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A theorem on the means of an entire Dirichlet series of order zero (**)

1 - Introduction

A Dirichlet series $f(s)=\sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ $(s=\sigma+it,\ \lambda_{n+1}>\lambda_n,\ \lambda_1\geqslant 0,\ \lambda_n\to\infty$ with $n)$, which we shall assume to be absolutely convergent everywhere in the complex plane C and is bounded in any left strip, and hence it defines an entire function. The order $\varrho(0\leqslant\varrho\leqslant\infty)$ of $f(s)$ is defined by Ritt [**5**], as the limit superior of $(\log\log M(\sigma)/\sigma)$, as $\sigma\to\infty$, with

$$M(\sigma)=\sup\left[|f(\sigma+it)|: -\infty<t<\infty\right].$$

For a class of functions of order zero, that is for which $\varrho=0$, we define logarithmic order ϱ^* as [**4**]

$$\limsup_{\sigma\to\infty}\frac{\log\log M(\sigma)}{\log\sigma}=\varrho^*,\quad 1\leqslant\varrho^*\leqslant\infty. \tag{1.1}$$

Let us consider the following means of $f(s)$

$$I_\delta(\sigma)=\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^{\delta}\,dt\}^{1/\delta}, \tag{1.2}$$

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where the integral in (1.2) exists on account of the absolute convergence of the series for $f(s)$

$$m^*_{\delta,k}(\sigma) = \sigma^{-k-1}\int_0^\sigma x^k I_\delta(x)\,\mathrm{d}x, \tag{1.3}$$

where $0 < \delta < \infty$ and $0 < k < \infty$.

Let $1 < \varrho^* < \infty$. Then, there exists a logarithmic proximate order $\varrho^*(\sigma)$ satisfying the conditions [1]

(i) $\lim\limits_{\sigma\to\infty} \varrho^*(\sigma) = \varrho^*$, (ii) $\lim\limits_{\sigma\to\infty} (\varrho^*(\sigma))'\sigma\log\sigma = 0$.

For functions of logarithmic order ϱ^* $(1 < \varrho^* < \infty)$, with a logarithmic proximate order $\varrho^*(\sigma)$, the logarithmic proximate type T^* and lower logarithmic proximate type t^* are defined as

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{\sigma^{\varrho^*(\sigma)}} = \begin{matrix}T^*\\ t^*\end{matrix}, \qquad 0 \leqslant t^* \leqslant T^* \leqslant \infty. \tag{1.4}$$

It is known that ([3], p. 274)

$$\limsup_{\sigma\to\infty} \left[\frac{I_\delta(\sigma)}{m^*_{\delta,k}(\sigma)}\right]^{1/\log\sigma} = e^{\varrho^*}, \qquad 1 \leqslant \varrho^* \leqslant \infty. \tag{1.5}$$

My purpose in this note is to prove a result which gives a refinement of the above result when $f(s)$ is of logarithmic order ϱ^* $(1 < \varrho^* < \infty)$.

2 - Theorem. *Let $f(s)$ be of logarithmic order ϱ^* $(1 < \varrho^* < \infty)$ and a logarithmic proximate order $\varrho^*(\sigma)$. Let T^* and t^* be the logarithmic proximate type and lower logarithmic proximate type as defined in* (1.4). *Then*

$$\varrho^* T^* \leqslant \limsup_{\sigma\to\infty} \frac{I_\delta(\sigma)}{m^*_{\delta,k}(\sigma)\,\sigma^{\varrho^*(\sigma)}} \leqslant e\,\varrho^* T^* \quad \textit{and} \tag{2.1}$$

$$\liminf_{\sigma\to\infty} \frac{I_\delta(\sigma)}{m^*_{\delta,k}(\sigma)\,\sigma^{\varrho^*(\sigma)}} \leqslant \varrho^* t^*. \tag{2.2}$$

Proof. It is seen from the definitions of $I_\delta(\sigma)$ and $m^*_{\delta,k}(\sigma)$, that

$$\frac{\mathrm{d}}{\mathrm{d}x}[(k+1)\log x + \log m^*_{\delta,k}(x)] = \frac{1}{x}\left(\frac{I_\delta(x)}{m^*_{\delta,k}(x)}\right), \quad \text{so that}$$

$$(k+1)\log\frac{\sigma}{\sigma_0}+\log m^*_{\delta,k}(\sigma)-\log m^*_{\delta,k}(\sigma_0)=\int_{\sigma_0}^{\sigma}\frac{I_\delta(x)}{m^*_{\delta,k}(x)}\frac{dx}{x}, \quad \sigma\geqslant\sigma_0,$$ that is

(2.3) $$\log m^*_{\delta,k}(\sigma)=\log m^*_{\delta,k}(\sigma_0)+\int_{\sigma_0}^{\sigma}\nu^*(x)\frac{dx}{x},$$ where

(2.4) $$\nu^*(x)=\frac{I_\delta(x)}{m^*_{\delta,k}(x)}-(k+1),$$

is an increasing function of x, for all large values of x, in view of the fact that $x^{k+1}I_\delta(x)$ is a convex function with respect to $x^{k+1}m^*_{\delta,k}(x)$ (see [**3**], lemma 2).

It is known that ([**2**], p. 15)

(2.5) $$\log I_\delta(\sigma)\sim\log M(\sigma)$$

as $\sigma\to\infty$, and also that ([**3**], p. 277)

(2.6) $$m^*_{\delta,k}(\sigma)\leqslant\frac{I_\delta(\sigma)}{k+1}\leqslant\Big(\frac{\Delta^{k+1}}{\Delta^{k+1}-\sigma^{k+1}}\Big)m^*_{\delta,k}(\Delta), \qquad \Delta>\sigma.$$

From (1.4), (2.5) and (2.6), we have $\limsup\limits_{\sigma\to\infty}\dfrac{\log m^*_{\delta,k}(\sigma)}{\sigma^{\varrho^*(\sigma)}}=T^*$. Let

(2.7) $$\limsup_{0\to\infty}\frac{\nu^*(\sigma)}{\sigma^{\varrho^*(\sigma)}}=\gamma^*.$$

Then from (2.3) and (2.7), we get

$$\log m^*_{\delta,k}(\sigma)<\log m^*_{\delta,k}(\sigma_0)+(\gamma^*+\varepsilon)\int_{\sigma_0}^{\sigma}x^{\varrho^*(x)-1}dx$$

$$\sim\frac{\gamma^*+\varepsilon}{\varrho^*}\sigma^{\varrho^*(\sigma)}(1+0(1)),$$

for $\sigma\geqslant\sigma_0$, $\varepsilon>0$. This implies

(2.8) $$\limsup_{\sigma\to\infty}\frac{\nu^*(\sigma)}{\sigma^{\varrho^*(\sigma)}}\geqslant\varrho^*T^*.$$

Further, for $\eta > 1$,

$$\nu^*(\sigma)\log\eta \leqslant \int_\sigma^{\eta\sigma} \frac{\nu^*(x)}{x}\, dx < \log m^*_{\delta,k}(\eta\sigma)\,,$$

which gives

$$\limsup_{\sigma\to\infty} \frac{\nu^*(\sigma)}{\sigma^{\varrho^*(\sigma)}} \leqslant \frac{\eta^{\varrho^*}}{\log\eta} T^*.$$

Putting $\eta = e^{1/\varrho^*}$, which is the best value of η here, in the above inequality one gets

$$\limsup_{\sigma\to\infty} \frac{\nu^*(\sigma)}{\sigma^{\varrho^*(\sigma)}} \leqslant e\varrho^* T^*. \tag{2.9}$$

Hence (2.1) follows from (2.8) and (2.9) through (2.4). Similarly, (2.2) is proved.

References

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