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Some remarks on periodic solutions of Carathéodory RFDES via Ważewski principle (**)

1 – Bound sets for RFDES with continuous right-hand side were introduced by Mawhin (cfr. e.g. [**3**]) to prove existence theorems for periodic solutions of RFDES. The similar concept of a regular polyfacial set was also independently considered in $[\mathbf{5}]_{1,2,3}$ with the purpose of generalizing Ważewski's principle to RFDES. This raises the question of how Ważewski's principle is related to Mawhin's topological degree method. More specifically, we ask if Ważewski's principle can be used to detect periodic solutions of RFDES. We show in these remarks that the answer is yes, in dimensions one and two, and no in any higher dimension. In the scalar case, this is trivial, for $n = 2$ it is a consequence of Hopf's extension theorem, and for $n \geqslant 3$ we answer the question by giving an example of an ordinary differential equation $\dot{x} = f(x)$, and a bound set U for this equation, with $\mathrm{Cl}\, U$ being homeomorphic to the closed unit ball in $\boldsymbol{R}^n$, $f(x) \neq 0$ for $x \in \mathrm{Cl}\, U$, and such that the retract properties of Ważewski's principle are satisfied with respect to U and f.

In these remarks we also generalize, in the spirit of [4] and $[\mathbf{5}]_3$, the concept of a bound set and that of a guiding function to retarded functional differential equations of Carathéodory type, so that the resulting method can be applied to such equations.

2 – Let us first explain some notation: if X is a topological space and A is a subset of X, then $\mathrm{Cl}\, A$ and ∂A denote the closure and the boundary

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of A, respectively. $C = C([-r, 0], \boldsymbol{R}^n)$, $r \geq 0$, is the space of all continuous maps $\varphi: [-r, 0] \to \boldsymbol{R}^n$ endowed with the sup-norm topology. If $t \in \boldsymbol{R}$, and $x: [-r+t, t] \to \boldsymbol{R}^n$ is a continuous map, then x_t is the element of C defined as $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$.

Let Ω be open in $\boldsymbol{R} \times C$, and let $f: \Omega \to \boldsymbol{R}^n$. f will be said to satisfy the weak Carathéodory condition, if properties (i) and (ii) below hold:

(i) $f(\cdot, \varphi)$ is measurable for every fixed φ, $f(t, \cdot)$ is continuous for a.e. fixed t;

(ii) for every $(t, \varphi) \in \Omega$ there is a neighborhood $V = V(t, \varphi)$ of (t, φ) and an integrable function $\mu: \boldsymbol{R} \to \boldsymbol{R}^+ \cup \{\infty\}$ such that $\|f(s, \psi)\| \leq \mu(s)$ for $(s, \psi) \in V$.

Suppose now that $f: \Omega \to \boldsymbol{R}^n$ satisfies (i) and the following property

(ii)′ for every bounded set $E \subset \Omega$ there is an integrable function $\mu: \boldsymbol{R} \to \boldsymbol{R}^+ \cup \{\infty\}$ such that $\|f(s, \psi)\| \leq \mu(s)$ for $(s, \psi) \in E$.

In this case, f will be said to satisfy the strong Carathéodory condition.

Remarks. If $r = 0$ and $\Omega = C$, then $C \cong \boldsymbol{R}^n$ and (ii) is equivalent to (ii)′, but for $r > 0$, (ii)′ is stronger than (ii). If f is continuous, then f satisfies the weak Carathéodory condition. If f is completely continuous (i.e. continuous and maps bounded sets into bounded sets), then f satisfies the strong Carathéodory condition.

3 – We shall define strict bound sets and strict guiding functions for RFDEs of Carathéodory type. For details concerning RFDEs, we refer to [**1**].

Let $f: \boldsymbol{R} \times C \to \boldsymbol{R}^n$ be a T-periodic mapping (i.e. $f(t+T, \varphi) = f(t, \varphi)$ for all (t, φ)), $T > 0$, satisfying the strong Carathéodory condition.

Consider the boundary value problem (1)

$$\dot{x} = f(t, x_t)\,, \qquad x \text{ is } T\text{-periodic}\,. \tag{1}$$

Let $X = \{x \mid x: \boldsymbol{R} \to \boldsymbol{R}^n,\ x \text{ is } T\text{-periodic}\}$.

Def. 1. An open bounded set $G \subset \boldsymbol{R}^n$ is called a *strict bound set for* (1), if there is a class $\mathscr{F}$ of C^1-functions $V: \boldsymbol{R}^n \to \boldsymbol{R}$ such that for every

$u \in \partial G$ there is a $V = V_u \in \mathcal{F}$ satisfying (i)-(iii) below:

(i) $G \subset \{v \in \boldsymbol{R}^n \mid V(v) < 0\}$;

(ii) $V(u) = 0$;

(iii) for every $t \in \boldsymbol{R}$ and every $x \in X$ for which $x[\boldsymbol{R}] \subset \mathrm{Cl}\, G$ and $x(t)=u$, there is an $\varepsilon > 0$ and a null-set $N \subset \boldsymbol{R}$ such that either

$$\langle \operatorname{grad} V(x(s)),\, f(s. x_s)\rangle > 0 \qquad \text{for } s \in [t-\varepsilon,\, t+\varepsilon]\setminus N\,,$$

or

$$\langle \operatorname{grad} V(x(s)),\, f(s, x_s)\rangle < 0 \qquad \text{for } s \in [t-\varepsilon,\, t+\varepsilon]\setminus N\,.$$

Remark. $\langle\,,\rangle$ is the scalar product in $\boldsymbol{R}^n$, grad $V(x)$ is the gradient of V at x.

Def. 2. A C^1-function $W: \boldsymbol{R}^n \to \boldsymbol{R}$ is called a *strict guiding function for* (1), if there is a $\varrho > 0$, such that for every $t \in \boldsymbol{R}$ and every $x \in X$ for which $\|x(t)\| \geqslant \varrho$ and $|W(x(t))| \geqslant |W(x(s))|$, $s \in \boldsymbol{R}$, there is an $\varepsilon > 0$ and a null set $N \subset \boldsymbol{R}$ such that $\langle \operatorname{grad} W(x(s)),\, f(s, x_s)\rangle < 0$, for $s \in [t-\varepsilon,\, t+\varepsilon]\setminus N$.

Remark. Definitions 1 and 2 generalize the corresponding concepts from definitions VII.1, VII.5 in [**3**] and definition 8.2 in [**5**]$_1$.

The following proposition, whose proof is analogous to that of proposition VII.6 in [**3**] relates guiding functions to bound sets.

Proposition 1. Let W be a strict guiding function for (1), such that $|W(v)| \to \infty$, as $\|v\| \to \infty$. Then there exists a strict bound set G for (1) such that $G \supset \{v \in \boldsymbol{R}^n \mid \|v\| \leqslant \varrho\}$.

The main applications of the concepts introduced above are Theorem 1 and Corollary 1.

Theorem 1. *Let G be a strict bound set for* (1). *Assume that* $\mathrm{d}[g, G, 0] \neq 0$, *where*

$$g(a) = (1/T) \cdot \int_0^T f(t, a)\, \mathrm{d}t\,, \qquad a \in \boldsymbol{R}^n\,.$$

Then there is a T-periodic solution x of $\dot{x} = f(t, x_t)$ such that $x(t) \in G$ for $t \in \boldsymbol{R}$.

Remark. $\mathrm{d}[g, G, 0]$ is the Brouwer-degree of g on G with respect to 0.

Theorem 1 is a consequence of the following lemma, which follows immediately from theorem IV.13 in [**3**].

Lemma 1. *Let f be as in the* BVP (1), *and Γ be an open bounded subset of X. Assume* (1)-(3) *below:*

(1) *for every λ, $0 < \lambda < 1$, there are no solutions in $\partial\Gamma$ of* (2_λ)

$$\dot{x} = \lambda f(t, x_t) \; ; \tag{2}$$

(2) *for all constant functions $x \in \partial\Gamma$ ($x(t) \equiv a$, $t \in \boldsymbol{R}$),*

$$\int_0^T f(t, a)\, \mathrm{d}t \neq 0 \; ;$$

(3) *if $U = \{a \in \boldsymbol{R}^n \,|\, x \in \Gamma$ for $x(t) \equiv a\}$, then* $\mathrm{d}[g, U, 0] \neq 0$, *$g$ being defined as in Theorem* 1.

Under the above assumptions there exists a T-periodic solution of $\dot{x} = f(t, x_t)$ such that $x \in \Gamma$.

Remark. Since Γ is open and bounded, the same is true for U. Since, by (2), $g(a) \neq 0$, for $a \in \partial U$, $\mathrm{d}[g, U, 0]$ *exists.*

Proof of Theorem 1. Let $\Gamma = \{x \in X \,|\, x(t) \in G,\ t \in \boldsymbol{R}\}$. Then Γ is open and bounded in X. Let us verify assumptions (1)-(3) Lemma 1.

(1) Suppose that for some $0 < \lambda < 1$, $x \in \partial\Gamma$ is a solution of (2). Then $x[\boldsymbol{R}] \subset \mathrm{Cl}\, G$, and $x(t) = u \in \partial G$ for some $t \in \boldsymbol{R}$. Let $V = V_u \in \mathscr{F}$ be a function satisfying (i)-(iii) of Def. 1. Assume first that $\langle \mathrm{grad}\, V(x(s)),\ f(s, x_s)\rangle > 0$ for a.e. $s \in [t, t+\varepsilon]$. Hence for a.e. $s \in [t, t+\varepsilon]$, $(\mathrm{d}/\mathrm{d}s)\, V(x(s)) > 0$, i.e. $V(x(t+\varepsilon)) = V(x(t)) + \int_0^{t+\varepsilon} (\mathrm{d}/\mathrm{d}s)\, V(x(s))\, \mathrm{d}s > 0$, which contradicts the fact that $x(t+\varepsilon) \in \mathrm{Cl}\, G$. The other case is dealt with similarly.

(2) If $x(t) \equiv a$, $t \in \boldsymbol{R}$, and $x \in \partial\Gamma$, then $a \in \partial G$. Choose $V = V_a$ satisfying (i)-(iii) of Def. 1. Let A^+ (resp. A^-) be the set of all $t \in \boldsymbol{R}$ for which there is an $\varepsilon > 0$ such that $\langle \mathrm{grad}\, V(a), f(s, a)\rangle > 0$, (resp. $\langle \mathrm{grad}\, V(a), \cap (s, a)\rangle < 0$), for a.e. $s \in (t-\varepsilon, t+\varepsilon)$. Both A^+ and A^- are open, $A^+ \bigcup A^- = \emptyset$, and, by Def. 1, $A^+ \cup A^- = \boldsymbol{R}$. Hence $A^+ = \boldsymbol{R}$ or $A^- = \boldsymbol{R}$. This implies $\langle \mathrm{grad}\, V(a), \int_0^T f(s, a)\, \mathrm{d}s > \neq 0$, which proves (2).

(3) If U is defined as in Lemma 1, then $U = G$ and (3) follows from the assumptions of Theorem 1.

This proves Theorem 1.

Corollary 1. *If* W *is a strict guiding function for* (1) *such that* $|W(v)| \to \infty$ *as* $\|v\| \to \infty$, *and if* $d[\text{grad } W, B_\varrho(0), 0] \neq 0$, *then the* BVP (1) *has a solution.*

Remark. $B_\varrho(0) = \{v \in \mathbf{R}^n \mid \|v\| < \varrho\}$.

Proof. By Proposition 1, there is a strict bound set for (1) such that $G \supset \text{Cl}\, B_\varrho(0)$. Hence it only needs to be proved that $d[g, G, 0] \neq 0$, where g is as in Theorem 1. It follows from Def. 2 that $\langle \text{grad } W(a), g(a) \rangle < 0$ for $a \notin B_\varrho(0)$, therefore $g(a) \neq 0$ for $a \in B_\varrho(0)$ and $d[g, B_\varrho(0), 0] = d[\text{grad } W, B_\varrho(0), 0] \neq 0$. Hence $d[g, G, 0] = d[g, B_\varrho(0), 0] \neq 0$, which proves Corollary 1.

Example 1 (cfr. eq. 2.6 in [6])

$$\dot{x}(t) = -\sigma x(t) + |x(t-r)|^s \exp(-|x(t-r)|) + e(t)\,, \tag{3}$$

where $x(t) \in \mathbf{R}$, $s > 0$, $\sigma \neq 0$, and $e(t)$ is a T-periodic, measurable and essentially bounded function. Then for $b > 0$ large enough, $G = (-b, b)$ is easily seen to be a strict bounded set for T-periodic solutions of (3) and hence, by Theorem 1, there is a T-periodic function $x(t) \in G$ solving (3).

Alternatively $W(y) = y^2$ (for $\sigma < 0$), or $W(y) = -y^2$ (for $\sigma > 0$) is a strict guiding function for (3), which satisfies the assumptions of Corollary 1.

Example 2. If $y \in \mathbf{R}^n$, let $y^k = (y_1^k, \ldots, y_n^k)^T$. Let $0 \leqslant r_i \leqslant r$, $i = 1, 2, 3$. Consider

$$\dot{x}(t) = Ax(t) + Bx(t-r_1) + Px^2(t) + Qx^2(t-r_2) + Cx^3(t) + Dx^3(t-r_3) + E(t)\,, \tag{4}$$

where A, B, P, Q, C, D are $n \times n$-matrices, $C = \text{diag}(c_1, \ldots, c_n)$, $D = \text{diag}(d_1, \ldots, d_n)$, $|c_i| > |d_i|$, $i = 1, \ldots, n$. $E\colon \mathbf{R} \to \mathbf{R}^n$ is T-periodic, measurable and essentially bounded.

It is a matter of trivial computation to verify that for $M > 0$ large enough, the cube $G = (-M, M)^n$ is a strict bound set for (4), and the hypotheses of Theorem 1 are satisfied. Hence there is a T-periodic solution of (4).

In Examples 1 and 2 and in many other examples of the application of the topological degree method to RFDEs, one can also use Ważewski's principle ($[\mathbf{5}]_{1,2,3}$) to prove existence of solutions of $\dot{x} = f(t, x_t)$ such that $x(t) \in G$ for all $t \geqslant 0$. To this end, we do not need the T-periodicity of f, and the strong Carathéodory condition can be replaced by the weak one. However, using

Ważewski's principle in this way, one does not obtain any information about possible T-periodic solutions in G. As it was pointed out at the beginning of this paper, one may ask if Ważewski's principle alone can be used to locate T-periodic solutions of RFDEs.

Let us first consider a trivial example.

$$\dot{\theta} = 1\,, \qquad \dot{r} = r(r^2 - 1)\,, \tag{5}$$

(5) is an ODE in $\boldsymbol{R}^2$ (in polar coordinates).

Let G be the open annulus contained between the two concentric circles at zero with radii $\frac{1}{2}$ and $\frac{3}{2}$, respectively. Being autonomous, (5) can be viewed as a T-periodic equation for every $T > 0$. Also, G is a regular polyfacial set with respect to (5), satisfying all hypotheses of the classical theorem of Ważewski ([7]). Hence, if Ważewski's principle alone were sufficient for determining the existence of T-periodic solutions of ODEs, then it would follow that, for every $T > 0$, there is a T-periodic solution of (5) in G. But every periodic solution of (5) in G is 2π-periodic, a contradiction. Hence Ważewski's principle «fails» even in this trivial example. The situation is different, if G is simply connected. E.g. if $G = \{x \in \boldsymbol{R}^2 \mid \|x\| < \frac{3}{2}\}$, then G contains 0, hence there is a T-periodic solution of (5) in G.

The situation just described holds generally for RFDEs in dimensions one and two.

Theorem 2 (cfr. theorem 8.3 in [**5**]$_1$). *Suppose $n = 1$ or $n = 2$. Let $f\colon \boldsymbol{R}\times C \to \boldsymbol{R}^n$ be a T-periodic mapping satisfying the strong Carathéodory condition. Consider the* BVP (1) *together with the ordinary differential equation* (6)

$$\dot{x} = g(x)\,, \tag{6}$$

where $g(a) = (1/T)\int_0^T f(t, a)\,\mathrm{d}t$, $a \in \boldsymbol{R}^n$.

Let G be a strict bound set for the BVP (1) *and let $\mathscr{F}$ be the associated class of functions from Def.* 1. *Then G satisfies the classical assumption of Ważewski's theorem* ([7]) *that every egrees point of G and g is also a strict egress point for G and g. Assume* (1)-(3):

(1) *eq.* (6) *satisfies the uniqueness property of solutions or else $\mathscr{F}$ is finite*;

(2) *the retract properties of Ważewski's principle* ([7]) *are satisfied for G and eq.* (6);

(3) *there is a homeomorphism $h\colon \mathrm{Cl}\,G \to \mathrm{Cl}\,B_1(0)$ such that $h\,[G] = B_1(0)$.*

Under the above hypotheses there exists a T-periodic solution x, *of* $\dot{x} = f(t, x_t)$ *such that* $x(t) \in G$ *for* $t \in \mathbf{R}$.

Proof. That every egress point of G and g is a strict egress point of G and g follows from proof of Theorem 1.

$n = 1$. By assumption (3), $G = (a, b)$, $a, b \in \mathbf{R}$, and by assumption (2), either both points a and b are strict egress points for eq. (6) or they are both strict ingress points for this equation, i.e., either $g(a) > 0$ and $g(b) < 0$ or $g(a) < 0$ and $g(b) > 0$. Hence, $\mathrm{d}[g, G, 0] \neq 0$, and the application of Theorem 1 proves the result for $n = 1$.

$n = 2$. Suppose that $\mathrm{d}[g, G, 0] = 0$. Then, by Hopf's theorem, $g|\partial G$ is null-homotopic. Hence $g|\partial G$ can be extended to a continuous mapping $\tilde{g}$: $\mathrm{Cl}\, G \to \mathbf{R}^2$ such that $\tilde{g}(x) \neq 0$ for all $x \in \mathrm{Cl}\, G$. Using assumption (1) it is easily proved that there exists a C^1-function $F\colon \mathbf{R}^2 \to \mathbf{R}^2$, $F(x) \neq 0$ for $x \in \mathrm{Cl}\, G$, and such that all assumptions of the classical principle of Ważewski ([7]) hold for G and the equation $\dot{x} = F(x)$. Hence, by Ważewski's principle, there exists a solution $x(t)$ of $\dot{x} = F(x)$, such that $x(t) \in G$ for all $t \geqslant 0$. By the Poincaré Bendixson theorem there follows the existence of a nonconstant periodic solution $z(t)$ of $\dot{x} = F(x)$ such that $z(t) \in G$ for $t \in \mathbf{R}$. Since $\mathrm{Cl}\, G$ is homeomorphic to the closed unit ball, it follows that there is an equilibrium point of $\dot{x} = F(x)$ in G, which yields a contradiction. Hence $\mathrm{d}[g, G, 0] \neq 0$ and Theorem 1 completes the proof for $n = 2$.

We shall now show by means of an example that Theorem 2 is not true for $n \geqslant 3$.

Proposition 2. For every $n \geqslant 3$, there is a C^∞-mapping $g\colon \mathbf{R}^n \to \mathbf{R}^n$ and a finite family of polynomial functions $V\colon \mathbf{R}^n \to \mathbf{R}^n$ such that the following properties hold:

(1) $G = \{x \in \mathbf{R}^n \mid V(x) < 0, \text{ for every } V \in \mathscr{F}\}$ is a strict bound set for the BVP (1), where $f(t, \varphi) \overset{\text{def}}{=} g(\varphi(0))$;

(2) all assumptions of Ważewski's principle ([7]) are satisfied for G and $\dot{x} = g(x)$;

(3) $g(x) \neq 0$ for $x \in \mathrm{Cl}\, G$;

(4) there is a homeomorphism $h\colon \mathrm{Cl}\, G \to \mathrm{Cl}\, B_1(0)$ such that $h[G] = B_1(0)$.

Remark. Proposition 2 obviously implies that Theorem 2 is not true for $n \geqslant 3$, because otherwise we would obtain a sequence of T_n-periodic solutions x_n of $\dot{x} = g(x)$, $T_n \to 0$, $x_n(t) \in G$, but this would contradict property (3).

Proof. Let $x = (x_1, \ldots, x_n)^T \in \boldsymbol{R}^n$. Define $\tilde{f}_i\colon \boldsymbol{R}^n \to \boldsymbol{R}$, $i = 1, \ldots, n$, as

$$\tilde{f}_1(x) = x_1^2 - 1/9\,,\ \tilde{f}_2(x) = -x_1 x_2\,,\ \tilde{f}_3(x) = -x_1 x_3\,,\ \tilde{f}_k(x) = -x_k\,, \quad \text{for } k \geqslant 4.$$

Let $\tilde{f}(x) = (\tilde{f}_1(x), \ldots, \tilde{f}_n(x))^T$.

Define functions $V_i\colon \boldsymbol{R}^n \to \boldsymbol{R}$, $i = 1, \ldots, n+1$, as follows:

$$\begin{aligned} V_1(x) &= x_1 - \sqrt{5}/3\,, & V_2(x) &= -(x_1/\sqrt{5}+1)^2 + x_2^2 + x_3^2\,,\\ V_3(x) &= -x_1 - \sqrt{5}/3\,, & V_4(x) &= -(-x_1/\sqrt{5}+1)^2 + x_2^2 + x_3^2\,,\\ V_k(x) &= x_{k-1}^2 - 1 & &\text{for } k = 5, \ldots, n+1\,. \end{aligned}$$

Let $\mathscr{F} = \{V_i \mid i = 1, \ldots, n+1\}$, $G = \{x \in \boldsymbol{R}^n \mid V_i(x) < 0 \text{ for } i = 1, \ldots, n+1\}$. It is a matter of a simple computation to show that property (1) holds. Property (4) follows from the fact that G is bounded and convex. Also, property (2) is easily proved. Now, $f(x)=0$ if and only if $x = (\pm 1/3, 0, \ldots, 0)^T$, hence $\tilde{f}(x) = 0$ implies $x \in G$. Moreover, $\det D\tilde{f}(x) = (-1)^{n-3} x_1^3$, hence $\mathrm{d}[\tilde{f}, G, 0] = 0$. Now Hopf's theorem implies that $\tilde{f}$ is null-homotopic, i.e. $\tilde{f} \mid \partial G$ can be extended to a continuous mapping $f^1\colon \mathrm{Cl}\, G \to \boldsymbol{R}^n$ such that $f^1(x) \neq 0$ for $x \in \mathrm{Cl}\, G$. The approximation of f^1 on $\mathrm{Cl}\, G$ by a suitable C^∞-mapping $g\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ completes the proof of Proposition 2.

Remark. In [2], Jones and Yorke construct an ODE $\dot{x} = H(x)$ for $n \geqslant 3$, such that $H(x) \neq 0$ for $x \in \boldsymbol{R}^n$ and $\dot{x} = H(x)$ has only bounded solutions. Their example serves a different purpose than ours, and it cannot be used in our situation, since the solutions of $\dot{x} = H(x)$ move on torus surfaces, and hence the retract properties of Ważewski's principle are not satisfied.

References

[1] J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag 1977.

[2] G. S. Jones and J. A. Yorke, *The existence and nonexistence of critical points in bounded flows*, J. Differential Equations **6** (1969), 238-246.

[3] J. Mawhin, *Topological degree methods in nonlinear boundary value problems*, CBMS 40, Providence, Rhode Island.

[4] P. K. Palamides, Y. G. Sficas and V. A. Staikos, *Asymptotic behavior of Carathéodory systems. Anti-Lyapunov method*, J. Differential Equations **36** (1980), 442-457.

[5] K. P. RYBAKOWSKI: $[\cdot]_1$ *On Ważewski's principle for retarded functional differential equations*, Ph. D. Dissertation, Brown University, June 1979; $[\cdot]_2$ *Ważewski's principle for retarded functional differential equations*, J. Differential Equations **36** (1980), 117-138; $[\cdot]_3$ *A topological principle for* RFDES *of Carathéodory type*, J. Differential Equations **39** (1981), 131-150.

[6] K. SCHMITT, *Periodic solutions of delay-differential equations*, Proceedings of Summerschool & Conference on FDEs & Approximation of Fixed Points held at University of Bonn, Germany, July 12-22, 1978 (Ed. by H-O. Peitgen & H-O. Walther), Lecture Notes in Mathematics 730, Springer-Verlag.

[7] T. WAŻEWSKI, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. **20** (1947), 279-313.

Sunto

In questa nota si esamina la relazione tra il metodo topologico di Mawhin e il principio di Ważewski per le equazioni differenziali funzionali $(*)$ $\dot{x} = f(t, x_t)$, $x \in \mathbf{R}^n$ *del tipo di Carathéodory. Più precisamente, ci si chiede se le ipotesi del principio di Ważewski siano sufficienti per l'esistenza di una soluzione periodica di* $(*)$. *Si dimostra, che la risposta è affermativa se* $n = 1, 2$ *e negativa se* $n \geqslant 3$, *così chiarificando la relazione tra l'indice di Brouwer e le proprietà ritrattive di Ważewski degli insiemi in* $\mathbf{R}^n$.

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