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Biorthogonal systems in Banach spaces (**)

1. - Introduction.

In this Note we examine the biorthogonal systems in Banach spaces, from the point of view of the convergence of their block sequences; precisely we point out, in § 7, as the progressive loss of these properties of convergence, accompanies the acquisition of better properties, from the general case till to the bases.

In § 3 we give a few constructions, in order to pass from a minimal and not M-basic sequence to an M-basis, and viceversa; we also lightly improve a remark of Singer. Moreover we examine the sequences which can be approximated as we want by minimal sequences.

§ 4 regards the sequences of elements with unitary norm and without convergent subsequences, that we characterize by the existence of subsequences which belong to bibounded biorthogonal systems. Again we examine the sequences that can be approximated as we want by bibounded biorthogonal systems; moreover we consider sequences with intermediate properties between the general minimal sequence and a bibounded biorthogonal system. We raise also two open problems.

In § 5 we examine the structure of the sequences of elements with unitary norm and without subsequences weakly convergent to elements $\neq 0$, connected to the existence of basic subsequences.

§ 6 regards the sequences without block sequences convergent to elements $\neq 0$: we show that these sequences are union of two M-basic sequences. Then

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we compare a few characterizations of the M-bases among the minimal sequences; precisely the sequences $\{x_n\}$ such that, for every infinite complementary subsequences $\{x_{n_1}\}$ and $\{x_{n_2}\}$ of $\{x_n\}$, $[\{x_{n_1}\}] \cap [\{x_{n_2}\}] = \{0\}$; moreover the sequences $\{x_n\}$ such that $\bigcap_{m=1}^{\infty} [\{x_n\}_{n>m}] = \{0\}$: we find that the sequences of the first case, by removing a finite number of elements at the most, always are M-bases: while the sequences of the 2-*nd* case are much more general. We also examine the properties of the sequences sufficiently «near» to the preceding sequences.

In § **7** we give also a general scheme of all the sequences that we consider in this Note, with all the interdependences.

We report, in § **3***, **4***, **5***, **6*** and **7***, the proofs of all theorems, remarks and examples, stated in the preceding paragraphs **3**, **4**, **5**, **6** and **7**.

2. - Notations, definitions and recalls.

Theorems are enumerated with roman figures, lemmas with arabic figures and theorems of recalls with starred roman figures. We shall use $\{n\}$ for the sequence of the natural numbers, R^+ for the positive real semiaxis, $\mathscr{C}$ for the complex field, B for the general separable Banach space, B' for the dual of B, moreover $S_B = \{x \in B;\ \|x\| = 1\}$, $\forall$ = «for every», $\exists$ = «it exists». Let P and Q be subspaces of B, we shall say that P and Q are *quasi complementary subspaces* of B, if $P \cap Q = \{0\}$ and $P + Q$ is dense in B. We shall use $\{x_n\}$ for the sequence $x_1, x_2, \ldots$ of B, $\operatorname{lin}\{x_n\}$ for the linear manifold described by $\{x_n\}$. $[\{x_n\}]$ for the closure of $\operatorname{lin}\{x_n\}$, and we shall say that $\{x_n\}$ is *complete* in B if $[\{x_n\}] = B$.

Let $\{x_n\} \subset B$, we shall say that $\{u_n\}$ is a *block sequence* of $\{x_n\}$ if $\exists$ an increasing sequence $\{t_n\}$ of natural numbers so that: $u_n \in \operatorname{lin}\{x_k\}_{k=t_n+1}^{t_{n+1}}$, $\forall n$. Let $\{x_n\} \subset B$, we shall say that $\{x_n\}$ is

(a) *Hamel basis* (*H-basis*) of $\operatorname{lin}\{x_n\}$, if every finite subsequence of $\{x_n\}$ is linearly independent;

(b) *ω-linearly independent* if $\sum_{1}^{\infty} {}_n\, \alpha_n x_n = 0$, for $\{\alpha_n\} \subset \mathscr{C}$, implies $\alpha_n = 0$, $\forall n$;

(c) *minimal* if $x_m \notin [\{x_n\}_{n \neq m}]$, $\forall m$.

Let $\{x_n\} \subset B$ and $\{f_n\} \subset B'$, we shall say that (x_n, f_n) is a *biorthogonal system*, and that $\{x_n\}$ *belongs to a biorthogonal system*, if $f_m(x_n) = \delta_{mn}$, $\forall m$ and n; moreover, if it is also

$$\|x_n\| \cdot \|f_n\|_{[\{x_k\}]} \leqslant M < +\infty, \quad \text{where} \quad \|f_n\|_{[\{x_k\}]} = \sup \left\{ |f_n(x)| / \|x\|;\ x \in [\{x_k\}] \right\}, \ \forall n,$$

we shall say that (x_n, f_n) is a *bibounded biorthogonal system*. Let (x_n, f_n) be a biorthogonal system, we shall say that $\{x_n\}$ is

(d) *Markuschevich basis* (*M-basis*) of B if $\{x_n\}$ is complete in B and $\{f_n\}$ is *total on* $[\{x_n\}]$ (that is $[\{f_n\}]_\perp \cap [\{x_n\}] = \{0\}$, where $[\{f_n\}]_\perp = \{x \in B;\ f_n(x) = 0,\ \forall n\})$;

(e) *basis with brackets* of B if $\exists$ an increasing sequence $\{q_n\}$ of natural numbers, so that, setting $q_0 = 0$, $x = \sum_{0}^{\infty}{}_n \big(\sum_{q_n+1}^{q_{n+1}}{}_k f_k(x)\, x_k\big)$, $\forall x \in B$;

(f) *basis* of B if we have e) with $q_n = n - 1$, $\forall n$;

(g) *M-basic* (*basic with brackets*) (*basic*) *sequence* if $\{x_n\}$ is M-basis (basis with brackets) (basis) of $[\{x_n\}]$.

Let $\{x_n\} \subset B$ and let $\bar{X}$ be a subspace of $[\{x_n\}]$, we shall say that $\{x_n\}$ is

(h) $\bar{X}$-*non contractive* if $\bigcap_{m=1}^{\infty} [\{x_n\}_{n>m}] = \bar{X}$;

(i) *totally non contractive* (*t-non contractive*) if we have h) with $\bar{X} = [\{x_n\}]$;

(l) *totally contractive* (*t-contractive*) if we have h) with $\bar{X} = \{0\}$;

(m) *M-basoidic* if, $\forall$ infinite complementary subsequences $\{x_{n_1}\}$ and $\{x_{n_2}\}$ of $\{x_n\}$, $[\{x_{n_1}\}] \cap [\{x_{n_2}\}] = \{0\}$.

Let $\{x_n\} \subset B$, with $x_n \neq 0$, $\forall n$, following the terminology of [**11**]$_2$ (p. 63 and p. 53), we shall call *norm* of $\{x_n\}$ the smallest number $K > 0$ so that

$$\Big\|\sum_{1}^{m}{}_n \alpha_n x_n\Big\| \leq K \Big\|\sum_{1}^{m+p}{}_n \alpha_n x_n\Big\| , \qquad \forall \{\alpha_n\}_{n=1}^{m+p} \subset \mathscr{C} ;$$

moreover we shall set

$$(1) \qquad \sigma_n = \big\{x \in [\{x_k\}_{k=1}^{n}];\ \|x\| = 1\big\} , \qquad P^{(n)} = [\{x_k\}_{k>n}] , \qquad \forall n .$$

Let us recall the following characterizations for minimal, M-basic and basic sequences.

I* – *Let* $\{x_n\} \subset B$, *then*

(a) ([**9**], see also [**11**]$_2$, p. 54) $\{x_n\}$ *minimal* $\Leftrightarrow$ $\{x_n\}$ *belongs to a biorthogonal system* $\Leftrightarrow$ *if* $\lim_{p\to\infty} \sum_{1}^{m_p}{}_n \alpha_{pn} x_n = 0$, *then* $\lim_{p\to\infty} \alpha_{pn} = 0$, $\forall n \Leftrightarrow \exists \{C_n\} \subset R^+$, *with* $1 \leq C_n < +\infty$, $\forall n$, *so that*

$$\Big\|\sum_{1}^{m} \alpha_n x_n\Big\| \leq C_m \Big\|\sum_{1}^{m+p}{}_n \alpha_n x_n\Big\| , \quad \forall \{\alpha_n\}_{n=1}^{m+p} \subset \mathscr{C} \quad \Leftrightarrow \operatorname{dist}(\sigma_n, P^{(n)}) > 0 , \qquad \forall n .$$

(b) $\{x_n\}$ *M-basic* $\Leftrightarrow$ ([**3**]$_2$, see also [**11**]$_1$, p. 171) $\{x_n\}$ *minimal and t-contractive* $\Leftrightarrow$ [**1**] $\{x_n\}$ *minimal and M-basoidic.*

(c) $\{x_n\}$ *basic* $\Leftrightarrow$ ([**4**], see also [**11**]$_2$, p. 58) $\{x_n\}$ *has a norm K with* $1 \leqslant K < +\infty \Leftrightarrow$ ([**3**]$_1$, see also [**11**]$_2$, p. 58) $\inf\{\text{dist}\,(\sigma_n, P^{(n)}); 1 \leqslant n < \infty\} = 1/K > 0$.

Let $F \subset B'$, we recall that F is *norming* on B if $\exists K \in R^+$ so that $K\|x\| \leqslant \sup\{|f(x)|/\|f\|; f \in F\}$, $\forall x \in B$. On the existence of basic subsequences we have

II* – ([**6**], see also [**10**], p. 128). *Let* $\{x_n\} \subset B$, *then*: $\{x_n\}$ *M-basic* ($\{x_n/\|x_n\|\}$ *without subsequences weakly convergent to elements* $\neq 0$) ($\lim f(x_n/\|x_n\|) = 0$, $\forall f \in F \subset B'$, *with F norming on B*) $\Rightarrow \{x_n\}$ *has an infinite basic subsequence.*

Let $P \subset B$, then $P^\perp = \{f \in B'; f(x) = 0, \forall x \in P\}$. On the extension of M-basic sequences we recall a few results of Singer.

III* – ([**11**]$_3$, p. 184-185). *Let* $\{x_n\}$ *be an M-basic sequence of B, with* (x_n, f_n) *biorthogonal system, then*

(a) $\exists\{h_n\} \subset B'$ *total on B with* (x_n, h_n) *biorthogonal system,*

(b) $\exists\{g_n\} \subset [\{x_n\}]^\perp$, *with* $\{f_n\} \cup \{g_n\}$ *total on B.*

Finally, about the properties of the «near» sequences, it is well known that

IV* – ([**8**] and [**9**], see also [**11**]$_2$, p. 98 and [**11**]$_1$, p. 171). *Let* $\{x_n\}$ *be a minimal sequence of B, then* $\exists\{\varepsilon_n\} \subset R^+$ *so that,* $\forall\{u_n\} \subset B$, *with* $\|u_n - x_n\| < \varepsilon_n$, $\forall n$, $\{u_n\}$ *is minimal; moreover* $\{u_n\}$ *is M-basic (basic) if* $\{x_n\}$ *is M-basic (basic).*

3. - Minimal and M-basic sequences.

A minimal sequence $\{y_n\}$ is in general $\overline{Y}$-non contractive; moreover, by (b) of Theorem I*, $\{y_n\}$ is M-basic if and only if $\overline{Y} = \{0\}$. Then, if $\overline{Y} \neq \{0\}$, Theorem I gives a construction of an M-basis of $[\{y_n\}]$, starting from $\{y_n\}$; while (a) of Theorem II allows the inverse passage.

I. *Let* (y_n, h_n) *be a biorthogonal system of B and let* $\{y_n\}$ *be* $\overline{Y}$*-non contractive, then* $\Rightarrow \exists$ *an M-basis* $\{\bar{y}_n\}$ *of* $\overline{Y}$, *with* $(\bar{y}_n, \bar{h}_n)$ *biorthogonal system, so that setting* $y_n^* = y_n - \sum_1^n{}_k \bar{h}_k(y_n)\bar{y}_k$, $\forall n$; $\{y_n^*\} \cup \{\bar{y}_n\}$ *is M-basis of* $[\{y_n\}]$, *with* $(y_n^*, h_n) \cup (\bar{y}_n, \bar{h}_n)$ *biorthogonal system.*

II. *Let* (y_n, h_n) *be a biorthogonal system of* B, *and let* (y_{n_1}, h_{n_1}) *be an infinite subsystem of* (y_n, h_n), *then*

(a) $\exists\{z_n\}$ *complete in* $[\{y_n\}]$ *with* (z_n, h_{n_1}) *biorthogonal system,*

(b) $\exists\{g_n\}$ *complete in* $[\{h_n\}]$ *with* (y_{n_1}, g_n) *biorthogonal system.*

Next corollary follows by precedent theorems; moreover, for $\overline{Y} = \{0\}$, we have Theorem III*.

Corollary 1. *Let* (y_n, h_n) *be a biorthogonal system of* B *and let* $\{y_n\}$ *be* $\overline{Y}$*-non contractive, then*

(a) $\exists\{f_n\} \subset B'$, *total on a subspace of* B *quasi complementary to* $\overline{Y}$, *with* (y_n, f_n) *biorthogonal system,*

(b) $\exists\{g_n\} \subset [\{y_n\}]^{\perp}$ *so that* $\{h_n\} \cup \{g_n\}$ *is total on a subspace of* B *quasi complementary to* $\overline{Y}$.

We shall now consider a few properties of the sequences sufficiently «near» to minimal sequences. In a precedent note **[12]**$_2$ we defined that a sequence $\{x_n\}$ of B has *property* P if, $\forall x \in [\{x_n\}]$, $\exists$ two infinite complementary subsequences $\{x_{n_1}\}$ and $\{x_{n_2}\}$ of $\{x_n\}$, which depend on x, so that $x \in [\{x_{n_1}\}] + [\{x_{n_2}\}]$. Moreover, in another note **[12]**$_4$, we defined that a sequence $\{x_n\}$ of B is $\overline{X}$*-overfilling* if every infinite subsequence of $\{x_n\}$ is $\overline{X}$-non contractive, where $\overline{X}$ is a subspace of $[\{x_n\}]$. Then, for $\overline{X} = [\{x_n\}]$, we have the already known (**[7]** p. 193, see also **[10]** p. 113) definition that $\{x_n\}$ is *overfilling*, that is every infinite subsequence of $\{x_n\}$ is complete in $[\{x_n\}]$. In the same Note we examined the properties of the «near» sequences for the general $\overline{X}$-non contractive and $\overline{X}$-overfilling sequences; then we consider now the same problem, with the further condition that the sequence is minimal.

III. *Let* $\{y_n\}$ *be a minimal sequence of* B, *then* $\exists\{\varepsilon_n\} \subset R^+$ *so that,* $\forall\{u_n\} \subset B$, *with* $\|u_n - y_n\| < \varepsilon_n$, $\forall n$, *we have the following equivalences*

(a) $\{y_n\}$ *is* $\overline{Y}$*-non contractive* $\Leftrightarrow \{u_n\}$ *is* $\overline{Y}$*-non contractive,*

(b) $\{y_n\}$ *is* $\overline{Y}$*-overfilling* $\Leftrightarrow \{u_n\}$ *is* $\overline{Y}$*-overfilling,*

(c) $\{y_n\}$ *has property* $P \Leftrightarrow \{u_n\}$ *has property* P.

Instead the general sequence with property P does not keep this property for sufficiently «near» sequences, in fact:

Example 1. Let $\{z_n\}$ be a t-non contractive sequence complete in a Banach space B_1. Let $\{x_n\}$ be a sequence of linearly independent vectors, with $x_n \notin B_1$, $\forall n$. Setting

$$u_m = \sum_1^{3^{2m-1}}{}_n x_n - \sum_{2\cdot 3^{2m-1}+1}^{3^{2m}}{}_n x_n , \qquad \forall m ,$$

we define in $\lim\{x_n\}$ the following norm

$$\|x\| = \min\{I(x, v);\ v \in \lim\{u_n\}\} ,$$

where, for $x = \sum_1^m{}_n \alpha_n x_n$ and $v = \sum_1^m{}_n \beta_n u_n$ (α_n and β_n can be $= 0$),

$$I(x, v) = \sum_1^m{}_n \left(|\alpha_n - \beta_n| \left(1 - \frac{1}{2^n}\right) + \frac{|\beta_n|}{2^n} \right) .$$

Moreover we define in $\lim\{x_n\} + \lim\{z_n\}$ the following norm, extension of the preceding norms: $\|x + z\| = \|x\| + \|z\|$, $\forall x \in \lim\{x_n\}$ and $z \in \lim\{z_n\}$. Then, if B_2 is the Banach space completion of $\lim\{x_n\} + \lim\{z_n\}$ in this norm, $B_2 = [\{x_n\}] + B_1$. Now, setting $y_{2n-1} = x_{2n-1}$ and $y_{2n} = z_n$, $\forall n$, $\{y_n\}$ has property P; but, $\forall\{\varepsilon_n\} \subset R^+$, $\exists\{v_n\} \subset B_2$ and without property P, with $\|v_n - y_n\| \leqslant \varepsilon_n$, $\forall n$.

Finally we consider the sequences which can be approximated as we want by minimal sequences of B.

IV. *Let $\{y_n\} \subset B$, with $[\{y_n\}]$ p-codimensional subspace of B, then*

(a) $p = +\infty \Rightarrow \{y_n\}$ *can be approximated as we want by minimal sequences of* B,

(b) $p < +\infty$ *and* $\{y_n\}$ *can be approximated as we want by minimal sequences of* $B \Rightarrow \{y_n\} = \{y_{n_1}\} \cup \{y_{n_2}\}_{n=1}^q$, *where* $0 \leqslant q \leqslant p$, *moreover* $\{y_{n_1}\}$ *is minimal and complete in* $[\{y_n\}]$.

4. - Sequences of elements of unitary norm and without convergent subsequences.

In a precedent note **[12]**$_3$ we called a sequence $\{y_n\}$ *convergent of order* p to an H-basic sequence $\{\bar{y}_n\}_{n=1}^p$ of S_B, with $1 \leqslant p \leqslant +\infty$, if

(2) $\quad y_n = \sum_1^{q_n}{}_k \alpha_{nk} \bar{y}_k + \bar{y}_n^*$, with $q_n = p$ if $p < +\infty$ and $q_n \geqslant n$ if $p = +\infty$, $\forall n$;

moreover $\quad \lim_{n\to\infty} (y_n - \sum_1^{m-1}{}_k \alpha_{nk}\bar{y}_k) / \|y_n - \sum_1^{m-1}{}_k \alpha_{nk}\bar{y}_k\| = \bar{y}_m$, for $1 \leqslant m < p + 1$.

In particular we called the sequence $\{y_n\}$ of (2) *elementary convergent of order* p if $\{\bar{y}_n\}_{n=1}^p$ was M-basic, with $(\bar{y}_n, \bar{h}_n)_{n=1}^p$ biorthogonal system, if $\{y_n^*\}$ was basic, or $y_n^* = 0$, $\forall n$, and if $\alpha_{nk} = \bar{h}_k(y_n)$, $\forall k$ and n. We recall

V*. *Let* $\{y_n\} \subset B$, *then*

(a) [**12**]$_4$. *Every infinite subsequence of* $\{y_n\}$ *has an infinite minimal subsequence* $\Leftrightarrow \{y_n\}$ *H-basic and without subsequences convergent of infinite order and overfilling* $\Leftrightarrow \forall \overline{Y}$*-overfilling subsequence* $\{y_{n_1}\}$ *of* $\{y_n\}$, $\overline{Y}$ *is an infinite codimensional subspace of* $[\{y_{n_1}\}]$,

(b) [**12**]$_3$. $\{y_n\} \subset S_B$ *has a convergent subsequence* $\Rightarrow \{y_n\}$ *has an elementary convergent subsequence.*

Therefore a convergent sequence of S_B can have an infinite minimal subsequence. On the other hand, if $\{x_n\}$ is weakly convergent to $\bar{x}$, for a well known theorem of Mazur $\bar{x} \in \bigcap_{m=1}^{\infty} [\{x_n\}_{n>m}]$. Hence, by (b) of Theorem I*, we have that Theorem II* affirms that

$$\text{(3)} \qquad \{y_n/\|y_n\|\} \textit{ is without subsequences weakly convergent to elements } \neq 0 \Leftrightarrow \textit{every infinite subsequence of } \{y_n\} \textit{ has an infinite basic subsequence}.$$

Consequently it remains to define the properties of the sequences of S_B weakly convergent to elements $\neq 0$, but without convergent subsequences; we can expect that these sequences will have subsequences with intermediate properties between the general minimal sequence and the basic sequence. We answer this question by next theorem

V. *Let* $\{y_n\} \subset B$, *then*

(a) $\{y_n/\|y_n\|\}$ *without convergent subsequences* $\Leftrightarrow$ *every infinite subsequence of* $\{y_n\}$ *has an infinite subsequence which belongs to a bibounded biorthogonal system,*

(b) $\{y_n\}$ *without convergent subsequences of infinite order* $\Rightarrow$ *every infinite subsequence of* $\{y_n\}$ *has an infinite minimal subsequence with property* P.

Preceding theorem involves that the minimal sequence with property P has intermediate properties between the general minimal sequence and the bibounded biorthogonal system (we observe that (see [**12**]$_2$) the sequence $\{x_n\}$ of Example 1 is minimal, but without property P). Then we precise this particular by next theorem, on the sequences with property P.

VI. *Let* $\{y_n\} \subset B$, *then*

(a) $\{y_n\}$ *belongs to a bibounded biorthogonal system* (*or* $\{y_n\}$ *is basic with brackets*) $\Rightarrow$ *every subsequence of* $\{y_n\}$ *has property* P,

(b) *every subsequence of* $\{y_n\}$ *has property* $P \Leftrightarrow$ *every minimal subsequence of* $\{y_n\}$ *has property* P.

In § **3** we examined the properties of the « near » sequences for the minimal sequences and for the sequences with property P; then we consider now the same problem for the bibounded biorthogonal systems.

VII. *Let* $\{y_n\} \subset S_B$ *and let* $\lambda \cup \{\varepsilon_n\} \subset R^+$, *with* $\sum_{1}^{\infty}{}_n \varepsilon_n = 1$ *and* $\lambda < 1$, *then*

(a) $\{y_n\}$ *belongs to a biorthogonal system* (y_n, h_n) *with* $\|h_n\|_{[\{y_k\}]} \leqslant M < +\infty$ $\forall n \Rightarrow \forall \{u_n\} \subset B$, *with* $\|u_n - y_n\| < \varepsilon_n$, $\forall_n$, *it follows that* $\{u_n\} = \{u_{n_1}\} \cup \{u_{n_2}\}_{n=1}^{p}$ *with* $0 \leqslant p < +\infty$, *moreover* $\{u_{n_1}\}$ *is complete in* $[\{u_n\}]$ *and belongs to a bibounded biorthogonal system* (u_{n_1}, g_n); *in particular, if*

$$\|u_n - y_n\| < \lambda\varepsilon_n / M \quad \forall n, \text{ then } p = 0 \quad \text{and} \quad \|g_n\|_{[\{u_k\}]} < \frac{M(M+\lambda)}{(1-\lambda)(M-\lambda)} \quad \forall n .$$

(b) $\exists \{v_n\} \subset B$ *with* $\|v_n - y_n\| < \varepsilon_n$, $\forall n$, *moreover* $\{v_n\}$ *belongs to a bibounded biorthogonal system* $\Rightarrow \{y_n\} = \{y_{n_3}\} \cup \{y_{n_4}\}_{n=1}^{q}$, *with* $0 \leqslant q < +\infty$, *moreover* $\{y_{n_3}\}$ *is complete in* $[\{y_n\}]$ *and belongs to a bibounded biorthogonal system.*

We complete this paragraph with two equivalent open problems.

Problem 1. Let $\{y_n\}$ be a convergent sequence of infinite order of B, has $\{y_n\}$ always an infinite subsequence with property P?

Problem 2. Let $\{y_n\} \subset B$, has $\{y_n\}$ always an infinite subsequence with property P?

5. - Sequences of elements of unitary norm and without subsequences weakly convergent to elements $\neq 0$.

Let $\{x_n\} \subset B$, we call *nucleus* of $\{x_n\}$ the set

$$N\{x_n\} = \{x \in B \text{ so that } \forall \{x_{n_1}\}_{n=1}^{\infty} \subseteq \{x_n\},\ x \in [\{x_{n_1}\}]\} . \tag{4}$$

Moreover we say that $\{x_n\}$ is *denucleated* if, $\forall$ infinite subsequence $\{x_{n_1}\}$ of $\{x_n\}$, $N\{x_{n_1}\} = \{0\}$. If $\{x_n\}$ is weakly convergent to $\bar{x}$, $\bar{x} \in N\{x_n\}$; on the other hand a t-contractive sequence is denucleated; therefore, by (3),

(5) $\{y_n/\|y_n\|\}$ *without subsequences weakly convergent to elements* $\neq 0 \Leftrightarrow \{y_n\}$ *denucleated*.

The structure of the denucleated sequences is quite general, in fact

Remark 1. (a) $\exists$ *a denucleated and complete* $\{x_n\}$ *of* B, *that is not union of a finite number of* H*-basic sequences.*

(b) $\exists$ *a denucleated,* H*-basic and complete* $\{y_n\}$ *of* B, *that is not union of a finite number of minimal sequences.*

(c) $\exists$ *a denucleated, minimal and complete* $\{z_n\}$ *of* B, *that is not union of a finite number of* M*-basic sequences.*

By (c) of precedent remark and by (b) of Theorem I* it is plain that the denucleated sequences are much more general than the t-contractive sequences. In order to complete precedent remark, we have to consider the particular denucleated and minimal sequences $\{x_n\}$, union of two M-basic sequences $\{y_n\}$ and $\{z_n\}$: a problem stated by Singer [**11**]$_3$ was if, with the further condition of $[\{y_n\}] \cap [\{z_n\}] = \{0\}$, $\{x_n\}$ becomes M-basic. We recall [**1**] that Courage-Davis answered this question, by means of a sequence $\{x_n\}$, union of two basic sequences $\{y_n\}$ and $\{z_n\}$, with $[\{y_n\}] \cap [\{z_n\}] = \{0\}$, moreover $\{x_n\}$ belongs to a biorthogonal system (x_n, f_n) with $[\{f_n\}]_\perp \cap [\{x_n\}] \neq \{0\}$. Next remark strengthens this solution.

Remark 2. $\exists \{x_n\} \subset B$, *complete and union of two basic sequences* $\{y_n\}$ *and* $\{z_n\}$ *with* $[\{y_n\}] \cap [\{z_n\}] = \{0\}$, *moreover* $\{x_n\}$ *belongs to a biorthogonal system* (x_n, f_n) *with* $[\{f_n\}]_\perp$ *infinite dimensional subspace of* $[\{x_n\}]$.

About the near sequences it is obvious, by (5), that if $\{x_n\} \subset S_B$ is denucleated, every sequence $\{u_n\}$ of B with $\lim_{n\to\infty} \|u_n - x_n\| = 0$ is also denucleated.

It is also obvious that, in a reflexive space B, (3) and (5) becomes (see Theorem II*).

$\{y_n\}$ *denucleated* $\Leftrightarrow$ $\{y_n/\|y_n\|\}$ *weakly convergent to* $0 \Leftrightarrow \exists$ *an* M*-basis* $\{x_n\}$ *of* B, *with* (x_n, f_n) *biorthogonal system and* $[\{f_n\}]$ *norming on* B, *so that*

$$\lim_{n\to\infty} f_k(y_n/\|y_n\|) = 0\,, \qquad \forall k\,.$$

6. - Sequences without block sequences convergent to elements $\neq 0$.

The following equivalence is immediate.

(6) $\{y_n\}$ *without block sequences convergent to elements* $\neq 0 \Leftrightarrow \{y_n\}$ t*-contractive.*

We consider now particular t-contractive sequences.

We say that a sequence $\{y_n\} \subset B$ is *norming* if $\exists$ a not decreasing sequence $\{l_n\}$ of natural numbers, and K with $1 \leqslant K < +\infty$, so that

$$\left\| \sum_{1}^{n} {}_k \alpha_k y_k \right\| \leqslant K \left\| \sum_{1}^{n} {}_k \alpha_k y_k + \sum_{l_n+1}^{l_n+m} {}_k \alpha_k y_k \right\| , \qquad \forall \{\alpha_k\}_{k=1}^{n} \cup \{\alpha_k\}_{k=l_n+1}^{l_n+m} \subset \mathscr{C} . \tag{7}$$

In fact, by (7), $\sup\{\text{dist}\,(y, [\{y_k\}_{k>n}])\,;\, 1 \leqslant n < +\infty\} \geqslant \|y\|/K$, $\forall y \in [\{y_n\}]$; therefore (see [**10**], p. 121-122 and Lemma I.11) if (y_n, h_n) is a biorthogonal system and if $\{y_n\}$ is complete in B, *$\{y_n\}$ is norming $\Leftrightarrow [\{h_n\}]$ is norming on B.* We recall ([**5**], see also [**10**], p. 123) that

(8) *$\{y_n\}$ minimal and norming $\Rightarrow \{y_n\}$ union of two basic with brackets sequences .*

We consider also particular norming sequences.

Generalizing the terminology of [**11**]$_2$ (p. 63) we say that $\{y_n\} \subset B$ has a *norm with brackets K*, with $1 \leqslant K < +\infty$, if $\exists$ an increasing sequence $\{q_n\}$ of natural numbers, so that we have (7) with $l_n = q_{m+1}$ for $q_m + 1 \leqslant n \leqslant q_{m+1}$, $\forall m$; that is, setting $q_0 = 0$

$$\left\| \sum_{0}^{p} {}_n \Big(\sum_{q_n+1}^{q_{n+1}} {}_k \alpha_k y_k \Big) \right\| \leqslant K \left\| \sum_{0}^{p+m} {}_n \Big(\sum_{q_n+1}^{q_{n+1}} {}_k \alpha_k y_k \Big) \right\| , \qquad \forall \{\alpha_k\}_{k=1}^{q_{p+m+1}} \subset \mathscr{C} . \tag{9}$$

About the structure of these sequences we have the following theorem, where (b) is a light improvement of (8).

VIII. *Let $\{y_n\}$ be an H-basic sequence of B, then*

(a) *$\{y_n\}$ t-contractive $\Rightarrow \{y_n\}$ is union of two M-basic sequences,*

(b) *$\{y_n\}$ norming $\Rightarrow \{y_n\}$ is union of two basic with brackets sequences,*

(c) *$\{y_n\}$ has a norm with brackets $\Leftrightarrow \{y_n\}$ is basic with brackets.*

Following remark completes preceding theorem.

Remark 3. $\exists$ *an H-basic and norming sequence $\{y_n\}$ of B, union of two M-bases of B.*

That is the sequence $\{y_n\}$ of precedent remark, by removing a finite number of elements, does never become minimal.

We consider now the t-contractive sequences under another point of view. We say in this Note that $\{y_n\} \subset B$ is *ω-b-linearly independent* (generalization of definition (b) of § **2**) if, $\forall$ series with brackets $\sum_{0}^{\infty} {}_m \Big(\sum_{t_m+1}^{t_{m+1}} {}_n \alpha_n y_n \Big)$ convergent to 0, it follows that $\alpha_n = 0$, $\forall n$.

By (b) of Theorem I* we know that the t-contractive sequences, with the M-basoidic sequences, characterize the M-basic among the minimal sequences. Then next theorem affirms that the M-basoidic sequences not only are t-contractive, but already are M-basic, except particular cases connected with the lacking of property P; however in these cases, by removing a finite number of elements, the sequence again becomes M-basic.

IX. *Let* $\{y_n\}$ *be an infinite sequence of* B, *then*

(a) $\{y_n\}$ *M-basoidic and not minimal* $\Leftrightarrow \{y_n\} = \{y_{n_1}\} \cup \{y_{n_2}\}_{n=1}^{p}$, *with* $1 \leq p < +\infty$, *where* $\{y_{n_1}\}$ *is an M-basis of* $[\{y_n\}]$ *without property* P; *precisely*, $\forall y \in \operatorname{lin}\{y_{n_2}\}_{n=1}^{p}$, *with* $y \neq 0$, *there are never two infinite complementary subsequences* $\{y_{n_3}\}$ *and* $\{y_{n_4}\}$ *of* $\{y_{n_1}\}$ *so that* $y \in [\{y_{n_3}\}] + [\{y_{n_4}\}]$;

(b) $\{y_n\}_{n\neq m}$ *has property* P $\forall m$: $\{y_n\}$ *M-basoidic* $\Leftrightarrow \{y_n\}$ *M-basic*;

(c) $\{y_n\}$ *M-basoidic* $\Rightarrow \{y_n\}$ ω*-b-linearly independent.*

The following characterization is immediate consequence of definition and of (a) of Theorem I*.

$\{y_n\}$ *M-basic* $\Leftrightarrow \forall$ *convergent sequence* $\{\sum_{1}^{q_m} {}_n \alpha_{mn} y_n\}_{m=1}^{\infty}$, $\lim_{m\to\infty} \sum_{1}^{q_m} {}_n \alpha_{mn} y_n = 0$ *if and only if* $\lim_{m\to\infty} \alpha_{mn} = 0$, $\forall n$.

We pass now to consider the «near» sequences.

The norming, the t-contractive and the M-basoidic sequences do not keep their properties for «near» sequences, in fact:

Remark 3′. *Let* $\{y_n\}$ *be the sequence of Remark* 3, *and let* $\{\varepsilon_n\} \subset R^+$ *then*: $\Rightarrow \exists$ *a t-non contractive sequence* $\{u_n\}$ *complete in* B, *with* $\|u_n - y_n\| \leq \varepsilon_n$, $\forall n$.

Example 1′. Let $\{x_n\}$ and $\{u_n\}$ be the sequences of Example 1, we have that $\lim_{n\to\infty} u_n = \bar{x}$, moreover $\bar{x} \cup \{x_n\}$ is M-basoidic but not minimal. Then, $\forall \{\varepsilon_n\} \subset R^+$, $\exists \{v_n\} \subset [\{x_n\}]$, with $\|v_1 - \bar{x}\| < \varepsilon_1$ and $\|v_{n+1} - x_n\| < \varepsilon_{n+1}$, $\forall n$, so that $\{v_n\}$ is not M-basoidic.

About the sequences which can be approximated as we want by t-contractive (M-basic) sequences we have

X. *Let* $\{y_n\} \subset B$, *with* $[\{y_n\}]$ *p-codimensional subspace of* B, *then*

(a) $p \leq +\infty$ *and*, $\forall \{\varepsilon_n\} \subset R^+$, $\exists$ *a t-contractive sequence* $\{u_n\}$ *of* B *with* $\|u_n - y_n\| < \varepsilon_n$, $\forall n \Rightarrow \{y_n\}$ *is t-contractive*,

(b) $p < +\infty$ *and*, $\forall \{\varepsilon_n\} \subset R^+$, $\exists$ *an M-basic* (*norming and M-basic*) (*basic with brackets*) (*basic*) *sequence* $\{v_n\}$ *of* B *with* $\|v_n - y_n\| < \varepsilon_n$, $\forall n \Rightarrow \{y_n\} = \{y_{n_1}\} \cup \{y_{n_2}\}_{n=1}^{q}$, *with* $0 \leq q \leq p$, *moreover* $\{y_{n_1}\}$ *is M-basis* (*norming and M-basis*) (*basis with brackets*) (*basis*) *of* $[\{y_n\}]$.

Following example completes a) of precedent theorem, where we cannot replace « t-contractive » by « M-basic », in fact:

Example 2. Let $\{y_n\}$ be the sequence of Remark 3 and let B_3 be another Banach space, with an M-basis $\{z_n\} \subset S_{B_3}$ and $B_3 \cap B = \{0\}$. We define in $\operatorname{lin}\{y_n\} + \operatorname{lin}\{z_n\}$ the following norm (extension of the norms in $\operatorname{lin}\{y_n\}$ and $\operatorname{lin}\{z_n\}$)

$$\|y + z\| = \|y\| + \|z\|\,, \qquad \forall y \in \operatorname{lin}\{y_n\} \text{ and } z \in \operatorname{lin}\{z_n\}\,.$$

Let B_4 be the Banach space completion of $\operatorname{lin}\{y_n\} + \operatorname{lin}\{z_n\}$ in this norm (hence $B_4 = B + B_3$). Then, $\forall \{\varepsilon_n\} \subset R^+$, setting $u_n = y_n + \varepsilon_n z_n$, $\{u_n\}$ is M-basic and $\|u_n - y_n\| = \varepsilon_n$, $\forall n$.

7. - General view and synthesis.

It is now opportune to give a comprehensive look at all the sequences considered in this Note, then

in Table 1, *for an H-basic $\{y_n\}$ of B, a flow of strict implications (that is the inverse implications do not hold) connects all the properties that we till now considered, with all the main interdependences.*

In order to complete the flow of Table 1 we remark that the ω-b linearly independent sequences are quite different from the ω-linearly independent sequences; in fact (1) every infinite H-basic sequence (hence a t-non contractive sequence) has an infinite ω-linearly independent subsequence. Our aim is also to point out as the property of a sequence, to approach a basic sequence, is strictly connected with the properties of convergence of the block sequences. Precisely the more the elements of the block sequences are « distant » on S_B, the more the sequence has better properties towards the basic sequence.

In a precedent note [**11**]$_3$ we defined that $\{y_n\}$ is *perturbation of order p of* $\{x_n\}$ by the elements $u_1, \ldots, u_p$ $(1 \leqslant p < +\infty)$, if $y_m = \alpha_m x_m + \sum_{1}^{p}{}_n \alpha_{mn} u_n$ with $\alpha_m \cup \{\alpha_{mn}\}_{n=1}^{p} \subset \mathscr{C}$, $\forall m$. Moreover, for the sake of convenience, in this paragraph we say that

(a) $\{y_n\}$ is an *almost-Cauchy sequence* if, $\forall p \in \{n\}$, $\exists \{y_{n_p}\}_{n=1}^{\infty} \subseteq \{y_n\}$, with $\|y_{n_p} - y_{(n+m)_p}\| < 1/p$, $\forall n$ and m;

(b) $\{y_n\}$ has a *quasi norm* $M \in R^+$ if, $\forall \{\alpha_n\}_{n=1}^{m+p} \subset \mathscr{C}$ and $\forall$ permutation $\{\tilde{y}_n\}$ of $\{y_n\}$, $\|\sum_{1}^{m}{}_n \alpha_n \tilde{y}_n\| \leqslant mM \|\sum_{1}^{m+p}{}_n \alpha_n \tilde{y}_n\|$.

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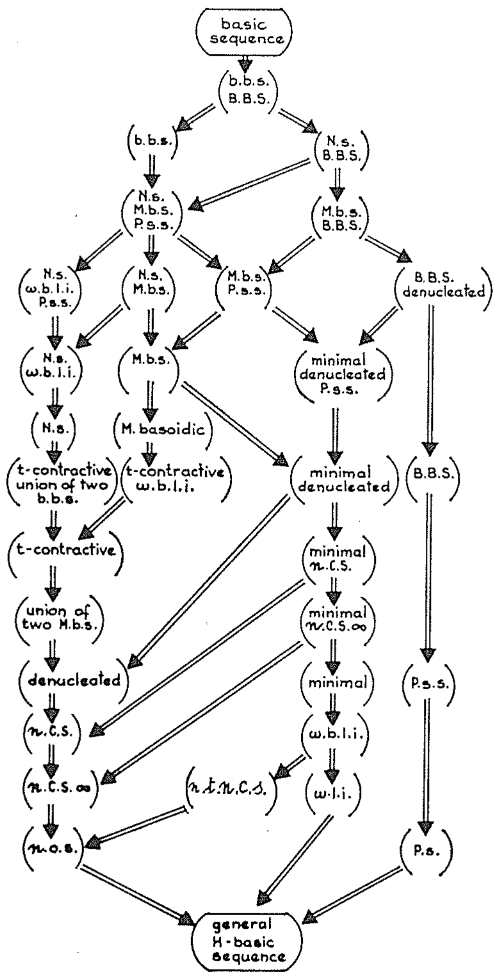
(1) Erdos-Strauss, *On linear independence of sequences in a Banach space,* Pacific J. Math. 3 (1953), 689-694.

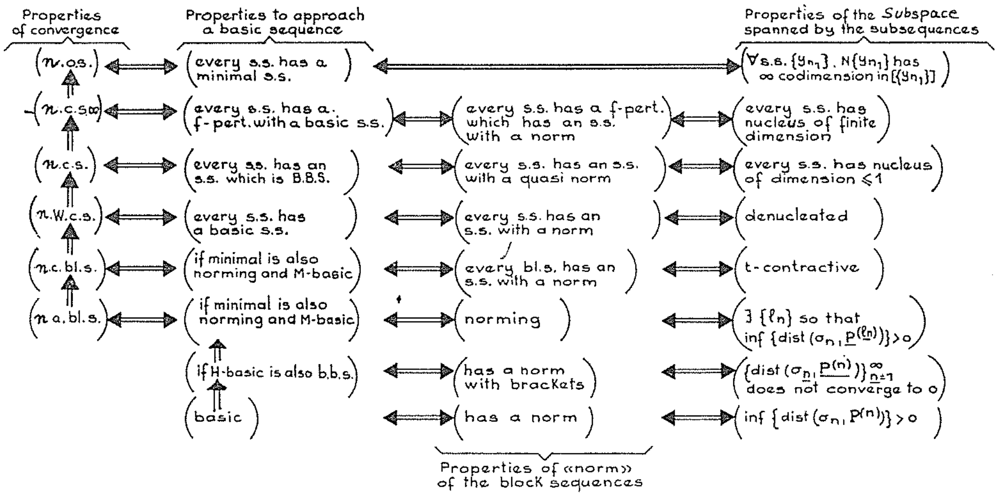
The sequences with a quasi norm (see Lemma 5 in § 7*) characterize the bibounded biorthogonal systems. Then

Let $\{y_n\} \subset S_B$, Table 2 *describes as, for the block sequences of* $\{y_n\}$, *the decrease of the properties of convergence and of the spanned space, accompanies the increase of the properties of « norm » and to approach a basic sequence.*

Table 1



where (we are concerned with an H-basic sequence $\{y_n\}$ of B):
b.b.s. = basic with brackets sequence; M.b.s. = M-basic sequence; N.s. = norming sequence; B.B.S. = sequence that belongs to a bibounded biorthogonal system; P.s. = sequence with property P; P.s.s. = every subsequence has property P; ω.b.l.i. = ω-b-linearly independent; ω.l.i. = ω-linearly independent; n.c.s. = $\{y_n/\|y_n\|\}$ has not convergent subsequences; n.c.s.∞ = $\{y_n\}$ has not convergent subsequences of infinite order; n.o.s. = $\{y_n\}$ has not overfilling subsequences; n.t.n.c.s. = $\{y_n\}$ has not t-non contractive subsequences. We point out that the inverse implications do not hold.



where (we are concerned with an $\{y_n\}$ of S_B):
s.s. = infinite subsequence; bl.s. = infinite block sequence made of elements of S_B; n.o.s. = $\{y_n\}$ has not overfilling s.s.; n.c.s.∞ = $\{y_n\}$ has not s.s. convergent of infinite order; n.c.s. = $\{y_n\}$ has not convergent s.s.; n.w.c.s. = $\{y_n\}$ has not s.s. weakly convergent to elements $\neq 0$; n.c.bl.s. = $\{y_n\}$ has not convergent bl.s.; n.a.bl.s. = $\{y_n\}$ has not almost-Cauchy bl.s.; f.pert. = perturbation of finite order; b.b.s. = basic with brackets sequence; B.B.S. = sequence that belongs to a bibounded biorthogonal system; nucleus of $\{y_n\} = N\{y_n\} = \cap\{[\{y_{n_1}\}]; \{y_{n_1}\}_{n=1}^{\infty} \subseteq \{y_n\}\}$; $\sigma_n = \{y \in [\{y_k\}_{k=1}^{n}];\ \|y\| = 1\}$; $P^{(n)} = [\{y_k\}_{k>n}]$.

About the «near» sequences we expound Theorem IV* again by means of next corollary, which follows by Theorems III, IV, VII and X.

Corollary 2. *For* $\{y_n\} \subset B$, $\exists\{\varepsilon_n\} \subset R^+$ *so that*, $\forall\{u_n\} \subset [\{y_n\}]$ *with* $\|u_n - y_n\| < \varepsilon_n$, $\forall n$, $\{u_n\}$ *is minimal* (*is minimal with property P*) (*belongs to a bibounded biorthogonal system*) (*is M-basic*) (*is norming and M-basic*) (*is basic with brackets*) (*is basic*) *if and only if* $\{y_n\}$ *has the same property.*

3*. - Proofs of § 3.

Let (x_n, f_n) be a biorthogonal system of B and $\{y_n\} \subset B$, moreover $\forall n$ let $q_n \in \{n\}$ so that $f_{q_n}(y_n) \neq 0$ and $f_k(y_n) = 0$ for $1 \leqslant k < q_n$; then we say that $\{y_n\}$ is *weakly ordered in* $\{x_n\}$, and we write $\{y_n\}_{w-in\{x_n\}}$, if $q_{n+1} > q_n$, $\forall n$. We recall

VI* – **[12]**$_1$. (a) $\{x_n\}$ *M-basis of* B *and* $\{u_n\}$ *H-basic sequence of* $B \Rightarrow \exists\{y_n\}$, *permutation of a sequence* $\{\hat{y}_n\}$ *with* $\lim\{\hat{y}_k\}_{k=1}^{n} = \lim\{u_k\}_{k=1}^{n}$, $\forall n$, *so that* $\{y_n\}_{w-in\{x_n\}}$;

(b) *(x_n, f_n) biorthogonal system of B and $\{y_n\}_{w-in\{x_n\}} \Rightarrow (y_n, h_n)$ is a biorthogonal system, with $h_n = (f_{q_n} - \sum_1^{n-1} {}_k h_k f_{q_n}(y_k))/f_{q_n}(y_n)$, $\forall n$, moreover $\{y_n\}$ is M-basic if $\{x_n\}$ is M-basis of B.*

Moreover for the M-bases we recall

VII* – *Let P and Q be two quasi complementary subspaces of B, then:*

(a) [**11**]$_3$ *Let $\{x_n\}$ be an M-basis of P with (x_n, f_n) biorthogonal system and $Q \subseteq [\{f_n\}]_\perp \Rightarrow \exists$ an M-basis $\{y_n\}$ of Q, with $\{x_n\} \cup \{y_n\}$ M-basis of B;*

(b) ([**10**], p. 121) *Let $\{z_n\}$ be an M-basis of $P \Rightarrow \exists \{w_n\} \subset Q$, with $\{z_n\} \cup \{w_n\}$ M-basis of B;*

(c) ([**13**], see also [**11**]$_3$, corollary 4) *$\exists$ an M-basis $\{u_n\}$ of P and an M-basis $\{v_n\}$ of Q so that $\{u_n\} \cup \{v_n\}$ is M-basis of B.*

Proof of Theorem I. Let $\{\tilde{y}_n\}$ be an M-basis of $\overline{Y}$, with $(\tilde{y}_n, \tilde{h}_n)$ biorthogonal system, and let us set

(10) $$y_n = \sum_1^n {}_k \tilde{h}_k(y_n)\tilde{y}_k + y_n^*, \qquad \forall n.$$

Firstly we affirm that $\{y_n^*\}$ is t-contractive. In fact let $y^* \in \bigcap_{m=1}^\infty [\{y_n^*\}_{n>m}]$, $\forall m$; $\exists y_1^{(m)} \in \operatorname{lin} \{y_n^*\}_{n>m} = \operatorname{lin} \{y_n - \sum_1^n {}_k \tilde{h}_k(y_n)\tilde{y}_k\}_{n>m}$ with $\|y_1^{(m)} - y^*\| < 1/(2m)$, hence $y_1^{(m)} = y_2^{(m)} + y_3^{(m)}$ with $y_2^{(m)} \in \operatorname{lin}\{y_n\}_{n>m}$ and $y_3^{(m)} \in \operatorname{lin}\{\tilde{y}_n\} \subseteq \overline{Y}$, that is $\exists y_4^{(m)} \in \operatorname{lin}\{y_n\}_{n>m}$, with $\|y_4^{(m)} - y_3^{(m)}\| < 1/(2m)$; consequently, setting $y_2^{(m)} + y_4^{(m)} = y^{(m)} \in \operatorname{lin}\{y_n\}_{n>m}$, $\|y^* - y^{(m)}\| < 1/m$, whence $y^* \in \overline{Y}$. On the other hand, by (10), $\forall m$ and $\forall y \in [\{y_n^*\}_{n>m}]$ it follows that $\tilde{h}_k(y) = 0$ for $1 \leqslant k \leqslant m$, therefore $\tilde{h}_n(y^*) = 0$, $\forall n$, hence $y^* = 0$ because $\{\tilde{h}_n\}$ is total on $\overline{Y}$. Now we affirm that $h_n(y_m^*) = \delta_{nm}$, $\forall n$ and m. In fact $\forall y \in [\{y_n\}_{n>m}]$ it is $h_k(y) = 0$ for $1 \leqslant k \leqslant m$, hence $\forall \bar{y} \in \overline{Y}$, $h_n(\bar{y}) = 0$, $\forall n$, therefore $h_n(\tilde{y}_k) = 0$, $\forall n$ and k; that is, by (10), $h_n(y_m^*) = h_n(y_m)$, $\forall n$ and m. Consequently by (b) of Theorem I* it follows that

(11) $\{y_n^*\}$ is M-basic, with (y_n^*, h_n) biorthogonal system.

Now we affirm that $[\{y_n^*\}] \cap \overline{Y} = \{0\}$. In fact, $\forall y \in [\{y_n^*\}] \cap \overline{Y}$, $h_n(y) = 0$, $\forall n$ because $y \in \overline{Y}$, hence $y = 0$ because $y \in [\{y_n^*\}]$ and $\{h_n\}$ is total on $[\{y_n^*\}]$ by (11). Therefore, by (11) and by (a) of Theorem VII*, $\exists \{\hat{y}_n\}$ M-basis of $\overline{Y}$ with $\{y_n^*\} \cup \{\hat{y}_n\}$ M-basis of $Y = [\{y_n\}]$ and with $(y_n^*, h_n) \cup (\hat{y}_n, \hat{h}_n)$ biorthogonal system. Consequently, by (a) of Theorem VI*, $\exists \{\bar{y}_n\} \subset \overline{Y}$ with $\operatorname{lin}\{\bar{y}_k\}_{k=1}^n =$

$=\operatorname{lin}\{\tilde{y}_k\}_{k=1}^n$, $\forall n$ and so that $\exists$ a permutation $\{y_n^+\}$ of $\{\bar{y}_n\}$ with $\{y_n^+\}_{w\text{-in}\{\hat{y}_n\}}$, hence $\{y_n^*\}\cup\{y_n^+\}=\{y_1^*, y_1^+, y_2^*, y_2^+, \ldots\}$ is weakly ordered in $\{y_n^*\}\cup\{\hat{y}_n\}$; that is, by (b) of Theorem VI*, $\{y_n^*\}\cup\{\bar{y}_n\}$ is an M-basis of Y with $(y_n^*, h_n)\cup(\bar{y}_n, \bar{h}_n)$ biorthogonal system. Finally

$$\operatorname{lin}\{\tilde{y}_k\}_{k=1}^n = \operatorname{lin}\{\bar{y}_k\}_{k=1}^n \Rightarrow \sum_1^n{}_k \tilde{h}_k(y_n)\tilde{y}_k = \sum_1^n{}_k \alpha_{nk}\bar{y}_k\,, \qquad \forall n\,. \tag{12}$$

By (10) and (12) we have for $1\leqslant m\leqslant n$

$$\alpha_{nm} = \bar{h}_m\Big(\sum_1^n{}_k \alpha_{nk}\bar{y}_k\Big) = \bar{h}_m\Big(\sum_1^n{}_k \tilde{h}_k(y_n)\tilde{y}_k\Big) = \bar{h}_m(y_n - y_n^*) = \bar{h}_m(y_n)\,, \qquad \forall n\,.$$

This completes the proof of Theorem I.

Proof of Theorem II. Let (y_{n_2}, h_{n_2}) be the subsystem of (y_n, h_n) complementary to (y_{n_1}, h_{n_1}) (we suppose (y_{n_2}, h_{n_2}) infinite, if is finite the proof is little different).

(a) We can suppose $\{y_n\}\subset S_B$, then let us set

$$z_n = \sum_1^n{}_k 10^{n^3-n(k-1)} y_{k_2} + y_{n_1}\,, \qquad \forall n\,. \tag{13}$$

By (13) $\forall m$ it follows that

$$\lim_{n\to\infty} \frac{z_n - \sum_1^{m-1}{}_k 10^{n^3-n(k-1)} y_{k_2}}{\big\|z_n - \sum_1^{m-1}{}_k 10^{n^3-n(k-1)} y_{k_2}\big\|} = \lim_{n\to\infty} \frac{\sum_m^n{}_k 10^{n^3-n(k-1)} y_{k_2} + y_{n_1}}{\big\|\sum_m^n{}_k 10^{n^3-n(k-1)} y_{k_2} + y_{n_1}\big\|} =$$

$$= \lim_{n\to\infty} \frac{y_{m_2} + \sum_{m+1}^n{}_k (y_{k_2}/10^{n(k-m)}) + y_{n_1}/10^{n^3-n(m-1)}}{\big\|y_{m_2} + \sum_{m+1}^n{}_k (y_{k_2}/10^{n(k-m)}) + y_{n_1}/10^{n^3-n(m-1)}\big\|} = y_{m_2}\,.$$

Therefore, by (2), $\{z_n\}$ is convergent of infinite order to $\{y_{n_2}\}$, hence $\{y_{n_2}\}\subset [\{z_n\}]$, that is $\{z_n\}$ is complete in $[\{y_n\}]$. Moreover, by hypothesis and by (13), $h_{n_1}(z_m) = \delta_{nm}$, $\forall n$ and m.

(b) Let us call $T(y)$ the element of B'' (the dual of B') so that $T(y)(h) = h(y)$, $\forall h\in B'$, for $y\in B$. Then $(h_n, T(y_n))$ is a biorthogonal system of B', therefore by (a) $\exists\{g_n\}$ complete in $[\{h_n\}]$ with $(g_n, T(y_{n_1}))$ (that is (y_{n_1}, g_n)) biorthogonal system. This completes the proof of Theorem II.

Proof of Corollary 1. (a) By hypothesis and by Theorem I $\exists\ \{\bar{y}_n\}$ so that

$$(14)\qquad [\{\bar{y}_n\}] = \overline{Y}\,, \qquad \text{moreover } \{\tilde{y}_n\}\cup\{\bar{y}_n\} \text{ is } M\text{-basis of } [\{y_n\}]\,,$$

$$\text{where } \tilde{y}_n = y_n - \sum_1^n{}_k\, \alpha_{nk}\bar{y}_k\,,\ \forall n\,.$$

Moreover by (b) of Theorem VII* $\exists\{z_n\}\subset B$ so that

$$(15)\qquad \{\tilde{y}_n\}\cup\{\bar{y}_n\}\cup\{z_n\} \text{ is } M\text{-basis of } B\,,$$

$$\text{with } (\tilde{y}_n, \tilde{h}_n)\cup(\bar{y}_n, \bar{h}_n)\cup(z_n, \tilde{g}_n) \text{ biorthogonal system}\,.$$

Let us set

$$(16)\qquad f_n = \tilde{h}_n + \sum_1^n{}_k\, 10^{n^3-n(k-1)}\|\tilde{h}_n\|\tilde{g}_k/\|\tilde{g}_k\|\,, \qquad \forall n\,.$$

By (16) proceeding as for (13), we find that $\{f_n\}$ is complete in $[\{\tilde{h}_n\}\cup\{\tilde{g}_n\}]$; therefore by (15) $\{f_n\}$ is total on a subspace of B quasi complementary to $\overline{Y}$. Moreover by (15) and (16) $f_m(\tilde{y}_n) = \delta_{mn}$ and $f_m(\bar{y}_n) = 0$, $\forall m$ and n, consequently by (14) $f_m(y_n) = f_m(\tilde{y}_n + \sum_1^n{}_k\,\alpha_{nk}\bar{y}_k) = f_m(\tilde{y}_n) = \delta_{mn}$, $\forall m$ and n.

(b) Following **[11]**$_3$ (p. 184) it is sufficient to set $g_n = h_n - f_n$, $\forall n$. This completes the proof of Corollary 1.

Proof of Theorem III. If (y_n, h_n) is a biorthogonal system let us set

$$(17)\qquad \varepsilon_n = 1/(10^{n+1}\|h_n\|)\,, \qquad \forall n\,.$$

$\forall\{u_n\}\subset B$, with $\|u_n - y_n\| < \varepsilon_n$, $\forall n$ and $\forall\{\alpha_n\}_{n=m}^{m+p}\subset\mathscr{C}$, by (17) we have

$$\Big|\|\sum_m^{m+p}{}_n\,\alpha_n y_n\| - \|\sum_m^{m+p}{}_n\,\alpha_n u_n\|\Big| \leqslant \|\sum_m^{m+p}{}_n\,\alpha_n(y_n - u_n)\| = \|\sum_m^{m+p}{}_n\,(y_n - u_n)\,\big(h_n(\sum_m^{m+p}{}_k\,\alpha_k y_k)\big)\| \leqslant$$

$$\leqslant \sum_m^{m+p}{}_n\,\|y_n - u_n\|\big(\|h_n\|\cdot\|\sum_m^{m+p}{}_k\,\alpha_k y_k\|\big) \leqslant \|\sum_m^{m+p}{}_k\,\alpha_k y_k\|\,\big(\sum_m^{m+p}{}_n\,\|h_n\|\varepsilon_n\big) =$$

$$= \|\sum_m^{m+p}{}_k\,\alpha_k y_k\|\sum_m^{m+p}{}_n\,\frac{1}{10^{n+1}} \leqslant \|\sum_m^{m+p}{}_n\,\alpha_n y_n\|\,\frac{1}{10^m}\,.$$

Consequently

$$(18)\quad \|\sum_{m}^{m+p}{}_n\alpha_n y_n\| \left(1-\frac{1}{10^m}\right) \leqslant \|\sum_{m}^{m+p}{}_n \alpha_n u_n\| \leqslant \left(1+\frac{1}{10^m}\right) \|\sum_{m}^{m+p}{}_n \alpha_n y_n\|, \quad \forall\{\alpha_n\}_{n=m}^{m+p} \subset \mathscr{C} .$$

(a) Suppose that $\bar{y} \in \bigcap_{m=1}^{\infty}[\{y_n\}_{n>m}]$. Then $\exists\{\bar{\alpha}_{mn}\}_{n=m}^{m+r_m} \subset \mathscr{C}$ so that

$$(19)\quad \|\bar{y}-\sum_{m}^{m+r_m}{}_n \bar{\alpha}_{mn} y_n\| < \frac{\|\bar{y}\|}{10^m}, \qquad \forall m .$$

By (19) and by proof of (18) it follows that

$$\|\sum_{m}^{m+r_m}{}_n \bar{\alpha}_{mn}(u_n-y_n)\| \leqslant \|\sum_{m}^{m+r_m}{}_n \bar{\alpha}_{mn} y_n\| \frac{1}{10^m} \leqslant (\|\bar{y}\| +$$

$$+ \|\bar{y}-\sum_{m}^{m+r_m}{}_n \bar{\alpha}_{mn} y_n\|) \frac{1}{10^m} < \frac{\|\bar{y}\|}{10^m}\left(1+\frac{1}{10^m}\right) .$$

Consequently, by (19)

$$\|\bar{y}-\sum_{m}^{m+r_m}{}_n \bar{\alpha}_{mn} u_n\| \leqslant \|\bar{y}-\sum_{m}^{m+r_m}{}_n \bar{\alpha}_{mn} y_n\| + \|\sum_{m}^{m+r_m}{}_n \bar{\alpha}_{mn}(u_n-y_n)\| < \frac{\|\bar{y}\|}{10^m}\left(2+\frac{1}{10^m}\right), \quad \forall m .$$

That is $\bar{y} \in \bigcap_{m=1}^{\infty}[\{u_n\}_{n>m}]$, hence $\bigcap_{m=1}^{\infty}[\{y_n\}_{n>m}] \subseteq \bigcap_{m=1}^{\infty}[\{u_n\}_{n>m}]$. The proof of viceversa is the same.

(b) Let $\{y_n\}$ be $\bar{Y}$-overfilling, then $\bigcap_{m=1}^{\infty}[\{y_{n_1}\}_{n>m}] = \bar{Y}$, $\forall$ infinite subsequence $\{y_{n_1}\}$ of $\{y_n\}$, therefore $\bigcap_{m=1}^{\infty}[\{u_{n_1}\}_{n>m}] = \bar{Y}$ by (a) $\forall$ infinite subsequence $\{u_{n_1}\}$ of $\{u_n\}$, that is $\{u_n\}$ is $\bar{Y}$-overfilling. The proof of viceversa is the same.

(c) We recall (see § **9-10** of Chapter I of [**11**]$_2$) that, setting

$$T(\sum_{1}^{m}{}_n \alpha_n y_n) = \sum_{1}^{m}{}_n \alpha_n u_n , \qquad \forall\{\alpha_n\}_{n=1}^{m} \subset \mathscr{C} ,$$

by (18) T is an isomorphism (= linear homeomorphism) of $[\{y_n\}]$ onto $[\{u_n\}]$. Then suppose that $\{y_n\}$ has property P and let $u \in [\{u_n\}]$. If $y = T^{-1}(u)$, by hypothesis $\exists$ two infinite complementary subsequences $\{y_{n_1}\}$ and $\{y_{n_2}\}$ of $\{y_n\}$ so that $y = y_{01} + y_{02}$ with $y_{01} \in [\{y_{n_1}\}]$ and $y_{02} \in [\{y_{n_2}\}]$. On the other hand, if $u_{01} = T(y_{01})$ and $u_{02} = T(y_{02})$, by definition of T $u_{01} \in [\{u_{n_1}\}]$ and $u_{02} \in [\{u_{n_2}\}]$. Therefore $u = T(y) = T(y_{01} + y_{02}) = u_{01} + u_{02}$, that is $\{u_n\}$ has property P. The proof of viceversa is obvious. This completes the proof of Theorem III. About the «near» sequences we recall

VIII* – (a) (Gurarij, see [**10**], p. 163). *Let* $\{x_n\} \subset B$, *then* $\exists \{\varepsilon_n\} \subset R^+$ *so that,* $\forall \{u_n\} \subset [\{x_n\}]$ *with* $\|u_n - x_n\| < \varepsilon_n$, $\forall n$, $[\{u_n\}] = [\{x_n\}]$.

(b) [**12**]$_4$. *Let* $\{y_n\} \subset B$ *and suppose that* $\exists$ *a subspace* $\overline{Y}$ *of* $[\{y_n\}]$ *and a subspace* P *of* B *so that*

$$\{\textit{closure of } (\overline{Y} + P)\} = \bigcap_{m=1}^{\infty} \{\textit{closure of } ([\{y_n\}_{n>m}] + P)\} : \Rightarrow \exists \{\varepsilon_n\} \subset R^+$$

so that, $\forall \{u_n\} \subset B$ *with* $\|u_n - y_n\| < \varepsilon_n$, $\forall n$,

$$\{\textit{closure of } (\overline{Y} + P)\} \subseteq \bigcap_{m=1}^{\infty} \{\textit{closure of } ([\{u_n\}_{n>m}] + P)\} .$$

Proof of Example 1. By hypothesis and by [**12**]$_2$ (see example) we have that

(20) $\{x_n\}$ is M-basic and without property P; precisely $\lim_{n\to\infty} u_n = \overline{x}$ and, if $\{x_{n_1}\}$ and $\{x_{n_2}\}$ are two infinite complementary subsequences of $\{x_n\}$, $\forall m$ if $x_{01m} \in \operatorname{lin}\{x_{n_1}\}$ and $x_{02m} \in \operatorname{lin}\{x_{n_2}\}$ so that $\|\overline{x} - (x_{01m} + x_{02m})\| < 1/m$, it follows that $\|x_{01m}\| > m$ and $\|x_{02m}\| > m$.

It is obvious that $\{y_n\}$ has property P, because $[\{y_n\}] = [\{y_{2n-1}\}] + [\{y_{2n}\}]$, by the definition of norm. On the other hand $\{z_n\}$ is t-non contractive, hence by (b) of Theorem VIII* $\exists \{\eta_n\} \subset R^+$ so that

$$(21) \qquad \forall \{w_n\} \subset B_2 \text{ with } \|w_n - y_{2n}\| \leqslant \eta_n , \ \forall n , \quad [\{y_{2n}\}] = B_1 \subseteq \bigcap_{m=1}^{\infty} [\{w_n\}_{n>m}] .$$

Then let us set

$$(22) \qquad v_{2n-1} = y_{2n-1} = x_{2n-1} , \qquad v_{2n} = y_{2n} + x_{2n} \cdot \min\{\varepsilon_{2n}, \eta_n\} , \qquad \forall n .$$

By (21), (22) and by hypothesis it follows that $\{v_n\}$ is complete in B_2, hence $\overline{x} \in [\{v_n\}]$. Let now $\{v_{n_1}\}$ and $\{v_{n_2}\}$ be two infinite complementary subsequences of $\{v_n\}$ and suppose that

$$(23) \quad \exists v_{01m} \in \operatorname{lin}\{v_{n_1}\} \text{ and } v_{02m} \in \operatorname{lin}\{v_{n_2}\} \text{ so that } \|\overline{x} - (v_{01m} + v_{02m})\| < \frac{1}{m} , \ \forall m .$$

By (22) and (23) it follows that

$$(24) \quad \begin{cases} v_{01m} = y_{01m} + x_{01m} \text{ with } x_{01m} \in \operatorname{lin}\{x_{n_1}\} \text{ and } y_{01m} \in \operatorname{lin}\{y_{2n}\} \\ v_{02m} = y_{02m} + x_{02m} \text{ with } x_{02m} \in \operatorname{lin}\{x_{n_2}\} \text{ and } y_{02m} \in \operatorname{lin}\{y_{2n}\} . \end{cases}$$

On the other hand, by the definition of norm in B_2, by (23) and (24) it follows that

$$\|\bar{x} - (x_{01m} + x_{02m})\| \leqslant \|\bar{x} - (x_{01m} + x_{02m})\| + \|y_{01m} + y_{02m}\| =$$
$$= \|\bar{x} - (x_{01m} + x_{02m}) - (y_{01m} + y_{02m})\| = \|\bar{x} - (v_{01m} + v_{02m})\| < 1/m\ , \ \forall m\ .$$

Therefore by (20) $\|x_{01m}\| > m$ and $\|x_{02m}\| > m$, consequently by (24) $\|v_{01m}\| = \|x_{01m} + y_{01m}\| = \|x_{01m}\| + \|y_{01m}\| \geqslant \|x_{01m}\| > m$ and $\|v_{02m}\| > m$, $\forall m$. That is $\bar{x} \notin [\{v_{n_1}\}] + [\{v_{n_2}\}]$, hence $\{v_n\}$ has not property P. This completes the proof of Example 1.

Proof of Theorem IV. (a) By hypothesis and by Theorem VII* $\exists$ a biorthogonal system (z_n, g_n) of B with $\{z_n\} \subset S_B$ and $\{y_n\} \subset [\{g_n\}]_1$. Then, $\forall \{\varepsilon_n\} \subset R^+$, it is sufficient to set

$$(25) \qquad u_n = y_n + \varepsilon_n z_n \qquad \text{and} \qquad h_n = g_n/\varepsilon_n\ , \qquad \forall n\ .$$

By (25) $\|u_n - y_n\| = \varepsilon_n$, $\forall n$, moreover $h_m(u_n) = (g_m/\varepsilon_m)(y_n + \varepsilon_n z_n) = (\varepsilon_n/\varepsilon_m)\cdot g_m(z_n) = \delta_{mn}$, $\forall m$ and n, hence $\{u_n\}$ is minimal.

(b) Suppose by absurd that

$$(26) \qquad \{y_n\} = \{y_{n_1}\} \cup \{y_{n_2}\}_{n=1}^q \quad \text{with} \quad \{y_{n_2}\}_{n=1}^q \subset [\{y_{n_1}\}] \quad \text{and} \quad q > p\ .$$

By hypothesis $\exists \{w_n\}_{n=1}^p \subset B$ so that

$$(27) \qquad \{w_n\}_{n=1}^p \ \text{is } H\text{-basic and } \{y_n\} \cup \{w_n\}_{n=1}^p \ \text{is complete in } B\ .$$

By (26) and (27) $\{y_{n_1}\} \cup \{w_n\}_{n=1}^p$ also is complete in B; therefore, by hypothesis and by (a) of Theorem VIII*, $\exists \{\varepsilon_n\} \subset R^+$ and $\{v_n\} \subset B$ so that

$$(28) \qquad \{v_n\} = \{v_{n_1}\} \cup \{v_{n_2}\}_{n=1}^q \ \text{is minimal,} \qquad \|v_n - y_n\| < \varepsilon_n\ , \qquad \forall n\ ,$$

moreover $B = [\{v_{n_1}\}] + \text{lin}\{w_n\}_{n=1}^p$.

By (28) it follows that

$$(29) \qquad v_{n_2} = v_{01n} + w_{0n}\ , \qquad \text{with } v_{01n} \in [\{v_{n_1}\}]\ ,\ w_{0n} \in \text{lin}\{w_n\}_{n=1}^p\ , \qquad 1 \leqslant n \leqslant q\ .$$

By (28) and (29) $\{w_{0n}\}_{n=1}^q$ is H-basic, otherwise, if $w_{0\bar{n}} \in \text{lin}\{w_{0n}\}_{n(\neq\bar{n})=1}^q$, it would be $v_{\bar{n}_2} \in [\{v_{n_1}\}] + \text{lin}\{v_{n_2}\}_{n(\neq\bar{n})=1}^q$, while $\{v_n\}$ is minimal. On the other hand, by (26), it is impossible that $\{w_{0n}\}_{n=1}^q$ is H-basic, because $\{w_{0n}\}_{n=1}^q \subset \text{lin}\{w_n\}_{n=1}^p$ and $q > p$. This completes the proof of Theorem IV.

4*. - Proofs of § 4.

Firstly we recall

IX* – [**12**]$_3$. *Let* $\{x_n\}\subset S_B$ *and without convergent subsequences, then* $\Rightarrow\exists$ *an infinite subsequence* $\{y_n\}$ *of* $\{x_n\}$ *which is basic, otherwise*

(30) $y_n=\bar{\alpha}_n\bar{y}+y_n^*$ *with* $a<\|y_n^*\|<A$, $\forall n$, $\{a,A\}\subset R^+$ *and* $\bar{y}\neq 0$; *moreover* $\{\bar{\alpha}_n\}\subset\mathscr{C}$ *with* $\lim\limits_{n\to\infty}\bar{\alpha}_n=1$, *while* $\{y_n^*\}$ *is basic and weakly convergent to* 0.

X* – *Let* $\{x_n\}\subset S_B$ *and* $\{f_n\}\subset B'$, *we have the following equivalences*: (x_n, f_n) *biorthogonal system with* $\sup\limits_n\|f_n\|_{[\{x_k\}]}=M<+\infty\Leftrightarrow$ ([**3**]$_2$, see also [**11**]$_1$, p. 165) (x_n, f_n) *biorthogonal system and* $\lim\limits_{n\to\infty}f_n(x)=0$, $\forall x\in[\{x_k\}]\Leftrightarrow$ ([**3**]$_2$, see also [**11**]$_1$, p. 165) dist $(x_n, [\{x_k\}_{k\neq n}])\geqslant 1/M$, $\forall n\Leftrightarrow$ ([**11**]$_1$, p. 165) $\|\sum_1^m{}_n\,\alpha_n x_n\|\geqslant\max\{|\alpha_n|/M;\ 1\leqslant n\leqslant m\}$, $\forall\{\alpha_n\}_{n=1}^m\subset\mathscr{C}$.

1. (y_n, h_n) *biorthogonal system of* B *and* $\bar{y}\in[\{y_n\}]$, *then* $\Rightarrow\exists$ *an increasing sequence* $\{\bar{t}_n\}$ *of natural numbers and* $\{\bar{\alpha}_n\}\subset\mathscr{C}$ *so that*

(31) $$\|\bar{y}-(\sum_1^{\bar{t}_m}{}_n\,h_n(\bar{y})y_n+\sum_{\bar{t}_m+1}^{\bar{t}_{m+1}}{}_n\,\bar{\alpha}_n y_n)\|<\frac{1}{m}\,,\qquad\forall m\,.$$

Proof. By hypothesis we have that

(32) $$\bar{y}=\lim_{m\to\infty}(\sum_1^{q_m}{}_n\,\alpha_{mn}y_n)\qquad\text{with}\qquad\lim_{m\to\infty}\alpha_{mn}=h_n(\bar{y})\,,\ \forall n\,.$$

Suppose to have defined $\{\bar{t}_n\}_{n=1}^m$ and $\{\bar{\alpha}_n\}_{n=1}^{\bar{t}_m}$, by (32) $\exists\{\beta_{mn}\}_{n=1}^{\bar{t}_{m+1}}\subset\mathscr{C}$, with $\bar{t}_{m+1}>\bar{t}_m$,

$$\|\sum_1^{\bar{t}_{m+1}}{}_n\,\beta_{mn}y_n-\bar{y}\|<\frac{1}{2m}\quad\text{and}\quad\|\sum_1^{\bar{t}_m}{}_n\,(h_n(\bar{y})-\beta_{mn})\,y_n\|<\frac{1}{2m}\,.$$

Therefore, setting $\beta_{mn}=\bar{\alpha}_n$ for $\bar{t}_m+1\leqslant n\leqslant\bar{t}_{m+1}$, the thesis is proved.

2. (y_n, h_n) *bibounded biorthogonal system of* $B\Rightarrow$ *every subsequence of* $\{y_n\}$ *has property* P.

Proof. Suppose that $\bar{y}\in[\{y_n\}]$ and let $\{\bar{t}_n\}$ and $\{\bar{\alpha}_n\}$ be the sequences of (31). By Theorem X* $\lim\limits_{n\to\infty}h_n(\bar{y})y_n=0$, hence $\exists$ an increasing sequence $\{n_1\}$

of natural numbers so that

$$(33)\qquad (n+1)_1 > n_1 + 1, \quad \text{moreover, setting} \quad \bar{t}_{n_1} = n_2\,, \ \|h_{n_2}(\bar{y})y_{n_2}\| < \frac{1}{2^n},\ \forall n\,.$$

By (33) the series $\sum_1^\infty{}_n h_{n_2}(\bar{y})y_{n_2}$ converges to an element $\bar{y}_{02}$ of $[\{y_{n_2}\}]$. By (31) and (33) it follows that

$$\|(\sum_1^{m-1}{}_n\ (\sum_{n_2+1}^{(n+1)_2-1}{}_k\ h_k(\bar{y})y_k) + \sum_{m_2+1}^{\bar{t}_{m_1}+1}{}_n\ \bar{\alpha}_n y_n) - (y - \bar{y}_{02})\| = \|(\sum_1^{m_2}{}_n h_n(\bar{y})y_n + \sum_{m_2+1}^{\bar{t}_{m_1}+1}{}_n\ \bar{\alpha}_n\, y_n) -$$

$$- (\bar{y} - \sum_{m+1}^{\infty}{}_n h_{n_2}(\bar{y})y_{n_2})\| = \|(\sum_1^{\bar{t}_{m_1}}{}_n\ h_n(\bar{y})\, y_n + \sum_{\bar{t}_{m_1}+1}^{\bar{t}_{m_1}+1}{}_n\ \bar{\alpha}_n y_n - \bar{y}) + \sum_{m+1}^{\infty}{}_n h_{n_2}(\bar{y})y_{n_2}\|$$

$$< \frac{1}{m_1} + \|\sum_{m+1}^{\infty}{}_n h_{n_2}(\bar{y})y_{n_2}\| < \frac{1}{m} + \sum_{m+1}^{\infty}{}_n \frac{1}{2^n} = \frac{1}{m} + \frac{1}{2^m}\,,\ \forall m\,.$$

On the other hand, if $\{y_{n_3}\}$ is the infinite subsequence of $\{y_n\}$ complementary to $\{y_{n_2}\}$, by (33) $\{y_n\}_{n=m_2+1}^{\bar{t}_{m_1}+1} \subset \{y_{n_3}\}$, $\forall m$, therefore precedent inequality says that $\bar{y} - \bar{y}_{02} \in [\{y_{n_3}\}]$, that is $\bar{y} \in [\{y_{n_2}\}] + [\{y_{n_3}\}]$. This completes the proof of Lemma 2.

Proof of Theorem V. (a) The proof of implication $\Leftarrow$ is obvious, by 2-nd equivalence of Theorem X*. Let us prove the implication $\Rightarrow$: by Theorem IX* it is sufficient to prove the thesis when $\{y_n\}$ has an infinite subsequence, which we call $\{y_n\}$ again, as the sequence of (30). By (b) of Theorem I* $\bigcap_{m=1}^{\infty}[\{y_n^*\}_{n>m}] = \{0\}$; hence $\exists \bar{m} \in \{n\}$ so that $\bar{y} \notin [\{y_n^*\}_{n>\bar{m}}]$; therefore, without losing of generality, we can suppose

$$(34)\qquad \bar{y} \notin [\{y_n^*\}]\,.$$

Suppose by absurd that $\exists$ an infinite subsequence $\{n_1\}$ of $\{n\}$ so that

$$(35)\qquad \|y_{n_1} - y_{0n}\| < 1/n \qquad \text{with} \qquad y_{0n} \in \operatorname{lin}\{y_k\}_{k\neq n_1}\,, \qquad \forall n\,.$$

By (30) $y_{n_1} = \bar{\alpha}_{n_1}\bar{y} + y_{n_1}^*$ and $y_{0n} = \beta_n\bar{y} + y_{0n}^*$, with $y_{0n}^* \in [\{y_k^*\}_{k\neq n_1}]$, that is

$$(36)\qquad y_{n_1} - y_{0n} = \bar{y}(\bar{\alpha}_{n_1} - \beta_n) + (y_{n_1}^* - y_{0n}^*)\,, \qquad \forall n\,.$$

On the other hand $\{y_n^*\}$ is basic, then (see [11]$_2$, p. 20) by Theorem X* dist $(y_n^*/\|y_n^*\|, [\{y_k^*\}_{k\neq n}]) > C \in R^+$, $\forall n$; that is

$$\|y_{n_1}^* - y_{0n}^*\| = \|y_{n_1}^*\|(\|y_{n_1}^*/\|y_{n_1}^*\| - y_{0n}^*/\|y_{n_1}^*\|\|) > \|y_{n_1}^*\|C > aC \quad \text{by (30)}, \ \forall n;$$

therefore, by (35) and (36),

$$\|\bar{y}(\bar{\alpha}_{n_1} - \beta_n)\| = \|(y_{n_1} - y_{0n}) - (y_{n_1}^* - y_{0n}^*)\| \geqslant \|y_{n_1}^* - y_{0n}^*\| - \|y_{n_1} - y_{0n}\| > aC - \frac{1}{n}, \forall n;$$

consequently

$$(37) \qquad |\bar{\alpha}_{n_1} - \beta_n| > \left(aC - \frac{1}{n}\right)/\|\bar{y}\|, \qquad \forall n.$$

By (35), (36) and (37) it follows that

$$\left\|\bar{y} + \frac{y_{n_1}^* - y_{0n}^*}{\bar{\alpha}_{n_1} - \beta_n}\right\| = \frac{\|y_{n_1} - y_{0n}\|}{|\bar{\alpha}_{n_1} - \beta_n|} < \frac{1/n}{(aC - 1/n)/\|\bar{y}\|} = \frac{\|\bar{y}\|}{naC - 1}, \qquad \forall n.$$

That is $\bar{y} \in [\{y_n^*\}]$, in contradiction with (34); hence (35) is not possible; that is, by Theorem X*, the thesis is proved.

(b) If $\{y_n/\|y_n\|\}$ is without convergent subsequences the proof follows by (a) and by Lemma 2. Hence, by (b) of Theorem V*, it is sufficient to prove the thesis when $\{y_n\}$ has an infinite subsequence, which we call $\{y_n\}$ again, as the sequence of (2), with $q_n = p < +\infty$ $\forall n$, $\{y_n^*\}$ basic and $[\{y_n^*\}] \cap [\{\bar{y}_n\}_{n=1}^p] = \{0\}$. Therefore

$$(38) \qquad [\{y_{n_1}\}] = [\{y_{n_1}^*\}] + \operatorname{lin}\{\bar{y}_n\}_{n=1}^p, \qquad \forall \text{ infinite subsequence } \{y_{n_1}\} \text{ of } \{y_n\}.$$

Let $y \in [\{y_n\}]$, by (38) $y = y^* + \bar{y}$ with $y^* \in [\{y_n^*\}]$ and $\bar{y} \in \operatorname{lin}\{\bar{y}_n\}_{n=1}^p$. On the other hand $\{y_n^*\}$ has property P by Lemma 2, hence $\exists$ two infinite complementary subsequences $\{y_{n_2}^*\}$ and $\{y_{n_3}^*\}$ of $\{y_n^*\}$ so that

$$(39) \qquad y^* = y_{02} + y_{03} \qquad \text{with} \qquad y_{02} \in [\{y_{n_2}^*\}] \text{ and } y_{03} \in [\{y_{n_3}^*\}].$$

By (38) $\{y_{n_2}^*\} \subset [\{y_{n_2}\}]$, $\{y_{n_3}^*\} \subset [\{y_{n_3}\}]$ and $\bar{y} \in [\{y_{n_2}\}]$, therefore by (38) and (39) it follows that

$$y = y^* + \bar{y} = (y_{02} + \bar{y}) + y_{03} \qquad \text{with} \qquad y_{02} + \bar{y} \in [\{y_{n_2}\}] \text{ and } y_{03} \in [\{y_{n_3}\}].$$

This completes the proof of Theorem V.

Proof of Theorem VI. (a) By Lemma 2 it is sufficient to prove the thesis when $\{y_n\}$ is basic with brackets. Then let $y \in [\{y_n\}]$, by hypothesis $\exists \{\alpha_n\} \subset \mathscr{C}$ and an increasing sequence $\{q(n)\}$ of natural numbers, so that, setting $q(0) = 0$,

$$y = \sum_{0}^{\infty}{}_m \Big(\sum_{q(m)+1}^{q(m+1)}{}_n \alpha_n y_n \Big) . \tag{40}$$

By (40) $\exists$ an infinite subsequence $\{n_1\}$ of $\{n\}$ so that

$$\Big\| \sum_{q(m_1)+1}^{q(m_1+1)}{}_n \alpha_n y_n \Big\| < \frac{1}{2^m} \qquad \text{and} \qquad (m+1)_1 > m_1 + 1 , \qquad \forall m . \tag{41}$$

Therefore, setting $\{y_{n_2}\} = \bigcup_{m=1}^{\infty} \{y_n\}_{n=q(m_1)+1}^{q(m_1+1)}$, by (41) $\sum_{1}^{\infty}{}_m \big(\sum_{q(m_1)+1}^{q(m_1+1)}{}_n \alpha_n y_n \big)$ converges to an element y_{02} of $[\{y_{n_2}\}]$; hence, if $\{y_{n_3}\}$ is the subsequence of $\{y_n\}$ complementary to $\{y_{n_2}\}$, by (40) $y - y_{02} \in [\{y_{n_3}\}]$. On the other hand, by (41), $\{y_{n_3}\}$ is infinite, consequently $y \in [\{y_{n_2}\}] + [\{y_{n_3}\}]$.

(b) We have only to prove the implication $\Leftarrow$. Let us consider an infinite subsequence of $\{y_n\}$, which we call $\{y_n\}$ again, and let us set $y_1 = y_{1_4}$ if $y_1 \in [\{y_n\}_{n>1}]$, otherwise $y_1 = y_{1_5}$. So proceeding we have that

$$\{y_n\} = \{y_{n_4}\} \cup \{y_{n_5}\} \qquad \text{with} \qquad y_{m_4} \in [\{y_n\}_{n>m_4}] \text{ and } y_{m_5} \notin [\{y_n\}_{n>m_5}] , \ \forall m . \tag{42}$$

By (42) $\{y_{n_5}\}$ is minimal: in fact $y_{m_5} \notin [\{y_{n_5}\}_{n>m}]$, hence $\exists g_m \in B'$, with $g_m(y_{m_5}) = 1$ and $g_m(y) = 0$ for $y \in [\{y_{n_5}\}_{n>m}]$, $\forall m$. Therefore, setting $h_m = g_m - \sum_{1}^{m-1}{}_n g_m(y_{n_5}) h_n$, $\forall m$, (y_{n_5}, h_n) is a biorthogonal system. If $\{y_{n_4}\}$ is finite, by (42) $[\{y_n\}] = [\{y_{n_5}\}]$; moreover, by hypothesis, $\{y_{n_5}\}$ has property P; consequently $\{y_n\}$ has property P.

Suppose now that $\{y_{n_4}\}$ is infinite; then by (42), $\{y_{n_4}\} \subset \bigcap_{m=1}^{\infty} [\{y_{n_5}\} \cup \{y_{n_4}\}_{n>m}]$. Therefore $\exists$ an increasing sequence $\{r_n\}$ of natural numbers, moreover $\forall m$ $\exists \big\{ \{\alpha_{nk}\}_{k=r_m+1}^{r_{m+1}} \big\}_{n=1}^{m} \subset \mathscr{C}$ and $\{y_{5nm}\}_{n=1}^{m} \subset \operatorname{lin} \{y_{n_5}\}$, so that

$$\Big\| y_{n_4} - \Big(\sum_{r_m+1}^{r_{m+1}}{}_k \alpha_{nk} y_{k_4} + y_{5nm} \Big) \Big\| < \frac{1}{m} \qquad \text{for } 1 \leqslant n \leqslant m , \ \forall m . \tag{43}$$

Then, setting $\{y_{n_6}\} = \bigcup_{m=1}^{\infty} \{y_{n_4}\}_{n=r_{2m}+1}^{r_{2m+1}}$, if $\{y_{n_7}\}$ is the subsequence of $\{y_n\}$ complementary to $\{y_{n_6}\}$, by (43) $\{y_{n_6}\} \subset [\{y_{n_7}\}]$, therefore $[\{y_n\}] = [\{y_{n_7}\}] = [\{y_{n_6}\}] + [\{y_{n_7}\}]$, that is $\{y_n\}$ has property P. This completes the proof of Theorem VI.

Proof of Theorem VII. (a) We recall that $M \geqslant 1$; firstly suppose

$$\|u_n - y_n\| < \lambda\varepsilon_n/M\,, \qquad \forall n\,. \tag{44}$$

Proceeding as for (17) and (18) we find

$$\Big\|\sum_1^m {}_n\, \alpha_n y_n\Big\|(1-\lambda) \leqslant \Big\|\sum_1^m {}_n\, \alpha_n u_n\Big\| \leqslant (1+\lambda)\Big\|\sum_1^m {}_n\, \alpha_n y_n\Big\|\,, \qquad \forall\{\alpha_n\}_{n=1}^m \subset \mathscr{C}\,. \tag{45}$$

By hypothesis $\{y_n\} \subset S_B$, moreover by (44) $\|u_n - y_n\| < \lambda/M$, $\forall n$, therefore

$$\begin{cases} \|u_n\| \leqslant \|u_n - y_n\| + \|y_n\| < 1 + \lambda/M\,, & \forall n\,, \\ \|u_n\| \geqslant \big|\,\|u_n - y_n\| - \|y_n\|\,\big| = 1 - \|u_n - y_n\| > 1 - \lambda/M\,, & \forall n\,. \end{cases} \tag{46}$$

By hypothesis and by Theorem X*, $\forall\{\alpha_n\}_{n=1}^m \subset \mathscr{C}$, we have that

$$\Big\|\sum_1^m {}_n\, \alpha_n y_n\Big\| \geqslant \max\{|\alpha_n|/M;\ 1 \leqslant n \leqslant m\}\,;$$

therefore, by (45) and (46),

$$\Big\|\sum_1^m {}_n\, \alpha_n(u_n/\|u_n\|)\Big\| \geqslant (1-\lambda)\Big\|\sum_1^m {}_n\, (\alpha_n/\|u_n\|)\,y_n\Big\| \geqslant$$
$$\geqslant (1-\lambda)\cdot\max\{|\alpha_n|/(\|u_n\|M);\ 1 \leqslant n \leqslant m\} \geqslant \frac{1-\lambda}{M+\lambda}\cdot\max\{|\alpha_n|;\ 1 \leqslant n \leqslant m\}\,;$$

consequently, by Theorem X*, $\exists\{g_n\} \subset B'$ so that $(u_n/\|u_n\|,\ g_n\|u_n\|)$ is biorthogonal system, with

$$\|u_n\|\cdot\|g_n\|_{[\{u_k\}]} \leqslant \frac{M+\lambda}{1-\lambda}\,, \qquad \forall n\,;$$

hence, by (46), (u_n, g_n) is biorthogonal system, with

$$\|g_n\|_{[\{u_k\}]} \leqslant \frac{M+\lambda}{(1-\lambda)\|u_n\|} < \frac{M(M+\lambda)}{(1-\lambda)(M-\lambda)}\,, \qquad \forall n\,.$$

Suppose now that $\|u_n - y_n\| < \varepsilon_n$, $\forall n$. By hypothesis $\sum_1^\infty {}_n\, \varepsilon_n = 1$, hence $\exists \bar{n} \in \{n\}$ so that $\sum_{\bar{n}+1}^\infty {}_n\, \varepsilon_n \leqslant \lambda/M$; then, setting $\eta_n = M\varepsilon_n/\lambda$, $\forall n$, we have that:

$$\|u_n - y_n\| < \lambda\eta_n/M\,, \qquad \forall n > \bar{n} \qquad \text{with} \qquad \sum_{\bar{n}+1}^\infty {}_n\, \eta_n \leqslant 1\,.$$

Therefore, proceeding as for (44), we find that $\{u_n\}_{n>\bar{n}}$ belongs to a bibounded biorthogonal system. On the other hand $\{u_n\}_{n=1}^{\bar{n}} = \{u_{n_2}\}_{n=1}^{p} \cup \{u_{n_3}\}_{n=1}^{r}$ so that, setting $\{u_{n_1}\} = \{u_{n_3}\}_{n=1}^{r} \cup \{u_n\}_{n>\bar{n}}$, $\{u_{n_1}\}$ is minimal and complete in $[\{u_n\}]$. It is easy now to verify that $\{u_{n_1}\}$ belongs to a bibounded biorthogonal system.

(b) The proof follows by (a), if we replace $\{y_n\}$ by $\{v_n\}$ and $\{u_n\}$ by $\{y_n\}$. This completes the proof of Theorem VII.

5*. - Proofs of § 5.

Firstly we recall

XI* – $[\mathbf{12}]_4$. *$\exists$ a t-non contractive sequence $\{x_n\} = \{y_n\} \cup \{z_n\}$ complete in B, with $\{y_n\}$ and $\{z_n\}$ basic, moreover $[\{y_n\}] \cap [\{z_n\}] = \{0\}$.*

Proof of Remark 1. Let $\{u_n\}$ be an M-basis of B with $\{u_n\} \subset S_B$, let moreover $\{u_{n_1}\}$ and $\{u_{n_2}\}$ be two infinite complementary subsequences of $\{u_n\}$ with $\{u_{n_1}\}$ basic. Then let us set

$$v_n = \Big(\sum_{1}^{n}{}_k\, u_k/10^{n(k-1)}\Big) \Big/ \Big\| \sum_{1}^{n}{}_k\, u_k/10^{n(k-1)} \Big\| , \qquad \forall n . \tag{47}$$

By (47) $\{v_n\}$ is H-basic, moreover, proceeding as for (13) it follows that

(48) $\forall$ infinite subsequence $\{v_{n_3}\}$ of $\{v_n\}$ and $\forall\{w_n\} \subset B$ with $\|w_n - v_{n_3}\| < 1/10^{(n_3)^3}$, $\forall n$, $\{w_n\}$ is overfilling and complete in B.

By Theorem IV* $\exists\{\eta_n\} \subset R^+$ so that

$$\forall\{w_n\} \subset B \quad \text{with} \quad \|w_n - u_{n_1}\| < \eta_n , \qquad \forall n , \ \{w_n\} \text{ is basic} . \tag{49}$$

Finally let us set

$$p_{mn} = 10^m + n \quad \text{and} \quad \varepsilon_{mn} = 1/10^{p_{mn}^{3n}} , \qquad \forall m \text{ and } n . \tag{50}$$

(a) It is sufficient to set $\{x_n\} = \bigcup_{m=1}^{\infty} \{x_{mn}\}_{n=1}^{m}$, with $x_{mn} = u_m$ for $1 \leqslant n \leqslant m$, $\forall m$.

(b) We set

$$\{y_n\} = \bigcup_{m=1}^{\infty} \{y_{mn}\}_{n=1}^{m}, \quad \text{with} \quad y_{mn} = u_{m_1} + \eta_m \varepsilon_{mn} v_{p_{mn}} , \text{ for } 1 \leqslant n \leqslant m , \ \forall m . \tag{51}$$

By (47), (50) and (51) $\{y_n\}$ is H-basic, moreover by (49) and (51) $\{y_n\}$ is denucleated. It is now sufficient to prove that

(52) $\{y_n\} = \bigcup_{r=1}^{q} \{y_{n_r}\}_{n=1}^{\infty}$ with $1 \leqslant q < +\infty \Rightarrow \exists \bar{r}$, with $1 \leqslant \bar{r} \leqslant q$, so that $\{y_{n_{\bar{r}}}\}_{n=1}^{\infty}$ is t-non contractive and complete in B.

In fact, in hypothesis of (52), $\exists \bar{r}$, with $1 \leqslant \bar{r} \leqslant q$, so that $\{y_{n_{\bar{r}}}\}_{n=1}^{\infty}$ has a subsequence $\{\tilde{y}_k\} = \{y_{m(k),\, n(k)}\}_{k=1}^{\infty} \cup \{y_{m(k),\, n(k)+s(k)}\}_{k=1}^{\infty}$, with $s(k) \geqslant 1$, $\forall k$.

We observe that, by (50), $p_{m,n+s} > p_{m,n}$, hence $\varepsilon_{m,n+s}/\varepsilon_{mn} < 10^{-p_{mn}^3}$, $\forall$ natural numbers m, n and s; therefore, setting $\tilde{v}_k = (y_{m(k),n(k)} - y_{m(k),n(k)+s(k)})/(\eta_{m(k)}\, \varepsilon_{m(k),n(k)})$, $\forall k$, by (51) it follows that

$$\|\tilde{v}_k - v_{p_{m(k),\, n(k)}}\| = \Big\| \Big(v_{p_{m(k),\, n(k)}} - \frac{\varepsilon_{m(k),n(k)+s(k)}}{\varepsilon_{m(k),n(k)}} v_{p_{m(k),n(k)+s(k)}}\Big) - v_{p_{m(k),\, n(k)}} \Big\|$$

$$= \frac{\varepsilon_{m(k),n(k)+s(k)}}{\varepsilon_{m(k),\, n(k)}} < 1/10^{p_{m(k),n(k)}^3}, \qquad \forall k\,.$$

Consequently, by (48), $\{\tilde{v}_k\}$ is overfilling and complete in B; therefore $\{\tilde{y}_k\}$ is t-non contractive and complete in B, hence (52) is proved.

(c) By (51) we have that $\forall \{w_n\} = \bigcup_{m=1}^{\infty} \{w_{mn}\}_{n=1}^{m} \subset B$, with $\|w_{mn} - y_{mn}\| < \eta_m^2 \varepsilon_{mn}^2$, $\forall m$ and n, $\{w_n\}$ has the property of (52). Consequently, if $\{\tilde{u}_n\}$ is a sequence of $[\{u_{n_2}\}]$ with the same properties of $\{y_n\}$ of (51) for B, by (a) of Theorem IV $\exists$ a sequence $\{\tilde{z}_n\}$ of B so that

(53) $\{\tilde{z}_n\} = \bigcup_{r=1}^{q} \{\tilde{z}_{n_r}\}_{n=1}^{\infty}$ with $1 \leqslant q < +\infty \Rightarrow \exists \bar{r}$ with $1 \leqslant \bar{r} \leqslant q$ so that $\{\tilde{z}_{n_{\bar{r}}}\}_{n=1}^{\infty}$ is Q-non contractive with $[\{u_{n_2}\}] \subseteq Q$; moreover $\{\tilde{z}_n\}$ is minimal.

By (b) of Theorem VII* $\exists$ a denucleated sequence $\{\hat{z}_n\}$ of B so that $\{\tilde{z}_n\} \cup \{\hat{z}_n\}$ is minimal and complete in B; hence, by (53), it is sufficient to set $\{z_n\} = \{\tilde{z}_n\} \cup \{\hat{z}_n\}$. This completes the proof of Remark 1.

Proof of Remark 2. Let P be a subspace of B, of infinite dimension and of infinite codimension. By Theorem XI* $\exists \{u_n\} \subset B$ so that

(54) $\{u_n\} = \{u_{n_1}\} \cup \{u_{n_2}\}$ is t-non contractive and complete in P; moreover $\{u_{n_1}\}$ and $\{u_{n_2}\}$ are basic with $[\{u_{n_1}\}] \cap [\{u_{n_2}\}] = \{0\}$.

By (c) and (b) of Theorem VII* $\exists \{v_n\} \subset B$ and $\{g_n\} \subset B'$ so that

(55) $\{v_n\} = \{v_{n_3}\} \cup \{v_{n_4}\} \cup \{v_{n_5}\} \cup \{v_{n_6}\}$ is M-basis of B, with (v_n, g_n) biorthogonal system; moreover $\{v_{n_3}\}$ is M-basis of $[\{u_{n_1}\}]$ and $\{v_{n_4}\}$ is M-basis of $[\{u_{n_2}\}]$.

On the other hand, by Theorem IV* and by (b) of Theorem VIII*, $\exists\{\varepsilon_n\}\subset R^+$ so that

(56) $\forall\{w_n\}=\{w_{n_1}\}\cup\{w_{n_2}\}\subset B$ with $\|w_n-u_n\|\leqslant\varepsilon_n$, $\forall n$, $\{w_n\}$ is Q-non contractive with $P\subseteq Q$; moreover $\{w_{n_1}\}$ and $\{w_{n_2}\}$ are basic.

Therefore let us set

(57) $\{x_n\}=\{y_n\}\cup\{z_n\}$ with $y_n=u_{n_1}+\varepsilon_{n_1}v_{n_5}/\|v_{n_5}\|$ and $z_n=u_{n_2}+\varepsilon_{n_2}v_{n_6}/\|v_{n_6}\|$, $\forall n$.

By (55) and (57) $(y_n, g_{n_5}\|v_{n_5}\|/\varepsilon_{n_1})\cup(z_n, g_{n_6}\|v_{n_6}\|/\varepsilon_{n_2})=(x_n, f_n)$ is a biorthogonal system; moreover by (56) and (57) $\{y_n\}$ and $\{z_n\}$ are basic and $\{x_n\}$ is Q-non contractive with $P\subseteq Q$; consequently by (54), (55) and (57) $\{x_n\}$ is complete in B and $P\subseteq[\{f_n\}]_\perp$. Finally, by (57) and (55), $\{y_n\}\subset[\{u_{n_1}\}\cup\{v_{n_5}\}]=[\{v_{n_3}\}\cup\cup\{v_{n_5}\}]$ and $\{z_n\}\subset[\{u_{n_2}\}\cup\{v_{n_6}\}]=[\{v_{n_4}\}\cup\{v_{n_6}\}]$, hence $[\{y_n\}]\cap[\{z_n\}]=\{0\}$. This completes the proof of Remark 2.

6*. - Proofs of § 6.

3. *$\{y_n\}$ H-basic sequence of S_B with the property of* (9): $\Rightarrow\{y_n\}$ *is basic with brackets.*

Proof. $\{y_n\}$ is minimal: in fact $\forall m$, if $r_m\in\{n\}$ so that $q_{r_m-1}<m\leqslant q_{r_m}$, by (9) it follows that $\operatorname{dist}(y_m, [\{y_n\}_{n\neq m}])\geqslant\operatorname{dist}(y_m, \operatorname{lin}\{y_n\}_{n(\neq m)=1}^{q_{r_m}})/K>0$ because $\{y_n\}$ is H-basic. Therefore $\exists\{h_n\}\subset B'$ with (y_n, h_n) biorthogonal system. Let now $\bar y\in[\{y_n\}]$ and let us set, $\forall n$,

(58) $$\bar H_n=\sum_{q_n+1}^{q_{n+1}}{}_k\, h_k(\bar y)y_k,\qquad \bar A_{mn}=\sum_{q_n+1}^{q_{n+1}}{}_k\,\bar\alpha_{mk}y_k \text{ so that, setting}$$

$\bar y_m=\sum_0^{t(m)}{}_n\bar A_{mn}$ where $t(m+1)>t(m)$ it follows that $\|\bar y-\bar y_m\|<\dfrac{1}{m}$, $\qquad\forall m$.

It is sufficient to verify that

(59) $$\forall m,\ \exists s_m \text{ so that } \Big\|\bar y-\sum_1^p{}_n\bar H_n\Big\|<\frac{1}{m},\qquad \forall p>s_m.$$

Now $h_k(\bar y)=h_k(\lim_{m\to\infty}\bar y_m)=\lim_{m\to\infty}\bar\alpha_{mk}$, $\forall k$; consequently $\forall m$ $\exists l(m)\in\{n\}$ so that

(60) $$\Big\|\sum_0^{t(m)}{}_n\bar A_{l(m)n}-\tilde y_m\Big\|<\frac{1}{m}\quad\text{where } \tilde y_m=\sum_0^{t(m)}{}_n\bar H_n \text{ and } l(m)>m,\ \forall m.$$

We observe that, by (58) and (60), $t(l(p)) > t(p)$, $\forall p$; therefore, $\forall m$ and $\forall p > 2m(K+1)$, by (9), (58) and (60) it follows that

$$\|\bar{y}-\tilde{y}_p\| \leqslant \|\bar{y}-\bar{y}_p\| + \|\bar{y}_p-\tilde{y}_p\| \leqslant \|\bar{y}-\bar{y}_p\| + \|\bar{y}_p - \sum_0^{t(p)}{}_n \bar{A}_{l(p),n}\| + \| \sum_0^{t(p)}{}_n \bar{A}_{l(p),n} - \tilde{y}_p\| < \frac{2}{p} + \|\bar{y}_p - \sum_0^{t(p)}{}_n \bar{A}_{l(p),n}\| \leqslant \frac{2}{p} + K\|\bar{y}_p - \bar{y}_{l(p)}\| \leqslant \frac{2}{p} + K\|\bar{y}_p - \bar{y}\| + K\|\bar{y}-\bar{y}_{l(p)}\| < \frac{2}{p}(1+K) < \frac{1}{m}.$$

Consequently, $\forall m\ \exists s(m) > m$ so that $\|\bar{y}-\tilde{y}_p\| < 1/(m+2Km)$ for $p \geqslant s(m)$. Hence $\forall m$, setting $s_m = t(s(m))$, $\forall p > s_m$ (then $p > s(m)$), by (58) and (60) it follows that

$$\|\bar{y} - \sum_1^p{}_n \bar{H}_n\| \leqslant \|\bar{y}-\tilde{y}_{s(m)}\| + \|\sum_{s_m+1}^p{}_n \bar{H}_n\| \leqslant \|\bar{y}-\tilde{y}_{s(m)}\| + K\|\sum_{s_m+1}^{t(p)}{}_n \bar{H}_n\| = \|\bar{y}-\tilde{y}_{s(m)}\| + K\|\tilde{y}_p - \tilde{y}_{s(m)}\| \leqslant \|\bar{y}-\tilde{y}_{s(m)}\| + K\|\tilde{y}_p-\bar{y}\| + K\|\bar{y}-\tilde{y}_{s(m)}\| < 1/m,$$

that is (59) is proved. This completes the proof of Lemma 3.

Proof of Theorem VIII. (a) By hypothesis $\bigcap_{m=1}^\infty [\{y_n\}_{n>m}] = \{0\}$; hence $\forall m\ \exists r(m) \in \{n\}$ so that

$$\text{lin}\{y_n\}_{n=1}^m \cap [\{y_n\}_{n>r(m)}] = \{0\}. \tag{61}$$

Let us set

$$s(0)=0, \quad s(1)=1, \quad s(n+1) = r(s(n)) \quad \text{for } n \geqslant 1; \tag{62}$$
$$\{y_{n_1}\} = \bigcup_{m=0}^\infty \{y_n\}_{n=s(2m)+1}^{s(2m+1)}, \qquad \{y_{n_2}\} = \bigcup_{m=1}^\infty \{y_n\}_{n=s(2m-1)+1}^{s(2m)}.$$

It is sufficient to prove that $\{y_{n_1}\}$ is M-basic, indeed that $\{y_{n_1}\}$ is minimal, because $\{y_{n_1}\}$ is t-contractive by hypothesis. Then $\forall m\ \exists$ two natural numbers p_m and d_m so that

$$y_{m_1} \in \bigcup_{n=0}^{p_m} \{y_k\}_{k=s(2n)+1}^{s(2n+1)} = \{y_{n_1}\}_{n=1}^{d_m}. \tag{63}$$

By (63) $(d_m)_1 = s(2p_m+1)$; moreover by (62) $(d_m+1)_1 = s(2p_m+2)+1 = r(s(2p_m+1))+1$; consequently by (61) it follows that

$$\{y_{n_1}\}_{n=1}^{d_m} \subseteq \{y_n\}_{n=1}^{s(2p_m+1)}, \quad \{y_{n_1}\}_{n>d_m} \subset \{y_n\}_{n>r(s(2p_m+1))} \text{ moreover } \text{lin}\{y_n\}_{n=1}^{s(2p_m+1)} \cap [\{y_n\}_{n>r(s(2p_m+1))}] = \{0\}. \tag{64}$$

Suppose by absurd that $y_{m_1} \in [\{y_{n_1}\}_{n\neq m}]$, by (63) $\exists y_{0m1}$ so that

$$(65) \qquad y_{0m1} \in \text{lin}\,\{y_{n_1}\}_{n(\neq m)=1}^{d_m} \qquad \text{with} \qquad y_{m_1} + y_{0m1} \in [\{y_{n_1}\}_{n>d_m}] \,.$$

But $\{y_{n_1}\}$ is H-basic by hypothesis, hence $y_{m_1} + y_{0m1} \neq 0$; therefore by (65)

$$\text{lin}\,\{y_{n_1}\}_{n=1}^{d_m} \cap [\{y_{n_1}\}_{n>d_m}] \ni y_{m_1} + y_{0m1} \neq 0 \,,$$

absurd by (64); consequently $y_{m_1} \notin [\{y_{n_1}\}_{n\neq m}]$, $\forall m$.

(b) By hypothesis $\{y_n\}$ has the property of (7). Then let us set

$$(66) \qquad t(0) = 0 \,, \qquad t(1) = 1 \,, \qquad t(n+1) = l(t(n)) \qquad \text{for } n \geqslant 1 \,;$$

$$\{y_{n_3}\} = \bigcup_{m=0}^{\infty} \{y_n\}_{n=t(2m)+1}^{t(2m+1)} \,, \quad \{y_{n_4}\} = \bigcup_{m=1}^{\infty} \{y_n\}_{n=t(2m-1)+1}^{t(2m)} \,.$$

It is sufficient to prove that $\{y_{n_3}\}$ is basic with brackets. By (7) and (66), setting $\{y_n\}_{n=t(2m)+1}^{t(2m+1)} = \{y_{n_3}\}_{n=q_m+1}^{q_{m+1}}$, $\forall m$, we have that

$$\Big\|\sum_{0}^{p}{}_m \Big(\sum_{q_m+1}^{q_{m+1}}{}_n \alpha_{n_3} y_{n_3}\Big)\Big\| = \Big\|\sum_{0}^{p}{}_m \Big(\sum_{t(2m)+1}^{t(2m+1)}{}_n \alpha_n y_n\Big)\Big\| \leqslant K\Big\|\sum_{0}^{p+r}{}_m \Big(\sum_{t(2m)+1}^{t(2m+1)}{}_n \alpha_n y_n\Big)\Big\|$$
$$= K\Big\|\sum_{0}^{p+r}{}_m \Big(\sum_{q_m+1}^{q_{m+1}}{}_n \alpha_{n_3} y_{n_3}\Big)\Big\| \,, \qquad \forall \bigcup_{m=0}^{p+r} \{\alpha_n\}_{n=t(2m)+1}^{t(2m+1)} \subset \mathscr{C} \,.$$

Therefore, by Lemma 3, $\{y_{n_3}\}$ is basic with brackets.

(c) The proof follows by Lemma 3. This completes the proof of Teorem VIII.

Proof of Remark 3. Let $\{x_n\}$ be an M-basis of B with the property of (7) (where we put x_n for y_n $\forall n$) and with $\{x_n\} \subset S_B$. By Theorem IV* $\exists \{\varepsilon_n\} \subset R^+$ so that

$$(67) \qquad \forall \{u_n\} \subset B \text{ with } \|u_n - x_n\| < \varepsilon_n \,, \ \forall n \,, \ \{u_n\} \text{ is } M\text{-basis of } B \,.$$

Let us set

$$(68) \qquad y_{2n-1} = x_n \qquad \text{and} \qquad y_{2n} = x_n + \varepsilon_n \sum_{n+1}^{\infty}{}_k \frac{x_k}{10^{nk}} \,, \qquad \forall n \,.$$

By (68) $\{y_n\}$ is H-basic; moreover $\|y_{2n} - x_n\| < \varepsilon_n$, $\forall n$, hence by (67) $\{y_n\}$ is union of two M-bases of B. On the other hand $[\{y_k\}_{k>2n}] = [\{x_k\}_{k>n}]$, $\forall n$, consequently

$$\sup_n \operatorname{dist}(y, [\{y_k\}_{k>n}]) = \sup_n \operatorname{dist}(y, [\{x_k\}_{k>n}]) \geqslant \|y\|/K, \qquad \forall y \in B;$$

that is $\{y_n\}$ is norming. This completes the Proof of Remark 3.

Proof of Theorem IX. Proceeding as for (42) we find that

$$\{y_n\} = \{y_{n_1}\} \cup \{y_{n_2}\}_{n=1}^{p}, \qquad \text{with } \{y_{n_1}\} \text{ minimal and } y_{m_2} \in [\{y_n\}_{n>m_2}], \quad \forall m.$$

Now, if $\{y_n\}$ is M-basoidic $p < +\infty$, otherwise, by the proof of (b) of Theorem VI, $\exists$ two infinite complementary subsequences $\{y_{n_6}\}$ and $\{y_{n_7}\}$ of $\{y_n\}$ with $[\{y_n\}] = [\{y_{n_7}\}]$, that is $[\{y_{n_7}\}] \cap [\{y_{n_6}\}] = [\{y_{n_6}\}]$, absurd because $\{y_n\}$ is M-basoidic. Therefore, by (b) of Theorem I*, we have that

$$\{y_n\} = \{y_{n_1}\} \cup \{y_{n_2}\}_{n=1}^{p}, \qquad \text{with } \{y_{n_1}\}\ M\text{-basis of } [\{y_n\}] \text{ and } p < +\infty. \tag{69}$$

In this proof we denote by $\{y_{n_3}\}$ and $\{y_{n_4}\}$ two infinite complementary subsequences of $\{y_{n_1}\}$. It is obvious that $\{y_n\}$, M-basoidic, is also H-basic.

(b) It is sufficient to prove that

(70) $p \geqslant 1$ in (69) $\Rightarrow \forall m$, with $1 \leqslant m \leqslant p$, $\{y_n\}_{n \neq m_2}$ has not property P; precisely it is never possible that $y_{m_2} \in \operatorname{lin}\{y_{n_2}\}_{n(\neq m)=1}^{p} + [\{y_{n_3}\}] + [\{y_{n_4}\}]$.

In fact, otherwise, $y_{m_2} = y_{02} + y_{03} + y_{04}$, where $y_{02} \in \operatorname{lin}\{y_{n_2}\}_{n(\neq m)=1}^{p}$, $y_{03} \in [\{y_{n_3}\}]$ and $y_{04} \in [\{y_{n_4}\}]$. We can suppose $y_{03} \neq 0$, because $\{y_n\}$ is H-basic; consequently it would be $[\{y_{n_3}\}] \cap [\{y_{n_4}\} \cup \{y_{n_2}\}_{n=1}^{p}] \ni y_{03} \neq 0$, absurd because $\{y_n\}$ is M-basoidic.

(a) By (69) and (70) it is sufficient to prove the implication $\Leftarrow$: Suppose by absurd that $\exists \bar{y} \neq 0$ and two complementary subsequences $\{y_{n_5}\}$ and $\{y_{n_8}\}$ of $\{y_{n_2}\}_{n=1}^{p}$, so that $\bar{y} = y_{03} + y_{05} = y_{04} + y_{08}$, with $y_{03} \in [\{y_{n_3}\}]$, $y_{04} \in [\{y_{n_4}\}]$, $y_{05} \in \operatorname{lin}\{y_{n_5}\}$ and $y_{08} \in \operatorname{lin}\{y_{n_8}\}$. Now $y_{05} = y_{08}$, otherwise $[\{y_{n_3}\}] + [\{y_{n_4}\}] \ni y_{05} - y_{08} \neq 0$, absurd by hypothesis; moreover $y_{05} = y_{08} \Rightarrow y_{05} = y_{08} = 0$, because $\{y_{n_2}\}_{n=1}^{p}$ is H-basic; therefore $[\{y_{n_3}\}] \cap [\{y_{n_4}\}] \ni \bar{y} \neq 0$, absurd because $\{y_{n_1}\}$ is M-basic by (69).

(c) Suppose that $\exists \{\beta_n\} \subset \mathscr{C}$ and an increasing sequence $\{t_m\}$ of natural numbers so that, setting to $= 0$,

$$\sum_{0}^{\infty}{}_m \Big(\sum_{t_m+1}^{t_{m+1}}{}_n \beta_{n_1} y_{n_1}\Big) + \sum_{1}^{p}{}_n \beta_{n_2} y_{n_2} = 0. \tag{71}$$

By (71) $\exists$ an infinite subsequence $\{t_{m_q}\}$ of $\{t_m\}$ so that

$$(72) \qquad \left\| \sum_{t_{m_q}+1}^{t_{m_q+1}} {}_n \beta_{n_1} y_{n_1} \right\| < \frac{1}{2^m}, \qquad \forall m .$$

Let us set $\{y_{n_3}\} = \bigcup_{m=1}^{\infty} \{y_{n_1}\}_{n=t_{(2m)_q}+1}^{t_{(2m)_q+1}}$, then, if $\{y_{n_4}\}$ is the subsequence of $\{y_{n_1}\}$ complementary to $\{y_{n_3}\}$, by (71) and (72) it follows that $\sum_1^p {}_n \beta_{n_2} y_{n_2} \in [\{y_{n_3}\}] + [\{y_{n_4}\}]$, hence $\beta_{n_2} = 0$ for $1 \leqslant n \leqslant p$ by (a); consequently, by (71), $\beta_{n_1} = 0$, $\forall n$, because $\{y_{n_1}\}$ is minimal. This completes the proof of Theorem IX.

Proof of Remark 3′. Let $\{y_n\}$ be the sequence of (68). Let $\{v_n\}$ be a t-non contractive sequence complete in B, with $\{v_n\} \subset S_B$ and let us set

$$(73) \qquad u_{2n-1} = y_{2n-1}, \qquad u_{2n} = y_{2n} + \varepsilon_n v_n, \qquad \forall n .$$

By (68) and (73) $u_{2n-1} = x_n$, $\forall n$, hence $\{y_{2n}\}_{n>m} \subset [\{u_{2n-1}\}_{n>m}]$, that is $\{v_n\}_{n>m} \subset [\{u_n\}_{n>2m}]$, consequently $B = [\{u_n\}_{n>2m}]$, $\forall m$. Therefore $\{u_n\}$ is t-non contractive, moreover $\|u_n - y_n\| \leqslant \varepsilon_n$, $\forall n$. This completes the Proof of Remark 3′.

Proof of Example 1′. By (20) and by implication $\Leftarrow$ of (a) of Theorem IX it is sufficient to set

$$v_{n+1} = x_n, \qquad v_1 \in \operatorname{lin}\{x_n\}_{n=1}^m \qquad \text{so that } \|v_1 - \bar{x}\| < \varepsilon_1 .$$

Therefore $\{v_n\}_{n=1}^{m+1}$ is not H-basic, consequently by (c) of Theorem IX $\{v_n\}$ cannot be M-basoidic. This completes the Proof of Example 1′.

Proof of Theorem X. (a) The proof follows by (b) of Theorem VIII*.

(b) The proof follows by (b) of Theorem IV, by (a) of Theorem III, by (7) and (9) and by (17) and (18).

Proof of Example 2. Setting $x_{2n-1} = y_n$ and $x_{2n} = z_n$, $\forall n$, we have that $\{x_n\}$ is t-contractive; in fact $[\{x_n\}] = B_4 = [\{y_n\}] + [\{z_n\}]$, hence if $\bar{x} \in \bigcap_{m=1}^{\infty} [\{x_n\}_{n>m}]$, $\bar{x} = \bar{x}_1 + \bar{x}_2$ with $\bar{x}_1 \in [\{y_n\}]$ and $\bar{x}_2 \in [\{z_n\}]$; but $\bar{x}_1 = 0$ otherwise, if $\bar{x}_1 \neq 0$, $\exists \bar{n}$ so that $\bar{x}_1 \notin [\{y_n\}_{n>\bar{n}}]$, hence $\bar{x} \notin [\{x_n\}_{n>2\bar{n}}] = [\{y_n\}_{n>\bar{n}}] + [\{z_n\}_{n>\bar{n}}]$; for the same reason $\bar{x}_2 = 0$, that is $\bar{x} = 0$. On the other hand $\{u_n\}_{n>m} \subset \operatorname{lin}\{x_n\}_{n>2m}$, $\forall m$, therefore $\{u_n\}$ is t-non contractive. Finally by definition of norm $\exists \{g_n\} \subset B_4'$ with (z_n, g_n) biorthogonal system and $[\{y_n\}] \subseteq [\{g_n\}]_\perp$; then $(u_n, g_n/\varepsilon_n)$ is a biorthogonal system; that is $\{u_n\}$ is M-basic. This completes the Proof of Example 2.

7*. - Proofs of § 7.

It is known that ($\{y_n\}$ basic) $\not\Leftarrow$ ($\{y_n\}$ basic with brackets and belongs to a bibounded biorthogonal system) $\not\Leftarrow$ ($\{y_n\}$ basic with brackets) $\not\Leftarrow$ ($\{y_n\}$ norming and belongs to a bibounded biorthogonal system), however let us point out this with a few easy examples.

Example 3. Let $\{y_n\}$ be an infinite H-basic sequence of a linear space. We define in $\operatorname{lin}\{y_n\}$ the following two norms, $\forall\{\alpha_n\}_{n=1}^{m} \subset \mathscr{C}$ (if $p_m \in \{n\}$ so that $10^{p_m-1} < m \leqslant 10^{p_m}$, we set $\alpha_n = 0$ for $m+1 \leqslant n \leqslant 10^{p_m}$ if $m < 10^{p_m}$):

$$(74)\quad \Big\|\sum_{1}^{m}{}_n \alpha_n y_n\Big\|_1 = \Big\|\sum_{1}^{10^{p_m}}{}_n \alpha_n y_n\Big\|_1 = |\alpha_1| + \sum_{1}^{p_m}{}_n \Big\|\sum_{10^{n-1}+1}^{10^n}{}_k \alpha_k y_k\Big\|_1$$

$$= |\alpha_1| + \sum_{1}^{p_m}{}_n \left\{\frac{1}{2}\max\{|\alpha_k|;\, 10^{n-1}+1 \leqslant k \leqslant 10^n\} + \frac{1}{2}\Big|\sum_{10^{n-1}+1}^{10^n}{}_k \alpha_k\Big|\right\};$$

$$(75)\quad \Big\|\sum_{1}^{m}{}_n \alpha_n y_n\Big\|_2 = |\alpha_1| + \sum_{1}^{p_m}{}_n \left\{\sum_{10^{n-1}+1}^{10^n}{}_k \frac{|\alpha_k|}{n} + \left(1-\frac{1}{n}\right)\Big|\sum_{10^{n-1}+1}^{10^n}{}_k \alpha_k\Big|\right\}.$$

We call B_5 the completion of $\operatorname{lin}\{y_n\}$ in the norm of (74) and B_6 the completion in the norm of (75). It is obvious that $\{y_n\}$ is basic with brackets for both B_5 and B_6. Moreover, by last equivalence of Theorem X*, $\{y_n\}$ belongs to a bibounded biorthogonal system in B_5, while this does not occur in B_6, because

$$\|y_{10^n+1} - y_{10^n+2}\|_2 = \frac{2}{n+1}, \qquad \forall n.$$

Finally, by (c) of Theorem I*, $\{y_n\}$ is not basis for B_5, because

$$\Big\|\sum_{10^n+1}^{10^n+n}{}_k y_k\Big\|_1 = \frac{n+1}{2} = (n+1)\Big\|\sum_{10^n+1}^{10^n+n}{}_k y_k - \sum_{10^n+n+1}^{10^n+2n}{}_k y_k\Big\|_1, \qquad \forall n.$$

Example 4. Let $\{x_n\}$ be the natural basis of c_0, with (x_n, f_n) biorthogonal system. Then let us set $y_n = x_{2n}$ for $1 \leqslant n \leqslant 10^2$, moreover, for $m > 1$:

$$(76)\qquad y_n = \begin{cases} x_{2n} - x_{10^{m-1}-1} & \text{for } 10^m + 1 \leqslant n \leqslant 2\cdot 10^m, \\ x_{2n} + x_{10^m-1} & \text{for } 2\cdot 10^m + 1 \leqslant n \leqslant 3\cdot 10^m, \\ x_{2n} & \text{for } 3\cdot 10^m + 1 \leqslant n \leqslant 8\cdot 10^m, \\ x_{2n} + x_{10^m+1} & \text{for } 8\cdot 10^m + 1 \leqslant n \leqslant 9\cdot 10^m, \\ x_{2n} - x_{10^{m+1}+1} & \text{for } 9\cdot 10^m + 1 \leqslant n \leqslant 10^{m+1}. \end{cases}$$

By (76) (y_n, f_{2n}) is biorthogonal system, hence $\{y_n\}$ belongs to a bibounded biorthogonal system. Moreover by (76) it follows that

$$\{y_n\}_{n=10^m+1}^{10^{m+1}} \subset \lim\{x_n\}_{n=10^{m}-1-1}^{2\cdot 10^{m+1}}, \qquad \{y_n\}_{n\geq 10^{m+3}+1} \subset \lim\{x_n\}_{n\geq 10^{m+2}-1}, \qquad \forall m .$$

Therefore, setting $l_n = 10^{m+3}$ for $10^m+1 \leq n \leq 10^{m+1}$, $\forall m$, $\{y_n\}$ is norming, consequently is M-basic. On the other hand $\{y_n\}$ is not basic with brackets: in fact suppose by absurd that $\{y_n\}$ has the property of (9), let us fix m and let $p_m \in \{n\}$ so that

$$10^{p_m} + 1 \leq q_m \leq 10^{p_m+1} . \tag{77}$$

If $3\cdot 10^{p_m} < q_m$ by (76) it follows

$$\Big\|\sum_{2\cdot 10^{p_m}+1}^{3\cdot 10^{p_m}} {}_n\, y_n\Big\| = \Big\|\sum_{2\cdot 10^{p_m}+1}^{3\cdot 10^{p_m}} {}_n\, x_{2n} + 10^{p_m} x_{10^{p_m}-1}\Big\| = 10^{p_m} = 10^{p_m}\Big\|\sum_{2\cdot 10^{p_m}+1}^{3\cdot 10^{p_m}} {}_n\, x_{2n} + \sum_{10^{p_m+1}+1}^{10^{p_m+1}+10^{p_m}} {}_n\, x_{2n}\Big\| =$$

$$= 10^{p_m}\Big\|\sum_{2\cdot 10^{p_m}+1}^{3\cdot 10^{p_m}} {}_n\, y_n + \sum_{10^{p_m+1}+1}^{10^{p_m+1}+10^{p_m}} {}_n\, y_n\Big\| ;$$

but this is absurd by (9), because by (77) $3\cdot 10^{p_m} < q_m < 10^{p_m+1} + 1$. On the other hand if $q_m \leq 3\cdot 10^{p_m}$ by (76) it follows

$$\Big\|\sum_{9\cdot 10^{p_m-1}+1}^{10^{p_m}} {}_n\, y_n\Big\| = \Big\|\sum_{9\cdot 10^{p_m-1}+1}^{10^{p_m}} {}_n\, x_{2n} - 10^{p_m-1} x_{10^{p_m}+1}\Big\| = 10^{p_m-1}$$

$$= 10^{p_m-1}\Big\|\sum_{9\cdot 10^{p_m-1}+1}^{10^{p_m}} {}_n\, x_{2n} + \sum_{8\cdot 10^{p_m}+1}^{8\cdot 10^{p_m}+10^{p_m-1}} {}_n\, x_{2n}\Big\| = 10^{p_m-1}\Big\|\sum_{9\cdot 10^{p_m-1}+1}^{10^{p_m}} {}_n\, y_n + \sum_{8\cdot 10^{p_m}+1}^{8\cdot 10^{p_m}+10^{p_m-1}} {}_n\, y_n\Big\| ,$$

but this is absurd by (9), because by (77) $10^{p_m} < q_m < 8\cdot 10^{p_m}$. This completes Example 4. It is also known that, if $\{y_n\}$ belongs to a bibounded biorthogonal system of B, then ($\{y_n\}$ norming) $\not\Leftarrow$ ($\{y_n\}$ M-basic) $\not\Leftarrow$ ($\{y_n\}$ denucleated); we point out this again by following two examples.

Example 5. Let $\{y_n\}$ be an H-basic sequence of a linear space, setting $m_n = 2^{m-1}(2n-1)$ $\forall n$ and m (hence $\{y_n\} = \bigcup_{m=1}^{\infty}\{y_{m_n}\}_{n=1}^{\infty}$), we define in $\lim\{y_n\}$ the following norm

(78)
$$\|\sum_1^p{}_m\Big(\sum_1^{r_m}{}_n\alpha_{m_n}y_{m_n}\Big)\| = \sum_1^p{}_m\|\sum_1^{r_m}{}_n\alpha_{m_n}y_{m_n}\| =$$
$$= \sum_1^p{}_m\left\{\frac{1}{2}\max\{|\alpha_{m_n}|;\ 1\leqslant n\leqslant r_m\} + \frac{1}{2^{m+1}}\sum_1^{r_m}{}_n|\alpha_{m_n}| + \frac{1}{2}\left(1-\frac{1}{2^m}\right)|\sum_1^{r_m}{}_n\alpha_{m_n}|\right\},$$

$\forall\{\{\alpha_{m_n}\}_{n=1}^{r_m}\}_{m=1}^p\subset\mathscr{C}$. Therefore

$$\|\sum_1^{2^m}{}_n y_{m_n}\| = \frac{1+2^m}{2} = \frac{1+2^m}{3}\,\|\sum_1^{2^m}{}_n y_{m_n} - \sum_{2^m+p+1}^{2^{m+1}+p}{}_n y_{m_n}\|, \qquad \forall m \text{ and } p;$$

that is, by (7) and (78), two complementary subsequences of $\{y_n\}$ are never both norming. Moreover, by last equivalence of Theorem X* and by (78), $\{y_n\}$ belongs to a bibounded biorthogonal system. Finally, if B_7 is the completion of $\operatorname{lin}\{y_n\}$ in the norm of (78), it is easy to see that $\{y_n\}$ is an M-basis of B_7.

Example 6. Let $\{x_n\}$ be the natural basis of c_0, with (x_n, f_n) biorthogonal system; we set

(79)
$$y_n = x_{n+1} + x_1/m, \qquad \text{for} \qquad 10^m+1\leqslant n\leqslant 10^{m+1}, \qquad \forall m.$$

Then (y_n, f_{n+1}) is a bibounded biorthogonal system; moreover $\{x_{n+1}\}$ is denucleated and $\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0$, hence (see § 5) $\{y_n\}$ is denucleated. Finally by (79)

$$\left\|x_1 - \frac{1}{m}\sum_{10^m+1}^{10^m+m^2}{}_n y_n\right\| = \frac{1}{m}\left\|\sum_{10^m+1}^{10^m+m^2}{}_n x_{n+1}\right\| = \frac{1}{m}, \qquad \forall m,$$

consequently $x_1\in\bigcap_{m=1}^{\infty}[\{y_n\}_{n>m}]$, that is $\{y_n\}$ is not M-basic. Next example is continuation of Example 5 and regards (b) of Theorem VIII.

Example 7. Let $\{y_n\}$ and B_7 be the sequence and the Banach space of Example 5; let $\{y_{n_1}\}$ and $\{y_{n_2}\}$ be two infinite complementary subsequences of $\{y_n\}$, with $\{y_{n_1}\}$ basic; moreover let $\{\varepsilon_n\}\subset R^+$ so that, $\forall\{u_n\}$ of B_7 with $\|u_n - y_{n_1}\|\leqslant\varepsilon_n$, $\forall n$, $\{u_n\}$ is basic (see Theorem IV*). Then we set

(80)
$$z_{2n-1} = y_{n_1} \qquad \text{and} \qquad z_{2n} = y_{n_1} + \varepsilon_n y_{n_2}, \qquad \forall n.$$

By (80) $\{z_n\}$ is union of two basic sequences. Moreover, if (y_n, h_n) is biorthogonal system, setting $g_{2n} = h_{n_2}/\varepsilon_n$ and $g_{2n-1} = h_{n_1} - g_{2n}$, $\forall n$, (z_n, g_n) becomes a biorthogonal system with $\text{lin}\{z_n\} = \text{lin}\{y_n\}$ and $\text{lin}\{g_n\} = \text{lin}\{h_n\}$; hence $\{z_n\}$ is M-basis of B_7, because $\{g_n\}$ is total on B_7. On the other hand by hypothesis $\{y_{n_2}\}$ is not norming, moreover $\{y_{n_2}\}_{n=m}^{m+p} \subset \text{lin}\{z_n\}_{n=2m-1}^{2m+2p}$ by (80), $\forall m$ and p, hence by (7) $\{z_n\}$ is not norming.

We report now a few lemmas.

4. (a) (See also [7] p. 193 and [**10**] p. 113) $\exists$ *an H-basic and overfilling sequence $\{y_n\}$ complete in B*; (b) $\exists$ *a minimal sequence $\{z_n\}$ complete in B and convergent of infinite order.*

Proof. Let $\{x_n\}$ be an M-basis of B, with $\{x_n\} \subset S_B$ and (x_n, f_n) biorthogonal system, then we set

$$(81) \qquad y_n = \sum_1^n {}_k \frac{x_k}{10^{n(k-1)}} \qquad \text{and} \qquad z_n = \sum_1^n {}_k \frac{x_{2k}}{10^{n(k-1)}} + \frac{x_{2n-1}}{10^{n^3}}, \qquad \forall n .$$

Proceeding as in the proof of (a) of Theorem II, it is possible to verify that $\{y_n\}$ is overfilling and $\{z_n\}$ is convergent of infinite order to $\{x_{2n}\}$, moreover $\{y_n\}$ and $\{z_n\}$ are complete in B. On the other hand by (81) $\{y_n\}$ is H-basic and $(z_n, 10^{n^3} f_{2n-1})$ is a biorthogonal system. This completes the Proof of Lemma 4.

5. *Let $\{y_n\} \subset S_B$, then: $\{y_n\}$ belongs to a bibounded biorthogonal system $\Leftrightarrow$ $\Leftrightarrow \{y_n\}$ has a quasi norm.*

Proof. Suppose that (y_n, h_n) is a biorthogonal system with $\|h_n\|_{[\{y_k\}]} \leqslant M$, $\forall n$, then $\forall$ permutation $\{\tilde{y}_n\}$ of $\{y_n\}$ and $\forall \{\alpha_n\}_{n=1}^{m+p} \subset \mathscr{C}$ it follows that

$$\Big\|\sum_1^m {}_n \alpha_n \tilde{y}_n\Big\| \leqslant \sum_1^m {}_n |\alpha_n| = \sum_1^m {}_n \Big|\tilde{h}_n\Big(\sum_1^{m+p} {}_k \alpha_k \tilde{y}_k\Big)\Big| \leqslant \sum_1^m {}_n \|\tilde{h}_n\|_{[\{y_k\}]} \Big\|\sum_1^{m+p} {}_k \alpha_k \tilde{y}_k\Big\| \leqslant mM \Big\|\sum_1^{m+p} {}_n \alpha_n \tilde{y}_n\Big\| .$$

Suppose now that $\{y_n\}$ has a quasi norm M, then

$$1 = \|y_m\| \leqslant M \Big\|y_m + \sum_1^p {}_{n(\neq m)} \alpha_n y_n\Big\| , \qquad \forall \{\alpha_n\}_{n(\neq m)=1}^p \subset \mathscr{C} ;$$

therefore $\|h_m\|_{[\{y_k\}]} = 1/\text{dist}\,(y_m, [\{y_n\}_{n\neq m}]) \leqslant M$, $\forall m$. This completes the proof of Lemma 5.

For the sake of convenience we repeat Tables 1 and 2, where we denote by P_n, $1 \leqslant n \leqslant 34$, the properties of Table 1, and by Q_n, $1 \leqslant n \leqslant 29$, the properties of Table 2.

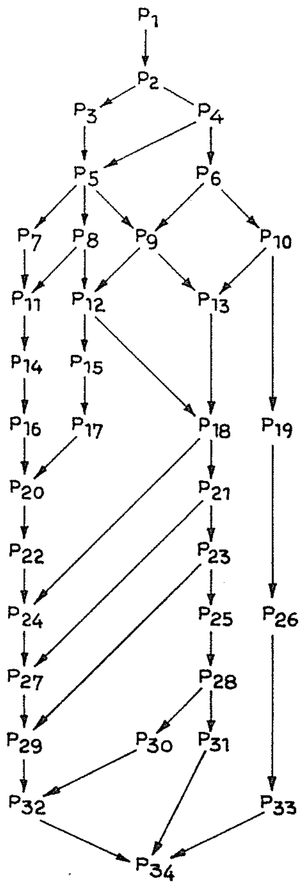


Table 1.

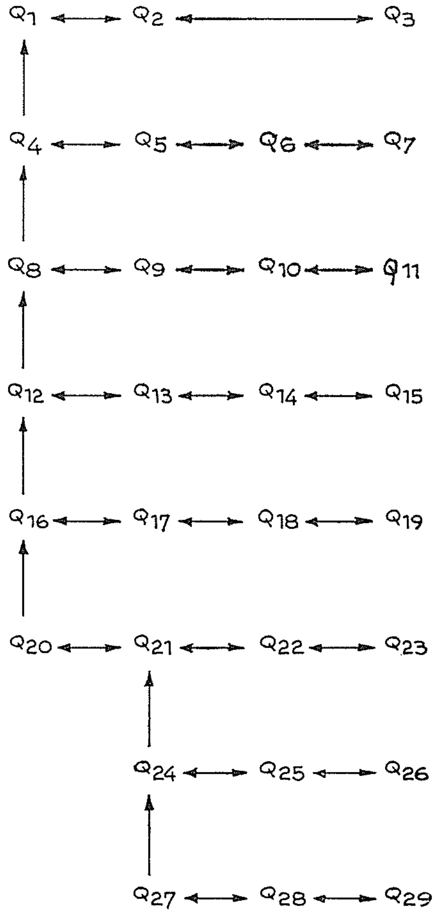


Table 2.

Proof of Table 1. $p_3 \to p_5$ and $p_{19} \to p_{26}$ are (a) of Theorem VI; $p_{14} \to p_{16}$ is b) of Theorem VIII; $p_{15} \to p_{17}$ follows by Theorem IX; $p_{20} \to p_{22}$ is (a) of Theorem VIII; $p_{28} \to p_{30}$ because, if $\{x_n\}$ is a t-non contractive sequence of B and if $\bar{x} \in [\{x_n\}]$, then (see Theorem VII of [**12**]$_4$) $\exists$ an increasing sequence $\{t_n\}$ of natural numbers and $\{\alpha_n\} \subset \mathscr{C}$ so that $\bar{x} = \sum_{1}^{\infty}{}_m \big(\sum_{t_m+1}^{t_{m+1}}{}_n \alpha_n x_n\big)$; the other implications are obvious.

Let us now verify that the inverse implications in Table 1 do not hold: $p_1 \not\leftarrow p_2 \not\leftarrow p_3$ (hence $p_4 \not\leftarrow p_5$, $p_6 \not\leftarrow p_9$, $p_{10} \not\leftarrow p_{13}$ and $p_{19} \not\leftarrow p_{26}$) follow by Example 3; $p_2 \not\leftarrow p_4$ (hence $p_3 \not\leftarrow p_5$) follow by Example 4; $p_4 \not\leftarrow p_6$ (hence $p_5 \not\leftarrow p_9$ and $p_8 \not\leftarrow p_{12}$) follow by Example 5; $p_5 \not\leftarrow p_8$ (hence $p_9 \not\leftarrow p_{12}$, $p_7 \not\leftarrow p_{11}$, $p_{13} \not\leftarrow p_{18}$ and $p_{33} \not\leftarrow p_{34}$) follow by [**12**]$_2$ (the sequence $\{x_n\}$ of example of [**12**]$_2$ is M-basic and without property P, moreover (see proof of (b) of § **5** of [**12**]$_2$) is norming); $p_6 \not\leftarrow p_{10}$ (hence $p_9 \not\leftarrow p_{13}$ and $p_{12} \not\leftarrow p_{18}$) follow by Example 6.

Let us prove that $p_{15} \not\leftarrow p_{17}$, $p_5 \not\leftarrow p_7$ (hence $p_8 \not\leftarrow p_{11}$, $p_{18} \not\leftarrow p_{24}$, $p_{21} \not\leftarrow p_{27}$, $p_{25} \not\leftarrow p_{28}$ and $p_{23} \not\leftarrow p_{29}$) and that (N.s., ω.b.l.i.) $\not\leftarrow$ (N.s., ω.l.i.) $\not\leftarrow$ (N.s.) (hence $p_{11} \not\leftarrow p_{14}$, $p_{17} \not\leftarrow p_{20}$, $p_{28} \not\leftarrow p_{30}$ and $p_{28} \not\leftarrow p_{31} \not\leftarrow p_{34}$): let $\{u_n\}$ be N.s., M.b.s. and P.s.s., but not b.b.s., let moreover $\{v_n\}$ be b.b.s. but not basic, then let $\bar{u} \in [\{u_{2n}\}]$ so that it is impossible to represent $\bar{u}$ by a series with brackets in terms of $\{u_n\}$, moreover $\exists \bar{v} \in [\{v_n\}]$ so that it is impossible to represent $\bar{v}$ by a series in terms of $\{v_n\}$. Hence, setting

$$w_1 = \bar{u},\ y_1 = \bar{v},\ z_1 = \sum_1^\infty{}_n \frac{v_n}{2^n \|v_n\|},\ w_{n+1} = u_n \quad \text{and} \quad y_{n+1} = z_{n+1} = v_n, \qquad \forall n,$$

it follows that $\{w_n\}$ is N.s., ω.b.l.i. and P.s.s., but not minimal and not M-basoidic; $\{y_n\}$ is N.s., ω.l.i. but not ω.b.l.i.; finally $\{z_n\}$ is N.s. but not ω.l.i.

Moreover $p_{10} \not\leftarrow p_{19}$ (hence $p_{18} \not\leftarrow p_{21}$ and $p_{24} \not\leftarrow p_{27}$) by a) of Theorem V (if we consider a sequence of S_B without convergent subsequences but weakly convergent to an element $\neq 0$); $p_{12} \not\leftarrow p_{15}$ by Example 1′; $p_{14} \not\leftarrow p_{16}$ by Example 7; $p_{16} \not\leftarrow p_{20}$ by Example 5 and by (7) and (9); $p_{20} \not\leftarrow p_{22}$ by Remark 2; $p_{22} \not\leftarrow p_{24}$ by Remark 1; $p_{23} \not\leftarrow p_{25}$ (hence $p_{29} \not\leftarrow p_{32}$) by (b) of Lemma 4; $p_{26} \not\leftarrow p_{33}$ by Example 1 (the sequence $\{x_n\} \cup \{z_n\}$ is P.s., but not P.s.s. by (20)); $p_{32} \not\leftarrow p_{34}$ by a) of Lemma 4; $p_{21} \not\leftarrow p_{23}$ (hence $p_{27} \not\leftarrow p_{29}$) by (a) of Theorem V*; $p_{30} \not\leftarrow p_{32}$ by Theorem XI*. This completes the Proof of Table 1.

Proof of Table 2. Firstly by (4) we observe that

$$(82) \qquad \{y_n\}\ \bar{Y}\text{-overfilling} \Rightarrow N(y_{n_1}) = \bar{Y}, \qquad \forall \{y_{n_1}\}_{n=1}^\infty \subseteq \{y_n\}.$$

$Q_1 \leftrightarrow Q_2 \leftrightarrow Q_3$ follows by a) of Theorem V*; $Q_4 \leftrightarrow Q_5$ follows by [**12**]$_3$ (Theorem X); $Q_5 \leftrightarrow Q_6$, $Q_{13} \leftrightarrow Q_{14}$ and $Q_{27} \leftrightarrow Q_{28} \leftrightarrow Q_{29}$ follow by (c) of Theorem I*; $Q_4 \rightarrow Q_7$ because, if $N\{y_{n_1}\}$ has infinite dimension, repeating the proof of theorem V of [**12**]$_4$ we find that $\{y_{n_1}\}$ has a subsequence convergent of infinite order; $Q_4 \leftarrow Q_7$ by (2) and (82); $Q_8 \leftrightarrow Q_9$ by (a) of Theorem V; $Q_9 \leftrightarrow Q_{10}$ by Lemma 5; $Q_8 \leftrightarrow Q_{11}$ by (4); $Q_{12} \leftrightarrow Q_{13} \leftrightarrow Q_{15}$ follow by (3) and (5); $Q_{16} \leftrightarrow Q_{19}$ by (6); $Q_{17} \leftrightarrow Q_{19}$ by (b) of Theorem I*; $Q_{16} \leftrightarrow Q_{18}$ follows by $Q_{12} \leftrightarrow Q_{14}$ and by the equivalence: $\{y_n\}$ without convergent bl.s. $\Leftrightarrow \{y_n\}$ without bl.s. weakly convergent to element $\neq 0$. Moreover $Q_{20} \leftrightarrow Q_{21} \leftrightarrow Q_{22} \leftrightarrow Q_{23}$ follow by (7); $Q_{24} \leftrightarrow Q_{25}$ by (c) of Theorem VIII and $Q_{25} \leftrightarrow Q_{26}$ by (9); the other implications are obvious. This completes the Proof of Table 2.

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Sommario

In questa Nota sono studiati in generale i sistemi biortogonali negli spazi di Banach, soprattutto dal punto di vista delle proprietà di convergenza delle loro blocco successioni, mettendo in risalto come la progressiva perdita di tali proprietà accompagni il miglioramento della successione, fino ad arrivare alle basi.

Si inizia con alcune costruzioni per ricavare una base di Markuscevich (M-base) da una successione minimale non M-basica, e viceversa; inoltre è specificato quando una successione può essere approssimata a piacere da una successione minimale.

Sono poi considerate le successioni di elementi con norma unitaria e senza sottosuccessioni convergenti, mettendo in evidenza il loro stretto legame con i sistemi biortogonali bilimitati; ancora è specificato quando una successione può essere approssimata a piacere da un sistema biortogonale bilimitato.

Quindi viene esaminata la struttura delle successioni di elementi con norma unitaria e senza sottosuccessioni debolmente convergenti ad elementi non nulli, collegate all'esistenza di sottosuccessioni basiche.

Infine sono considerate le successioni senza blocco successioni convergenti ad elementi non nulli, che caratterizzano le M-basi, studiandone la struttura nel caso generale ed in casi speciali; si fa inoltre un esame comparativo delle principali caratterizzazioni delle M-basi.

Il lavoro termina con uno schema generale, in cui compaiono tutte le successioni considerate, assieme alle loro interdipendenze.

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