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Semi local connectedness by a change of the topology (**)

A *space* means a topological space without any separation axioms. A space is *semi locally connected* provided that for each its point x and each open set U containing x there exists an open set V containing x and included in U such that the complement of V has a finite number of components (compare [4] p. 19).

We are concerned here with a pair (X, T) where X is a set and T is a topology on X. Throughout this paper we associate with X a topology, say T^*, defined by taking all closed connected subsets of (X, T) as a closed subbase. We will write sometimes X instead of (X, T) and X^* instead of (X, T^*).

Assertion 1. $T^* \subset T$.

Assertion 2. If (X, T) is compact, then (X, T^*) is compact also.

Assertion 3. If C is a closed connected subset of X, then C is a closed connected subset of X^*.

In fact, suppose that C is a closed subset of X and C fails to be connected in X^*. Then C is the union of two non-void disjoint and closed subsets P and Q of X^*. Since $T^* \subset T$, the sets P and Q are both closed in X and hence C fails to be connected in X.

Assertion 4. The space (X, T^*) is semi locally connected.

In fact, take a point $x \in X$ and a set $U \subset X$ with $U \in T^*$. By the definition of T^*, U includes a set V containing x and such that V is the com-

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plement of an element of the closed base of T^*. Thus $X \setminus V$ is the union of a finite number of closed connected subsets of X. The assertion follows now by Assertion 3.

Assertion 5. If (X, T) is T_1, then (X, T^*) is T_1 also.

In fact, any singleton is closed and connected.

In general, however, T^* is strictly included in T and if X is Hausdorff, then X^* may fail to be Hausdorff. To construct a simple example join each point of the set $\{(x, 0) \in E^2: x=0, 1, \frac{1}{2}, \ldots, 1/n, \ldots\}$ with the point $(0, 1)$ by a straight line segment obtaining so called *harmonic fan* X endowed with the Euclidean subspace topology T. The space X^* is compact but not Hausdorff and T^* is strictly contained in T. In fact, there exist no open disjoint subsets of X^* containing points $(0, 0)$ and $(0, \frac{1}{2})$ respectively. The set $\{(x, 0): x = 0, 1, \frac{1}{2}, \ldots, 1/n, \ldots\}$ is not closed in X^*.

Theorem 1. *Let (X, T) be a compact Hausdorff space. The following conditions are equivalent.*

(1) *(X, T) is semi locally connected.*
(2) *$T^* = T$.* (3) *T^* is Hausdorff.*

Proof. (1) implies (2). By Assertion 1 it is sufficient to prove that $T \subset T^*$. Take a set $U \in T$ and a point $x \in U$. Since (X, T) is semi locally connected there is an open set $V(x)$ containing x and included in U such that $X \setminus V(x)$ is the union of a finite number of components. Thus $V(x) \in T^*$. Hence $U = \bigcup \{V(x): x \in U\}$ is open in T^*.

That (2) implies (3) is trivial.

(3) implies (2) by Assertion 1, because compact topologies are minimal among all Hausdorff topologies.

(2) implies (1) by Assertion 4.

A space is *locally connected* provided that for each its point x and for each open set U containing x there exists an open connected set V such that $x \in V \subset U$. A space is *hereditarily finitely coherent* provided that the intersection of any two its closed connected subsets has a finite number of components (compare [**1**], p. 44).

Theorem 2. *Let (X, T) be a hereditarily finitely coherent compact Haus-*

dorff space. Then the following conditions are equivalent.

(1) (X, T) *is locally connected.*
(2) $T^* = T$. (3) T^* *is Hausdorff.*

Theorem 2 follows readily from Theorem 1 and the lemma below.

Lemma. *With (X, T) as in Theorem 2, (X, T) is semi locally connected if and only if it is locally connected.*

Proof. Take a point $x \in X$ and an open set $U \neq X$ containing x. Let y be a point of $X \setminus U$ and let V be an open set containing y such that $\mathrm{cl}\, V$ does not contain x. Since the space is semi locally connected, it follows that V includes an open subset W which contains y and such that $X \setminus W$ has a finite number of components. The component of $X \setminus W$ which contains x, say $C(y)$, includes an open subset of X which contains x. Obviously, $C(y)$ does not contain y. The family $\{X \setminus C(y)\colon y \in X \setminus U\}$ covers the compact set $X \setminus U$. Hence there exists a family $C(y_1), C(y_2), \ldots, C(y_n)$ such that $x \in \mathrm{int} \bigcap C(y_i) \subset \bigcap C(y_i) \subset U$. Since the space is hereditarily finitely coherent, the intersection $\bigcap C(y_i)$ has a finite number of components. The component of the set $\bigcap C(y_i)$ which contains x includes an open subset of X containing x. This readily follows that (X, T) is locally connected. The proof of the converse implication is easy.

A space is *hereditarily unicoherent* if the intersection of any two closed and connected subsets of it is connected. It is clear that in hereditarily unicoherent space the intersection of arbitrary family of closed and connected subsets is connected. A *continuum* is a compact connected Hausdorff space. A *tree* is a continuum in which each pair of points is separated by a third point. A continuum is *irreducible about a set A* if it includes A but no proper subcontinuum of the continuum includes A.

Theorem 3. *Let (X, T) be a Hausdorff continuum which is either hereditarily unicoherent or irreducible about a finite subset. Then the following conditions are equivalent.*

(1) (X, T) *is a tree.*
(2) $T^* = T$. (3) T^* *is Hausdorff.*

Proof. It is well known that a continuum is a tree if and only if it is hereditarily unicoherent and locally connected (see [**3**], p. 803). Thus the theorem follows for hereditarily unicoherent continua by Theorem 2. That a continuum irreducible about a finite subset is a tree if and only if it is semi locally connected is established in [2] (p. 256 the proof of theorem 1). Therefore the theorem follows for irreducible continua by Theorem 1.

Theorem 4. *Let (X, T) be a hereditarily unicoherent space. Then C is a closed connected subset of X if and only if C is a closed connected subset of X^*.*

Proof. Suppose that C is a closed connected subset of X^*. Then C being closed is the intersection

$$C = \bigcap \{F_\alpha : \alpha \in A\} ,$$

where each F_α is the finite union of mutually disjoint closed connected subsets of X, say $F_\alpha = F_\alpha^1 \cup F_\alpha^2 \cup \ldots \cup F_\alpha^{n(\alpha)}$. The set C being connected in X^* is contained in one of the sets $F_\alpha^1, \ldots, F_\alpha^{n(\alpha)}$ for each α. It follows by Assertion 3 that each F_α^i is closed and connected in X. Thus C is the intersection of closed connected subsets of X. Therefore C is connected itself because X is hereditarily unicoherent. The converse implication follows by Assertion 3.

Corollary. *Let (X, T) be a hereditarily unicoherent space. Then (X, T^*) is hereditarily unicoherent also.*

References

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Abstract

Let X be a topological space. Let X^* denote the set X endowed with the topology generated by taking the family of all closed connected subsets of X as the closed subbase. Then X is semi locally connected if and only if $X = X^*$. This result follows a characterization of trees (Theorem 3 below).

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