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Maximum modulus theorems for the elastic half-space (**)

A GIORGIO SESTINI per il suo 70° compleanno

1. - Introduction

It is well known that, except in special circumstances, the maximum principle does not hold for solution of systems of partial differential equations and in particular for the equations of linear elasticity.

Some years ago, Pólya [**6**] gave a pair of examples to show that in an elastic body, free of body forces, the maximum modulus of the displacement does not necessarily occur on the boundary. More recently Fichera [**2**] obtained a partial extension of the maximum principle in elasticity. He considered an elastic body B (not necessarily homogeneous and isotropic) free of body forces and subjected to displacements $\hat{u}_i$ on its surface C. If the solution u_i of the corresponding displacement boundary value problem is of class $C^2(B) \cap C^0(\bar{B})$, it is possible to find a constant H such that

$$\max_{\bar{B}} \sqrt{u_i u_i} \leqslant H \max_{C} \sqrt{\hat{u}_i \hat{u}_i}\,, \tag{1.1}$$

with H depending only on B and the elastic moduli. In general, as Pólya proved, H is greater than one.

In many cases, however, it is not only interesting to prove (1.1), but also

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to find explicit values for H. In this sense the above mentioned work by Fichera is a first attempt toward the calculation of H.

These considerations have led me to consider the problem of determining the constant H in a particular case. The body B is a homogeneous and isotropic half-space with prescribed displacements on its bounding plane and without body forces. This problem was solved in closed form by Boussinesq who found the Green function for the elastic half-space. Thus, by a simple application of algebraic and integral inequalities, it is possible to derive (1.1) and in particular to evaluate the constant H.

There exist, however, other forms for the maximum modulus theorem for the elastic half-space. Suppose in fact that the half-space is loaded only by surface tractions $\hat{p}_i$, continuous in a closed and bounded surface region $\mathscr{D}$. Then it is possible to calculate a constant H' such that

$$\max_{\bar{B}} \sqrt{u_i u_i} \leqslant H' \max_{\bar{\mathscr{D}}} \sqrt{\hat{p}_i \hat{p}_i}\,. \tag{1.2}$$

If instead the half-space is free of tractions on its bounding plane and loaded by body forces b_i, continuous in a closed and bounded region D entirely contained in B, it is possible to find a constant H'' such that

$$\max_{\bar{B}} \sqrt{u_i u_i} \leqslant H'' \max_{\bar{D}} \sqrt{b_i b_i}\,. \tag{1.3}$$

The evaluation of H' and H'' is rendered easy by the knowledge of the Green functions of the two problems. The Green function for the traction problem in the elastic half-space was given by Boussinesq [**1**]; the Green function for the body forces problem was obtained by Mindlin [**5**].

2. - The displacement problem for the elastic half-space

Let us consider a half-space B and its bounding plane $\mathscr{S}$, and choose a cartesian system of x_i-axes ($i = 1, 2, 3$) such that the origin O is placed on $\mathscr{S}$, the x_1, x_2-axes are contained in $\mathscr{S}$, and the x_3-axis is oriented so that $\bar{B}$ has the equation $x_3 \geqslant 0$.

Suppose that the material of which B is composed is homogeneous and isotropic with Lamé moduli μ, λ and the assumptions of linear theory of elasticity apply. On the boundary $\mathscr{S}$ a displacement field $\hat{u}_i(x_1, x_2)$, continuous and bounded, is prescribed and we want to find the corresponding elastic displacements $u_i(x_1, x_2, x_3)$, solutions of the boundary value problem

$$\begin{aligned} &\mu \Delta u_i + (\lambda + \mu)\theta_{,i} = 0 && \text{for } x_3 > 0\,, \\ &u_i(x_1, x_2, 0) = \hat{u}_i && \text{for } x_3 = 0\,, \end{aligned} \tag{2.1}$$

where $\theta = u_{j,i}$. An additional condition, imposed by physical considerations, requires that $u_i = O(1)$ as x_3 tends to infinity.

Problem (2.1) possesses a closed solution, due to Boussinesq [1], of the form

$$u_i = \frac{1}{2\pi}\iint_{\mathscr{S}} \frac{x_3\,\hat{u}_i(\xi_1, \xi_2)}{R^3}\,d\xi_1 d\xi_2 + \frac{x_3}{2\pi k}\iint_{\mathscr{S}} \left(\frac{\hat{u}_j(\xi_1, \xi_2)}{R}\right)_{,ji} d\xi_1 d\xi_2\,, \tag{2.2}$$

where $R^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + x_3^2$ and $k = 3 - 4\nu$. Since ν satisfies the inequality $0 < \nu < \frac{1}{2}$, it follows that $1 < k < 3$.

Our program is to bound $\sqrt{u_i u_i}$, the modulus of u_i, in terms of $\sqrt{\hat{u}_i \hat{u}_i}$.

For this purpose we differentiate the second integrand in (2.2) with respect to x_j and x_i

$$\left(\frac{\hat{u}_j(\xi_1, \xi_2)}{R}\right)_{,ji} = \hat{u}_j(\xi_1, \xi_2)\left(\frac{1}{R}\right)_{,ji} = \hat{u}_j(\xi_1, \xi_2)\left(-\frac{\delta_{ij}}{R^3} + 3\,\frac{(x_i - \xi_i)(x_j - \xi_j)}{R^5}\right),$$

and write (2.2) as

$$\begin{aligned} u_i &= \\ &= \frac{1}{2\pi}\iint_{\mathscr{S}} \frac{x_3\,\hat{u}_i(\xi_1, \xi_2)}{R^3}\,d\xi_1 d\xi_2 + \frac{x_3}{2\pi k}\iint_{\mathscr{S}}\left(-\frac{\delta_{ij}}{R^3} + \frac{3(x_i - \xi_i)(x_j - \xi_j)}{R^5}\right)\hat{u}_j(\xi_1,\xi_2)\,d\xi_1 d\xi_2 \\ &= \frac{1}{2\pi}\left(1 - \frac{1}{k}\right)\iint_{\mathscr{S}} \frac{x_3\,\hat{u}_i(\xi_1, \xi_2)}{R^3}\,d\xi_1 d\xi_2 + \frac{3x_3}{2\pi k}\iint_{\mathscr{S}} \frac{(x_i - \xi_i)(x_j - \xi_j)}{R^5}\,\hat{u}_j(\xi_1, \xi_2)\,d\xi_1\,d\xi_2\,. \end{aligned} \tag{2.3}$$

We now consider the scalar product $u_i\bar{v}_i$, where $\bar{v}_i$ is a constant vector, and, observing that $1 < k < 3$, we obtain from (2.3) the inequality

$$\begin{aligned} |u_i\bar{v}_i| \leqslant \max_{\mathscr{S}}|u_i\bar{v}_i|\,\frac{x_3}{2\pi}\left(1 - \frac{1}{k}\right)\iint_{\mathscr{S}} \frac{1}{R^3}\,d\xi_1 d\xi_2 \\ + \frac{3x_3}{2\pi k}\iint_{\mathscr{S}} \frac{|(x_i - \xi_i)\bar{v}_i|\,|(x_j - \xi_j)\,\hat{u}_j|}{R^5}\,d\xi_1\,d\xi_2\,. \end{aligned} \tag{2.4}$$

If $\sqrt{u_i u_i}$, the modulus of u_i, attains it maximum at a point $P(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ of $\bar{B}$, we can write (2.4) at P and choose $\bar{v}_i = u_i(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. A simple application of the Cauchy inequality permits us to derive

$$\begin{aligned} \max_{\bar{B}}\sqrt{u_i u_i} &\leqslant \max_{\mathscr{S}}\sqrt{\hat{u}_i\hat{u}_i}\,\frac{x_3}{2\pi}\left(1 - \frac{1}{k}\right)\iint_{\mathscr{S}} \frac{1}{R^3}\,d\xi_1\,d\xi_2 \\ &\quad + \max_{\mathscr{S}}\sqrt{\hat{u}_i\hat{u}_i}\,\frac{3x_3}{2\pi k}\iint_{\mathscr{S}} \frac{1}{R^3}\,d\xi_1 d\xi_2 \\ &\leqslant \max_{\mathscr{S}}\sqrt{\hat{u}_i\hat{u}_i}\,\frac{x_3}{2\pi}\left(1 + \frac{2}{k}\right)\iint_{\mathscr{S}} \frac{1}{R^3}\,d\xi_1\,d\xi_2\,. \end{aligned} \tag{2.5}$$

The integral in (2.5) is easily computable and is given by

$$\iint_{\mathscr{S}} \frac{1}{R^3}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\infty} \frac{\varrho\,\mathrm{d}\varrho}{(\varrho^2+x_3^2)^{3/2}} = \frac{2\pi}{x_3}\,,$$

and thus (2.5) becomes

$$\max_{\bar{B}} \sqrt{u_i u_i} \leqslant \left(1+\frac{2}{k}\right) \max_{\mathscr{S}} \sqrt{\hat{u}_i \hat{u}_i}\,,$$

which shows that *inequality* (1.1) *holds with* $H = 1 + 2/k$ *for the elastic half-space.*

It easy to show by simple examples that the modulus of u_i, the solution of the displacement problem for the half-space, may attain its maximum at an interior point of the half-space ([1]). Consider, for instance, the displacement field

$$u_i = \delta_{3i} B - \frac{1}{4(1-\nu)} (B x_3)_{,i}\,, \tag{2.6}$$

where B is a scalar harmonic function given by

$$B = \frac{1}{\sqrt{x_1^2 + x_2^2 + (x_3 + a)^2}} \qquad (a > 0)\,. \tag{2.7}$$

Since $a > 0$, $B(x_1, x_2, x_3)$ is regular in the half-space $x_3 \geqslant 0$, and so is the elastic state (2.6). This is a simple variant of the classic Boussinesq-Papkovich-Neuber solution ([2]).

By substituting (2.7) into (2.6)we obtain

$$u_3 = \left(1 - \frac{1}{4(1-\nu)}\right) \frac{1}{\sqrt{x_1^2 + x_2^2 + (x_3 + a)^2}} + \frac{3}{4(1-\nu)} \frac{x_3(x_3 + a)}{\left(x_1^2 + x_2^2 + (x_3 + a)^2\right)^{3/2}}\,,$$

$$u_\alpha = \frac{3}{4(1-\nu)} \frac{x_\alpha}{\left(x_1^2 + x_2^2 + (x_3 + a)^2\right)^{3/2}} \qquad (\alpha = 1, 2)\,.$$

At the boundary of the half-space, that is for $x_3 = 0$, $\sqrt{u_i u_i}$ attains its maximum at the origin, where we have

$$\max_{\mathscr{S}} \sqrt{u_i\, u_i} = |u_3|_{x_i=0} = \left(1 - \frac{1}{4\,(1-\nu)}\right)\frac{1}{a} \qquad (i = 1, 2, 3)\,. \tag{2.8}$$

([1]) Other examples of this type of behavior of the elastic solutions in a bounded domain have been given by Polya [**6**].

([2]) Cf., for instance, Gurtin [**3**].

On the other hand, for $x_1 = x_2 = 0$, $x_3 > 0$, $|u_i|$ becomes a maximum for $x_3 = (3 - 4(1-\nu))/(3 + 4(1-\nu))$, which is positive for $\nu > 1/4$. Under the assumption that $\nu > 1/4$, the value of $|u_i| = |u_3|$ at $x_1 = x_2 = 0$, $x_3 = (3 - 4(1-\nu))/(3 + 4(1-\nu))$ is

$$\sqrt{u_i u_i} = \Big(1 - \frac{1}{4(1-\nu)}\Big)\frac{1}{a}\,\frac{3 + 4(1-\nu)}{6}\Big(1 + \frac{1}{8(1-\nu)}\Big), \tag{2.9}$$

which is clearly greater than (2.8) for $\nu > 1/4$.

3. - The traction problem for the elastic half-space

We now examine the case in which the half-space $x_3 > 0$ is loaded by surface tractions $\hat{p}_i(x_1, x_2)$. We assume that the boundary data $\hat{p}_i(x_1, x_2)$ are defined on a bounded subdomain $\mathscr{D}$ of $\mathscr{S}$ and continuous there. The half-space is likewise unloaded outside $\mathscr{D}$.

The corresponding elastic state $u_i(x_1, x_2, x_3)$ is the solution to the boundary value problem

$$\begin{aligned} &\mu \Delta u_i + (\lambda + \mu)\theta_{,i} = 0 \qquad &&\text{for } x_3 > 0\,, \\ &\sigma_{3i}(x_1, x_2, 0) = \hat{p}_i &&\text{for } x_3 = 0 \text{ and } x_\alpha \in \mathscr{D}\,, \end{aligned} \tag{3.1}$$

where σ_{3i} is the stress vector on $x_3 = 0$.

It is well known that problem (3.1) has the solution (given by Boussinesq [**1**])

$$\begin{aligned} u_i = \frac{1}{4\pi\mu}\iint_{\mathscr{D}}\Big\{&\frac{\hat{p}_i}{R} + \frac{(x_i - \xi_i)(x_j - \xi_j)\,\hat{p}_j}{R^3} \\ &+ (1-2\nu)\,[\delta_{3i}\hat{p}_j \ln(R + x_3)_{,j} + \delta_{\alpha i}\hat{p}_j\Big(\frac{x_\alpha - \xi_\alpha}{R + x_3}\Big)_{,j}]\Big\}\,d\xi_1\,d\xi_2 \qquad (\alpha = 1, 2)\,, \end{aligned} \tag{3.2}$$

where, as before, $R^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + x_3^2$. A simple differentiation yields

$$\begin{aligned} &(\ln(R + x_3))_{,j} = \frac{1}{R + x_3}\Big(\delta_{3j} + \frac{x_j - \xi_j}{R}\Big)\,, \\ &\Big(\frac{x_\alpha - \xi_\alpha}{R + x_3}\Big)_{,j} = \frac{\delta_{\alpha j}}{R + x_3} - \frac{x_\alpha - \xi_\alpha}{(R + x_3)^2}\Big(\delta_{3j} + \frac{x_j - \xi_j}{R}\Big)\,. \end{aligned} \tag{3.3}$$

If we substitute (3.3) into (3.2) and multiply u_i by a constant vector $\bar{v}_i$ we obtain

$$u_i\bar{v}_i = \frac{1}{4\pi\mu}\iint_{\mathscr{D}}\{\frac{\hat{p}_i\bar{v}_i}{R} + \frac{(x_i-\xi_i)\bar{v}_i(x_j-\xi_j)\hat{p}_j}{R^3} + (1-2\nu)[(\hat{p}_3 \tag{3.4}$$
$$+\frac{\hat{p}_j(x_j-\xi_j)}{R})\frac{\bar{v}_3}{R+x_3} + \frac{\bar{v}_\alpha\hat{p}_\alpha}{R+x_3} - \frac{\bar{v}_\alpha(x_\alpha-\xi_\alpha)}{(R+x_3)^2}(\hat{p}_3 - \frac{\hat{p}_j(x_j-\xi_j)}{R})]\}d\xi_1 d\xi_2\,,$$

which implies the inequality

$$|u_i\bar{v}_i| \leqslant \frac{1}{4\pi\mu}\iint_{\mathscr{D}}\{\frac{|\hat{p}_i\bar{v}_i|}{R} + \frac{|(x_i-\xi_i)\bar{v}_i|\,|(x_j-\xi_j)\hat{p}_j|}{R^3} + (1-2\nu)[\frac{|\hat{p}_i\bar{v}_i|}{R+x_3}$$
$$+\frac{|\hat{p}_j(x_j-\xi_j)|}{R}(\frac{|\bar{v}_3|}{R+x_3} + \frac{|\bar{v}_\alpha(x_\alpha-\xi_\alpha)|}{(R+x_3)^2}) + \frac{|\hat{p}_3|\,|\bar{v}_\alpha(x_\alpha-\xi_\alpha)|}{(R+x_3)^2}]\}d\xi_1 d\xi_2.$$

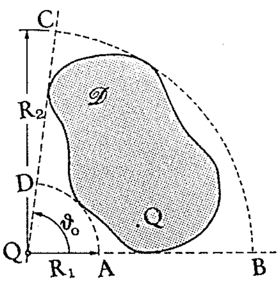
On choosing $\bar{v}_i = u_i(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, where $P(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is the point where the modulus of u_i is maximum, and applying the Cauchy inequality togheter with the inequality $R + x_3 \geqslant R$ we have

$$\max_{\mathscr{B}}\sqrt{u_i u_i} \leqslant \frac{1}{2\pi\mu}\max_{\mathscr{D}}\sqrt{\hat{p}_i\hat{p}_i}\iint_{\mathscr{D}}(\frac{2}{R} + (1-2\nu)\frac{2+\sqrt{2}}{R})d\xi_1 d\xi_2\,. \tag{3.5}$$

In order to bound the integral on the right-hand side of (3.5) we observe that we need only to find an upper bound for the integral

$$\iint_{\mathscr{D}}\frac{1}{R}d\xi_1 d\xi_2 = \iint_{\mathscr{D}}\frac{d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+x_3^2}} \leqslant \iint_{\mathscr{D}}\frac{d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}}\,.$$

When the point $Q(x_1, x_2, 0)$ belongs to $\mathscr{D}$ (fig. 3.1), and Q is taken as the



origin of a system of plane polar coordinates, we can write

$$\iint_{\mathscr{D}} \frac{d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} = \int_0^{2\pi} d\theta \int_0^{\varrho(\theta)} d\varrho \leqslant 2\pi d_M\,, \tag{3.6}$$

where d_M is the diameter of $\mathscr{D}$. When the point $Q(x_1, x_2, 0)$ is exterior to $\mathscr{D}$ and we denote by R_1 and R_2 the minimum and maximum distance from Q to any point of $\mathscr{D}$, we similarly obtain

$$\iint_{\mathscr{D}} \frac{d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} = \int_0^{\theta_c} d\theta \int_{\varrho_1(\theta)}^{\varrho_2(\theta)} d\varrho \leqslant \theta_0(R_2 - R_1)\,, \tag{3.7}$$

where θ_0 is the amplitude of the sector $ABCD$ $(\theta_0 \leqslant 2\pi)$ and $R_2 - R_1 \leqslant d_M$.

Thus, by using (3.6) or (3.7) in (3.5), we conclude that

$$\max_{\bar{B}} \sqrt{u_i\, u_i} \leqslant \frac{1}{\mu}\left[2 + (1-2\nu)(2+\sqrt{2})\right] d_M \max_{\bar{\mathscr{D}}} \sqrt{\hat{p}_i\, \hat{p}_i}\,, \tag{3.8}$$

which bounds *the maximum modulus of the displacement in the elastic half-space in terms of the surface tractions.*

4. - The elastic half-space under body forces

Let us now consider the case in which $\mathscr{S}$, the boundary of B, is free of body forces and the stresses in B are generated by body forces b_i applied on a bounded region D entirely contained in B. For sake of simplicity we assume that the body forces are parallel to the x_3-axis, that is $b_1 = b_2 \equiv 0$, $b_3 \neq 0$, and continuous in $\bar{D}$. The extension of the results to more general load distributions is not difficult.

We consider two points $P(x_1, x_2, x_3)$, $Q(\xi_1, \xi_2, \xi_3)$ and denote by R', R'' the distances

$$R'^2 = (x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2\,,$$

$$R''^2 = (x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3+\xi_3)^2\,,$$

which are the squares of the moduli of the vectors

$$p'_i \equiv (x_1-\xi_1,\, x_2-\xi_2,\, x_3-\xi_3)\,, \qquad p''_i \equiv (x_1-\xi_1,\, x_2-\xi_2,\, x_3+\xi_3)\,.$$

We next define the functions

$$(4.1)\qquad \Omega = -\frac{1}{4\pi\mu}\left(\frac{1}{R''} + \frac{\xi_3(x_3+\xi_3)}{2(1-\nu)R''^3}\right),$$

$$\omega = -\frac{1}{4\pi\mu}\left[\ln\left(R''+x_3+\xi_3\right] - \frac{\xi_3}{2(1-\nu)R''}\right],$$

and observe that, by a known solution due to Mindlin [**5**] the elastic displacement induced by body forces b_3 acting in D, can be written (³)

$$(4.2)\qquad u_i = \iiint_D \Big[\frac{1}{16\pi\mu(1-\nu)}\{(3-4\nu)\,\delta_{i3}\Big(\frac{1}{R'}-\frac{1}{R''}\Big) + [(x_3-\xi_3)\,p'_i\,\frac{1}{R'^3}$$

$$-(x_3+\xi_3)\,p''_i\,\frac{1}{R''^3}]\} + [(x_3-\xi_3)\,\Omega + (1-2\nu)\,\omega]_{,i}$$

$$-4(1-\nu)\,\delta_{i3}\Omega\Big]\,b_3(\xi_1,\xi_2,\xi_3)\,d\xi_1 d\xi_2 d\xi_3\,,$$

where, by using (4.1), we get

$$[(x_3-\xi_3)\,\Omega + (1-2\nu)\,\omega]_{,i} = -\frac{1}{4\pi\mu}\Big\{\frac{\delta_{i3}}{R''} + \frac{\xi_3(x_3+\xi_3)\,\delta_{i3}}{2(1-\nu)R''^3}$$

$$-p''_i\,\frac{x_3-\xi_3}{R''^3} + \delta_{3i}\xi_3\,\frac{x_3-\xi_3}{2(1-\nu)R''^3} - \frac{3}{2(1-\nu)}\,(x_3-\xi_3)\,\xi_3\,(x_3+\xi_3)\,p''_i\,\frac{1}{R''^5}$$

$$+(1-2\nu)(R''+x_3+\xi_3)^{-1}\Big[\frac{p''_i}{R''} + \delta_{i3}\Big] + \frac{1-2\nu}{2(1-\nu)}\,\xi_3 p''_i\,\frac{1}{R''^3}\Big\}.$$

We now take the scalar product of u_i with a constant vector $\bar{v}_i$ and obtain

$$(4.3)\qquad u_i\bar{v}_i = \iiint_D \Big[\frac{1}{16\pi\mu(1-\nu)}\{(3-4\nu)\,\bar{v}_3\Big(\frac{1}{R'}-\frac{1}{R''}\Big) + [(x_3-\xi_3)\,\bar{v}_i p'_i\,\frac{1}{R^3}$$

$$-(x_3+\xi_3)\,\bar{v}_i p''_i\,\frac{1}{R''^3}]\} - \frac{1}{4\pi\mu}\Big\{\frac{\bar{v}_3}{R''} + \frac{\bar{v}_3\xi_3(x_3+\xi_3)}{2(1-\nu)\,R''^3} - \bar{v}_i p''_i\,\frac{x_3-\xi_3}{R''^3}$$

$$+\bar{v}_3\xi_3\,\frac{x_3-\xi_3}{2(1-\nu)R''^3} - \frac{3}{2(1-\nu)}\,\xi_3(x_3-\xi_3)\,\bar{v}_i(x_3+\xi_3)\,p''_i\,\frac{1}{R''^5}$$

(³) Cf. also Solomon [**7**] (9, § 8).

$$+ (1-2\nu)(R''+x_3+\xi_3)^{-1}[p''_i \bar{v}_i \frac{1}{R''} + \bar{v}_3] + \frac{1-2\nu}{2(1-\nu)} \xi_3 \bar{v}_i p''_i \frac{1}{R''^3} \}$$

$$+ \frac{1-\nu}{\pi\mu} \bar{v}_3 \{ \frac{1}{R''} + \frac{\xi_3(x_3+\xi_3)}{2(1-\nu)} \frac{1}{R''^3} \}] b_3(\xi_1, \xi_2, \xi_3) \, d\xi_1 d\xi_2 d\xi_3 .$$

Choose now $\bar{v}_i = u_i(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, where $P(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is the point where $|u_i|$ becomes maximum, and apply the Cauchy inequality combined with the inequalities $R'' \geqslant R'$, $R''+x_3+\xi_3 \geqslant R''$, $\xi_3+x_3 \geqslant \xi_3$. A simple computation gives

$$\max_{\bar{B}} \sqrt{u_i u_i} \leqslant \max_{\bar{D}} |b_3| \iiint_D [\frac{1}{16\pi\mu(1-\nu)} \{ (3-4\nu) \frac{2}{R'} + \frac{2}{R'} \} \tag{4.4}$$

$$+ \frac{1}{4\pi\mu} \{ \frac{1}{R'} + \frac{1}{2(1-\nu)R'} + \frac{1}{R'} + \frac{1}{2(1-\nu)R'} + \frac{3}{2(1-\nu)R'}$$

$$+ \frac{2(1-2\nu)}{R'} + \frac{1-2\nu}{2(1-\nu)R'} \}] \, d\xi_1 d\xi_2 d\xi_3 .$$

In a system of spherical coordinates with origin at $P(x_1, x_2, x_3)$, contained in D, we easily find that

$$\iiint_D \frac{1}{R'} d\xi_1 d\xi_2 d\xi_3 = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\varrho(\varphi,\theta)} \varrho \sin\theta \, d\varrho \leqslant 2\pi d_M^2 , \tag{4.5}$$

where d_M is the diameter of D. When $P(x_1, x_2, x_3)$ is exterior to D it is always possible to choose the polar coordinates with origin in $P(x_1, x_2, x_3)$ so that D is entirely contained within the domain (fig. 4.1)

$$0 \leqslant \varphi \leqslant \varphi_0 , \quad 0 \leqslant \theta \leqslant \theta_0 , \quad 0 \leqslant \varrho \leqslant R_2 \quad (\varphi_0 \leqslant 2\pi, \theta_0 \leqslant \pi, R_2 \leqslant d_M) . \tag{4.6}$$

It follows that

$$\iiint_D \frac{1}{R'} d\xi_1 d\xi_2 d\xi_3 \leqslant \int_0^{\varphi_0} d\varphi \int_0^{\theta_0} d\theta \int_0^{R_2} \varrho \sin\theta \, d\varrho \leqslant \frac{1}{2} \varphi_0 (1-\cos\theta_0) R_2^2 , \tag{4.7}$$

or, better, if we use (4.6),

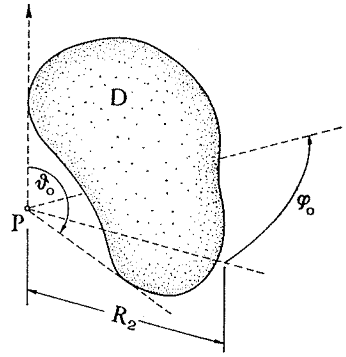
$$\iiint_D \frac{1}{R'} d\xi_1 d\xi_2 d\xi_3 \leqslant 2\pi d_M^2 . \tag{4.8}$$

Thus, since (4.5) and (4.8) give the same upper bound, the formula (4.4) can be replaced by the following

$$\max_{\overline{B}} \sqrt{u_i u_i} \leqslant \max_{\overline{D}} |b_3| \frac{d_M^2}{2\mu} \left\{ 2(3-2\nu) + \frac{3-\nu}{1-\nu} \right\} \tag{4.9}$$

which bounds *the maximum modulus of the displacement in the elastic half-space in terms of the body forces.*

With similar calculations it is easy to derive an analogous bound for $\max_{\overline{B}} \sqrt{u_i u_i}$ when $b_3 = 0$, $b_\alpha \neq 0$ $(\alpha = 1, 2)$.



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Abstract

In this paper we consider three boundary value problems for an elastic half-space: the displacement problem, the surface traction problem, and the case in which body forces are present and the surface is free of traction. In each case, under suitable assumptions of smoothness on the data, it is possible to bound the maximum modulus of the solution in terms of the maximum modulus of the corresponding datum.

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