = f(t, x, u) \qquad -\infty < t < +\infty, \quad 0 < x < \pi ,$$

$$(6.2)\qquad u(t, 0) = u(t, \pi) = 0 \qquad -\infty < t < +\infty ,$$

$$(6.3)\qquad u(t + 2\pi, x) = u(t, x) \qquad -\infty < t < +\infty, \quad 0 < x < \pi .$$

A weak formulation for this problem has been used by many authors ([**2**], [**5**], [**8**], [**9**]). We briefly describe this formulation and then compare it to the formulation given in this paper.

Let D be the set given by

$$D = \{\varphi \in C^\infty((-\infty, +\infty)\times(0, \pi)) : \varphi \text{ satisfies (6.2), (6.3)}\} .$$

Let $G = (0,2\pi)\times(0, \pi)$. We denote by $\mathscr{H}$ the Hilbert space obtained as the closure of D in the norm

$$\|u\|_{H^1(G)} = \left(\|u\|^2_{L^2(G)} + \|u_t\|^2_{L^2(G)} + \|u_x\|^2_{L^2(G)}\right)^{1/2} .$$

A function $u \in \mathscr{H}$ is a weak solution of (6.1), (6.2) if

$$\iint_G u(\varphi_{tt} - \varphi_{xx})\,dt\,dx = \iint_G f(u)\varphi\,dt\,dx \qquad \forall \varphi \in D . \tag{6.4}$$

In comparison we take $V = H^1_0(\Omega)$, $H = L^2(\Omega)$ and consider the problem

$$\frac{d^2u}{dt^2} + [-\partial^2/\partial x^2]u = f(u) \qquad 0 < t < 2\pi = T ,$$

$$B_1 u = u(0) - u(2\pi) = 0 , \quad B_2 u = u'(0) - u'(2\pi) = 0 .$$

Observe that the boundary condition coefficients $a_{ij}, b_{ij}, (i, j = 1, 2)$ appearing in (0.2), for the above problem, take the values $(a_{11}, a_{12}, b_{11}, b_{12}) = (1, 0, -1, 0)$ and $(a_{21}, a_{22}, b_{21}, b_{22}) = (0, 1, 0, -1)$. Hence $A_0 = a_{11}a_{22} - a_{12}a_{21} = 1 = b_{11}b_{22} - b_{12}b_{21} = B_0$. According to Lemma 3.1 we thus have

$$\Phi_{B^*} = \Phi_B = \{\varphi \in C^\infty[0, 2\pi] : \varphi(0) = \varphi(2\pi), \varphi'(0) = \varphi'(2\pi)\} .$$

Notice that $\|u\|_W = \|u\|_{H^1(G)}$. Since $C^\infty_0(0, \pi)$ is dense in $V = H^1_0(0, \pi)$ it is clear that $D \subset \Psi_B$ and $\mathscr{H} = W_{\text{per}}(0, 2\pi)$. Moreover (6.4) is equivalent to (4.3) which is easily seen after writing (4.3) in the equivalent form (4.4).

II) We consider the Cauchy problem of Lions-Magenes [7]

$$\frac{d^2u}{dt^2} + A(t)u = f \qquad 0 < t < T , \tag{6.5}$$

$$u(0) = u_0 \in V , \tag{6.6}$$

$$u'(0) = u_1 \in H . \tag{6.7}$$

The equations $u(0) = u_0$, $u'(0) = u_1$ have meaning in H and V^* respectively for u satisfying (6.5). This can be shown by applying Lemma 2.2 twice, once to $W(0, T)$ and once to the space $\{u \in L^2(0, T; H) : du/dt \in L^2(0, T; V^*)\}$. Notice that if $u(0) \in V$ then $u(0) = u_0$ in V since the inclusion map is injective.

In this case the boundary condition coefficients a_{ij}, b_{ij} $(i, j = 1, 2)$ take the values $(a_{11}, a_{12}, b_{11}, b_{12}) = (1, 0, 0, 0)$ and $(a_{21}, a_{22}, b_{21}, b_{22}) = (0, 1, 0, 0)$. Clearly $A_0 \neq B_0$. Using the above values in (3.4) we find that $\Phi_{B^*} = \{\varphi \in C^\infty[0,T]: \varphi(T) = \varphi'(T) = 0\}$. Since $B_1 u = u(0)$, $B_2 u = u'(0)$ we clearly have $\Phi_B = \{\varphi \in C^\infty[0, T]: \varphi(0) = \varphi'(0) = 0\}$, and $W_B(0, T) =$ closure $\{u \in C^\infty([0, T]; V): u(0) = u'(0) = 0\}$.

In addition to the previous assumptions on $a(t; u, v)$ we now further assume that the map $t \to a(t; u, v)$ is in $C^1[0, T]$, for all $u, v \in V$, and that there are constants λ, $\alpha \in \boldsymbol{R}$ with $\alpha > 0$ such that

$$a(t; u, u) + \lambda |u|^2 \geq \alpha \|u\|^2 \qquad \forall u \in V .$$

Let u_B be the unique solution of the problem

$$u'' + A(t)u = 0 , \quad u(0) = u_0 , \quad u'(0) = u_1 .$$

From the results of Lions-Magenes [7] we know that $u \in C([0, T]; V)$ and $u' \in C([0, T]; H)$. Notice that for any $\varphi \in \Phi_{B^*}$ we have $-(u_B, \varphi' v)\Big|_0^T + (u_B', \varphi v)\Big|_0^T = (u_0, \varphi'(0)v) - (u_1, \varphi(0)v)$. Thus u is a weak solution if $u \in u_B + W_B(0, T)$ and satisfies (5.3) which now takes the form

$$\int_0^T [(u, \varphi'' v) + a(t; u, \varphi v)]\, dt + (u_0, \varphi'(0)v) - (u_1, \varphi(0)v) = \int_0^T (f, \varphi v)\, dt$$

$$\forall \varphi \in \Phi_{B^*} , \quad v \in V .$$

Using the bilinear form $B(\cdot, \cdot)$ of **4** we obtain equivalently (after integration by parts)

$$(6.8) \quad B(u, \varphi v) + (u_0 - u(0), \varphi'(0)v) - (u_1, \varphi(0)v) = \int_0^T (f, \varphi v)\, dt \quad \forall \varphi \in \Phi_{B^*} , \quad v \in V .$$

Now if $u \in u_B + W_B(0, T)$ then there is a sequence $u_n \in C^\infty([0, T]; V)$ such that $u_n(0) = u_n'(0) = 0$ and $u_n \to u - u_B$ in $W(0, T)$. Thus it follows from Lemma 2.3 that $0 = u(0) - u_B(0) = u(0) - u_0$ in H. Therefore (6.8) reduces to

$$B(u, \varphi v) - (u_1, \varphi(0)v) = (f, \varphi v)_{L^2(0,T;H)} \qquad \forall \varphi \in \Phi_{B^*} , \quad v \in V .$$

This is precisely the weak formulation of the Cauchy problem given by Lions-Magenes [7], p. 265.

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