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The algebra of certain formal power series. (**)

1. - Introduction.

Formal power series with coefficients over a ring, and particularly those with coefficients over a commutative ring, occur in many problems in number theory, algebra and analysis, and for this reason have been extensively investigated (see Ch. 1 of [2], Ch. 7 of [20], [8], [9], [6] and [13] and its associated references). In this paper we show by use of quite elementary methods that for various types of ring $\boldsymbol{R}$ the system $\boldsymbol{P}\{\boldsymbol{R}\}$ of formal power series with coefficients over $\boldsymbol{R}$ shares many of the properties of $\boldsymbol{R}$.

2. - Formal power series with coefficients over a distributive ring.

Notation 1. *The symbol* $A \equiv \boldsymbol{R}$ *is used to indicate that an accompanying statement holds for every element* A *of a prescribed mathematical system* $\boldsymbol{R}$; *the symbols* $A, B \equiv \boldsymbol{R}$, $A, B, C \equiv \boldsymbol{R}$ *are used analogously.*

Definition 1. *A distributive ring* $\boldsymbol{R}$ *is an additive abelian group for which multiplication having a two sided distribution property with respect to addition is also defined*: *we have* $AB + AC = A(B + C)$, $BA + CA = (B + C)A$ $(A, B, C \equiv \boldsymbol{R})$.

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Definition 2. *A formal power series with coefficients over a distributive ring $\boldsymbol{R}$ is a series of the form*

$$A(z) = \mathfrak{p}\{a;\, A_\nu | z\} = \sum_{\nu=a}^{\infty} A_\nu z^\nu ,$$

where a $(-\infty < a < \infty)$ is an integer, $A_\nu \in \boldsymbol{R}$ $(\nu = a, a+1, \ldots)$ and z is undefined. The two power series $A(z)$ and $A'(z) = \mathfrak{p}\{a';\, A'_\nu | z\}$ where $a' \leqslant a$ and $A'_\nu = 0$ $(\nu = a', a'+1, \ldots, a-1)$, $A'_\nu = A_\nu$ $(\nu = a, a+1, \ldots)$ are not considered to be distinct.

If $B(z) = \mathfrak{p}\{b;\, B_\nu | z\}$, $C(z) = \mathfrak{p}\{c;\, C_\nu | z\}$, then $A(z) = B(z) + C(z)$ is defined by setting

$$A_\nu = \frac{B_{\nu\ (b<c)}}{C_{\nu\ (b>c)}} \quad (\nu = a, a+1, \ldots, a'-1) , \qquad A_\nu = B_\nu + C_\nu \quad (\nu = a', a'+1, \ldots),$$

where $a = \min(b, c)$, $a' = \max(b, c)$. Subtraction of formal power series is defined in the same way.

$A(z) = B(z)\,C(z)$ is defined by the formulae $a = b + c$,

$$A_{a+r} = \sum_{\nu=0}^{r} B_{b+r-\nu}\, C_{c+\nu} \qquad (r = 0, 1, \ldots) .$$

Notation 2. *We make consistent use of the series $A(z)$, $B(z)$, $C(z)$ of Definition 2 and tacitly assume that they have the form therein given.*

Notation 3. *The system of all possible formal power series with coefficients over a distributive ring $\boldsymbol{R}$ as described in Definition 2 (for which addition, subtraction and multiplication are as prescribed in that definition) is denoted by $\boldsymbol{P}\{\boldsymbol{R}\}$. The zero power series $A(z)$ for which $A_\nu = 0$ $(\nu = a, a+1, \ldots)$ is denoted by $O(z)$.*

According to the above definition, a reference to a power series $A(z)$ is also a reference to any one of a number of series belonging to an equivalence class defined by $A(z)$; all nonzero series of the same class have a unique representative $A(z)$ having a canonical form in which $A_a \neq 0$. This greatly facilitates the definition of operations upon such series, although the results of this paper can also be derived by dealing only with canonical series and associating with each operation a process of reduction to canonical form. Formulae such as those of Definition 2 used to define addition and multiplication mean only that given a sufficiently large numer of coefficients of the series $B(z)$, $C(z)$ an arbitrarily large number of those of $A(z)$ can be derived. Any relationship between formal power series is a system of relationships between sets of coefficients. The variable z is introduced merely to motivate the calculus of formal power series. Questions of convergence are not considered.

The following elementary results will be used in the proofs of the main theorems to be given later.

Lemma 1. *Let $\varphi(C, \ldots, B, A)$ be a product formed from the n numbers $(n < \infty)$ $C, \ldots, B, A$ and bracketed in a prescribed order, and let $C(z), \ldots, B(z)$, $A(z)$ be formal power series with coefficients over a distributive ring; then*

$$\varphi\big(C(z), \ldots, B(z), A(z)\big) = \sum_{\nu_n=0}^{\infty} G_{\nu_n} z^{a+b+\ldots+c+\nu_n}, \tag{1}$$

where

$$G_{\nu_n} = \sum_{\nu_{n-1}=0}^{\nu_n} \cdots \sum_{\nu_2=0}^{\nu_3} \sum_{\nu_1=0}^{\nu_2} \varphi(C_{c+\nu_n-\nu_{n-1}}, \ldots, B_{b+\nu_2-\nu_1}, A_{a+\nu_1}) \qquad (\nu_n = 0, 1, \ldots); \tag{2}$$

furthermore, the expression $\varphi(C_{c+\nu_n-\nu_{n-1}}, \ldots, B_{b+\nu_2-\nu_1}, A_{a+\nu_1})$ occurring on the right hand side of this relationship may be replaced by a similar expression in which the symbols $C_c, \ldots, B_b, A_a$ are retained in the same order but in which the partial suffices $\nu_n - \nu_{n-1}, \ldots, \nu_2 - \nu_1, \nu_1$ are subjected to a fixed permutation.

Proof. From the formula

$$B(z)A(z) = \sum_{\nu_2=0}^{\infty} \Big\{ \sum_{\nu_1=0}^{\nu_2} B_{b+\nu_2-\nu_1} A_{a+\nu_1} \Big\} z^{a+b+\nu_2}$$

we see that formula (1) is correct when $\varphi(C, \ldots, B, A)$ is a simple product of two numbers. Any compound product of the type under consideration involving r terms can be expressed as the simple product of two compound products of $i \geqslant 1$ and $j \geqslant 1$ terms respectively

$$\varphi(C, \ldots, B, A) = \varphi'(C, \ldots, B', A')\varphi''(C'', \ldots, B, A),$$

in which $i \leqslant r-1$, $j \leqslant r-1$. Assuming formula (1) to hold for the products φ' and φ'' we have

$$\varphi\big(C(z), \ldots, B(z), A(z)\big) = \sum_{r=0}^{\infty} G_r z^{c+\ldots+b'+a'+c''+\ldots+b+a+r},$$

where $(r = 0, 1, \ldots)$

$$G_r = \sum_{\nu_j=0}^{r} \{ \sum_{\tau_{i-1}=0}^{r-\nu_j} \cdots \sum_{\tau_2=0}^{\tau_3} \sum_{\tau_1=0}^{\tau_2} \varphi'(C_{c+r-\nu_j-\tau_{i-1}}, \ldots, B'_{b'+\tau_2-\tau_1}, A'_{a'+\tau_1}) \} \cdot$$

$$\cdot \{ \sum_{\nu_{j-1}=0}^{\nu_j} \cdots \sum_{\nu_2=0}^{\nu_3} \sum_{\nu_1=0}^{\nu_2} \varphi''(C''_{c''+\nu_j-\nu_{j-1}}, \ldots, B_{b+\nu_2-\nu_1}, A_{a+\nu_1}) \}$$

$$= \sum_{\nu_j=0}^{r} \sum_{\tau_{i-1}=0}^{r-\nu_j} \cdots \sum_{\tau_2=0}^{\tau_3} \sum_{\tau_1=0}^{\tau_2} \sum_{\nu_{j-1}=0}^{\nu_j} \cdots \sum_{\nu_2=0}^{\nu_3} \sum_{\nu_1=0}^{\nu_2}$$

$$\{\varphi'(C_{c+r-\nu_j-\tau_{i-1}}, \ldots, B'_{b'+\tau_2-\tau_1}, A'_{a'+\tau_1})\varphi''(C''_{c''+\nu_j-\nu_{j-1}}, \ldots, B_{b+\nu_2-\nu_1}, A_{a+\nu_1})\} .$$

We now change to a new system of suffices obtained by preserving the set $\nu_1, \nu_2, \ldots, \nu_j$ and thereafter adopting the substitutions $\tau_1 = \nu_{j+1} - \nu_j$, $\tau_2 = \nu_{j+2} - \nu_{j+1}, \ldots,$ $\tau_{i-1} - \tau_{i-2} = \nu_{i+j-1} - \nu_{i+j-2}$, $r - \nu_j - \tau_{i-1} = \nu_{i+j} - \nu_{i+j-1}$. As may easily be verified, we then have $\nu_r = \nu_{i+j}$ and find that formula (1) with n replaced by r holds true. The first result of the lemma now follows immediately.

With this result in hand we adopt a change of suffices to a dashed system according to a prescribed permutation (e.g. $\nu'_n = \nu_n - \nu_{n-1}, \ldots,$ $\nu'_{n-1} - \nu'_{n-2} = \nu_2 - \nu_1, \nu'_n - \nu'_{n-1} = \nu_1$) and find that the multiple sum of formula (2) can be replaced by a corresponding multiple sum involving dashed suffices and in which the partial suffices $\nu_n - \nu_{n-1}, \ldots, \nu_2 - \nu_1, \nu_1$ have been replaced by the prescribed permutation of dashed suffices. The dashes are now dropped from the new equation, and the second result of the lemma has been proved.

3. - Multiplicative properties.

Notation 4. *Assuming the numbers to the defined, we set* $[A, B] = AB - BA$, $(A, B, C) = A(BC) - (AB)C$.

Definition 3. *A distributive ring* $\mathbf{R}$ *with zero element* 0 *for which*

(i) $[A, B] = 0$ $(A, B \equiv \mathbf{R})$ *is a commutative distributive ring,*

(ii) $(A, B, C) = 0$ $(A, B, C \equiv \mathbf{R})$ *is a ring,*

(iii) $A^2 = 0$ $(A \equiv \mathbf{R})$ *and* $A(BC) + B(CA) + C(AB) = 0$ $(A, B, C \equiv \mathbf{R})$ *is a Lie ring,*

(iv) $(A, B, A) \equiv 0$ $(A, B \equiv \boldsymbol{R})$ *is a flexible ring,*

(v) $(A, A, B) = (A, B, A) = (B, A, A) = 0$ $(A, B \equiv \boldsymbol{R})$ *is an alternative ring,*

(vi) $A + A = 0$ *if and only if* $A = 0$ $(A \in \boldsymbol{R})$ *is a strong distributive ring,*

(vii) *a strong distributive ring* $\boldsymbol{R}$ *for which* $(A, A, A) = 0$ $(A \equiv \boldsymbol{R})$ *is a strong cube associative ring.,*

(viii) *a strong commutative distributive ring* $\boldsymbol{R}$ *for which* $A^2(AB) = A(A^2B)$ $(A, B \equiv \boldsymbol{R})$ *is a Jordan ring,*

(ix) *a strong flexible ring* $\boldsymbol{R}$ *for which* $A^2(AB) = A(A^2B)$ $(A, B \equiv \boldsymbol{R})$ *is a noncommutative Jordan ring.*

Reference may be made to Ch. 3 of [**18**] for the general theory of rings, and to Chs. 1 and 7 of [**3**] for the theories of Lie, flexible, alternative, Jordan and noncommutative Jordan rings.

Theorem 1. *If* $\boldsymbol{R}$ *is a distributive ring,* $\boldsymbol{P}\{\boldsymbol{R}\}$ *is also a distributive ring. The same holds true with* $\boldsymbol{R}$ *and* $\boldsymbol{P}\{\boldsymbol{R}\}$ *being:* (i) *commutative distributive rings,* (ii) *rings,* (iii) *Lie rings,* (iv) *flexible rings,* (v) *alternative rings,* (vi) *strong distributive rings,* (vii) *strong cube associative rings,* (viii) *Jordan rings,* (ix) *noncommutative Jordan rings.*

Proof. That the power series of Definition 2 satisfy the relationships occurring in Definition 1 is easily verified.

When proving the remaining clauses of the theorem we tacitly assume in each case that the assumptions relating to that clause hold.

(i) For all pairs of power series $A(z)$, $B(z)$ in question, we have from the commutative property of multiplication and Lemma 1:

$$A(z)B(z) = \sum_{j=0}^{\infty} \{ \sum_{i=0}^{j} A_{a+j-i}B_{b+i}\} z^{a+b+j} = \sum_{j=0}^{\infty} \{ \sum_{i=0}^{j} B_{b+i}A_{a+j-i}\} z^{b+a+j} = B(z)A(z) .$$

(ii) The proof is as above.

(iii) We have

$$A(z)^2 = \sum_{j=0}^{\infty} \{ \sum_{i=0}^{j} A_{a+j-i}\, A_{a+i}\}\, z^{2a+j} . \tag{3}$$

If A, B are elements of a Lie ring $A^2 = B^2 = (A+B)^2 = 0$, and hence $AB + BA = 0$.

Thus

$$(4) \qquad \begin{cases} A_{a+j-i}A_{a+i} + A_{a+i}A_{a+j-i} = 0 & (i = 0, 1, \ldots, [(j-1)/2]) \\ A_{a+j-i}A_{a+i} = 0 \quad \text{when } i = [j/2]\,, \ j = 2i\,, \end{cases}$$

where $[x]$ is an integer and $x = [x] + y$ $(0 \leqslant y < 1)$. The coefficient of z^{2a+j} in formula (3) is a sum of terms of the form (4). Hence $A(z)^2 = O(z)$. Again using Lemma 1, we have

$$A(z)\{B(z)\,C(z)\} + B(z)\{C(z)\,A(z)\} + C(z)\{A(z)\,B(z)\} =$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{i=0}^{j} \{A_{a+j-i}(B_{b+k-j}\,C_{c+i}) + B_{b+k-j}(C_{c+i}\,A_{a+j-i}) +$$

$$+ C_{c+i}(A_{a+j-i}B_{b+k-j})\}z^{a+b+c+k} = O(z)\,.$$

(iv) We have

$$A(z)\{B(z)\,A(z)\} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{i=0}^{j} A_{a+k-j}(B_{b+i}A_{a+j-i})z^{b+2a+k}\,,$$

$$\{A(z)\,B(z)\}A(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{i=0}^{j} (A_{a+k-j}B_{b+i})\,A_{a+j-i}z^{b+2a+k}\,.$$

The terms involving B_{b+i} in the coefficient of z^{b+2a+k} in these two compound sums are

$$A_{a+k-j}(B_{b+i}A_{a+j}) + A_{a+j}(B_{b+i}A_{a+k-j}) \qquad (j = 0, 1, \ldots, [(k-1)/2]),$$

$$A_{a+k-j}(B_{b+i}A_{a+j}) \qquad \text{when} \qquad j = [k/2]\,, \ k = 2j$$

and

$$(A_{a+k-j}B_{b+i})\,A_{a+j} + (A_{a+j}B_{b+i})\,A_{a+k-j} \qquad (j = 0, 1, \ldots, [(k-1)/2]),$$

$$(A_{a+k-j}B_{b+i})\,A_{a+j} \qquad \text{when} \qquad j = [k/2]\,, \ k = 2j\,.$$

If $\boldsymbol{R}$ is flexible and $A, B, C \in \boldsymbol{R}$, we have $A(BC) + C(BA) = (AB)\,C + (CB)\,A$. Hence the above two sets of terms are identical, and $A(z)\{B(z)A(z)\} = \{A(z)\,B(z)\}A(z)$.

(v) The proof is as for the preceding clause.

(vi) $A(z)+A(z)=O(z)$ if and only if $A(z)=O(z)$.

(vii) If $A_i, A_j \in \boldsymbol{R}$, where $\boldsymbol{R}$ is a distributive ring and

$$\acute{\theta}(A_i, A_j)=A_i^2A_j+(A_iA_j+A_jA_i)A_i\,, \qquad \grave{\theta}(A_i, A_j)=A_i(A_iA_j+A_jA_i)+A_jA_i^2,$$

then

$$2\acute{\theta}(A_i, A_j)=(A_i+A_j)^2(A_i+A_j)-(A_i-A_j)^2(A_i-A_j)-2A_j^2A_j\,,$$

$$2\grave{\theta}(A_i, A_j)=(A_i+A_j)(A_i+A_j)^2-(A_i-A_j)(A_i-A_j)^2-2A_jA_j^2,$$

where $2D$ is used as an abbreviation for $D+D$. Hence, if $\boldsymbol{R}$ is a strong cube associative ring, $\acute{\theta}(A_i, A_j)=\grave{\theta}(A_i, A_j)\ (A_i, A_j \equiv \boldsymbol{R})$. (It appears that the assumption that $\boldsymbol{R}$ be strong must be made for this relationship to hold.) Furthermore, we also have

$$\begin{aligned}\acute{\varphi}(A_i, A_j, A_k) &= (A_iA_j+A_jA_i)A_k+(A_iA_k+A_kA_i)A_j+(A_jA_k+A_kA_j)A_i\\ &= \acute{\theta}(A_j+A_k, A_i)-\acute{\theta}(A_j, A_i)-\acute{\theta}(A_k, A_i)\,,\\ \grave{\varphi}(A_i, A_j, A_k) &= A_i(A_jA_k+A_kA_j)+A_j(A_iA_k+A_kA_i)+A_k(A_iA_j+A_jA_i)\\ &= \grave{\theta}(A_j+A_k, A_i)-\grave{\theta}(A_j, A_i)-\grave{\theta}(A_k, A_i)\,,\end{aligned}$$

and hence again $\acute{\varphi}(A_i, A_k, A_j)=\grave{\varphi}(A_i, A_k, A_j)\ \ (A_i, A_j, A_k \equiv \boldsymbol{R})$ if $\boldsymbol{R}$ is strong cube associative.

By expansion, we have

$$\{A(z)^2\}A(z)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{m}(A_{a+n-m}A_{a+m-l})A_{a+l}z^{3a+n}\,.$$

In this expansion, we group together those products of the form $(A_iA_j)A_k$ in which the suffices i, j and k are unequal, those in which one equal pair occurs and, where they exist, those for which $i=j=k$. We derive

$$\{A(z)^2\}A(z)=\sum_{n=0}^{\infty}\acute{G}_n z^{3a+n}\,,$$

where

$$(5)\quad \acute{G}_n = \sum_{\nu=0}^{[(n-1)/3]} \acute{\theta}(A_{a+\nu}, A_{a+n-2\nu}) + \sum_{\nu=0}^{n_1} \acute{\theta}(A_{a+n_2-2\nu}, A_{a+n-2n_2+2\nu}) + \\ + \sum_{\nu=0}^{[(n-1)/2]} \sum_{\nu'=0}^{[(n-2\nu-1)/3]} \acute{\varphi}(A_{a+\nu'}, A_{a+n-\nu-2\nu'}, A_{a+\nu+\nu'}) + (n=3n_3)(A_{n_3})^2 A_{n_3}\,,$$

$$n_3 = [n/3],\; n_2 = [n/2],\; n_1 = [n/6] - \big(n-1 = 6[(n-1)/6]\big) - (n = 3n_3)$$

for $n = 0, 1, \ldots$, the numerical value of a Boolean expression such as $(n = 3n_3)$ being 1 if $n = 3n_3$ and 0 otherwise. (The reader may verify formula (5) at his leisure; its derivation has no special interest.)

Similarly we derive

$$A(z)\{A(z)^2\} = \sum_{n=0}^{\infty} \grave{G}_n z^{3a+n},$$

where $\grave{G}_n$ $(n = 0, 1, \ldots)$ is expressed in terms of the functions $\grave{\theta}(A_i, A_j)$ and $\grave{\varphi}(A_i, A_j, A_k)$ by a formula analogous to (5). It follows that if $\boldsymbol{R}$ is strong cube associative then $\acute{G}_n = \grave{G}_n$ $(n = 0, 1, \ldots)$ and hence that $\big(A(z), A(z), A(z)\big) = O(z)$.

We give the above method of proof because it can be extended to the following clause. The result of the present clause may, however, also be proved in the following way: we have $\big(A_n(z), A_n(z), A_n(z)\big) = 0$ for all polynomials of degree n when $n = 0$. Assume that this relationship holds with n replaced by $n-1$ for some $n \geqslant 1$. Writing out the three versions of this relationship with $A_n(z)$ replaced by $A_{n-1}(z)$, $A_{n-1}(z) + A_n$, $A_{n-1}(z) - A_n$ and judiciously combining them with powers of z^n, we find that the relationship in question holds for $A_n(z) = A_{n-1}(z) + A_n z^n$, i.e. for all polynomials of degree n.

(viii) If $\boldsymbol{R}$ is commutative and strong then, from clauses (i) and (vi), $\boldsymbol{P}\{\boldsymbol{R}\}$ is likewise. We set

$$\acute{\theta}(A_i, A_j | B) = A_i^2(A_j B) + (A_i A_j + A_j A_i)(A_i B)\,,$$

$$\grave{\theta}(A_i, A_j | B) = A_i\{(A_i A_j + A_j A_i)B\} + A_j(A_i^2 B)$$

and have

$$2\acute{\theta}(A_i, A_j | B) = (A_i + A_j)^2\{(A_i + A_j)B\} - (A_i - A_j)^2\{(A_i - A_j)B\} - 2A_j^2(A_j B)\,,$$

$$2\grave{\theta}(A_i, A_j | B) = (A_i + A_j)\{(A_i + A_j)^2 B\} - (A_i - A_j)\{(A_i - A_j)^2 B\} - 2A_j(A_j^2 B)\,.$$

If $A^2(AB) = A(A^2B)(A, B \equiv \boldsymbol{R})$ and $\boldsymbol{R}$ is strong, then $\acute{\theta}(A_i, A_j|B) = \grave{\theta}(A_i', A_j|B)$ $(A_i, A_j, B \equiv \boldsymbol{R})$. Furthermore

$$\begin{aligned}\acute{\varphi}(A_i, A_j, A_k|B) &= (A_iA_j + A_jA_i)(A_kB) + (A_iA_k + A_kA_i)(A_jB) + \\ &\quad + (A_jA_k + A_kA_j)(A_iB) = \\ &= \acute{\theta}(A_j + A_k, A_i|B) - \acute{\theta}(A_j, A_i|B) - \acute{\theta}(A_k, A_i|B),\end{aligned}$$

$$\begin{aligned}\grave{\varphi}(A_i, A_j, A_k|B) &= A_i\{(A_jA_k + A_kA_j)B\} + A_j\{(A_iA_k + A_kA_i)B\} + \\ &\quad + A_k\{(A_iA_j + A_jA_i)B\} = \\ &= \grave{\theta}(A_j + A_k, A_i|B) - \grave{\theta}(A_j, A_i|B) - \grave{\theta}(A_k, A_i|B)\end{aligned}$$

and we again have $\acute{\varphi}(A_i, A_j, A_k|B) = \grave{\varphi}(A_i, A_j, A_k|B)$ $(A_i, A_j, A_k, B \equiv \boldsymbol{R})$. As in the proof of the preceding clause, we have

$$A(z)^2\{A(z)B(z)\} = \sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{m=0}^{n}\sum_{l=0}^{m}(A_{a+r-n}A_{a+n-m})(A_{a+m-l}B_{b+l})z^{3a+b+r} = \sum_{r=0}^{\infty}\acute{G}_r z^{3a+b+r},$$

where the term involving B_{b+l} in $\acute{G}_r$ is

$$(6)\quad \left\{\begin{aligned}&\sum_{\nu=0}^{[(r-1)/3]}\acute{\theta}(A_{a+\nu}, A_{a+r-2\nu}|B_{b+l}) + \sum_{\nu=0}^{r_1}\acute{\theta}(A_{a+r_2-2\nu}, A_{a+r_2-2r+2\nu}|B_{b+l})\\ &+\sum_{\nu=0}^{[(r-1)/2]}\sum_{\nu'=0}^{[(r-2\nu-1)/3]}\acute{\varphi}(A_{a+\nu'}, A_{a+r-\nu-2\nu'}, A_{a+\nu+\nu'}|B_{b+l}) + (r = 3r_3)\, A^2_{a+r_3}(A_{a+r_3}B_{b+l}),\end{aligned}\right.$$

where $r_3, r_2, \ldots$ are defined by formulae similar to those defining $n_3, n_2, \ldots$ in equation (5). Again

$$A(z)\{A(z)^2B(z)\} = \sum_{r=0}^{\infty}\grave{G}_r z^{3a+b+r},$$

where the term involving B_{b+l} in $\grave{G}_r$ is obtained from expression (6) by replacing the accent grave by an accent acute, and the last term by $(r = 3r_3)\cdot A_{a+r_3}(A^2_{a+r_3}B_{b+l})$. It follows that if $\boldsymbol{R}$ is a Jordan ring, expression (6) and its companion are equal, i.e. $\acute{G}_r = \grave{G}_r$ $(r = 0, 1, \ldots)$, and hence that $A(z)^2\cdot\{A(z)B(z)\} = A(z)\{A(z)^2B(z)\}(A(z), B(z) \equiv \boldsymbol{P}\{\boldsymbol{R}\})$.

(ix) If $\boldsymbol{R}$ is flexible, $\boldsymbol{P}\{\boldsymbol{R}\}$ is likewise. The remainder of the proof is as above.

Once it has been established that $\boldsymbol{P}\{\boldsymbol{R}\}$ is a distributive ring, it immediately follows that a number of identities holding for elements of $\boldsymbol{R}$ also hold for elements of $\boldsymbol{P}\{\boldsymbol{R}\}$. For example, Zorn [**21**]$_1$ has derived the formulae

$$(AB, C, D) - (A, BC, D) + (A, B, CD) = A(B, C, D) + (A, B, C)D \qquad (A, B, C, D \equiv \boldsymbol{R}),$$

$$[[A, B], C] + [[B, C]A] + [[C, A], B] =$$

$$= (A, B, C) + (B, C, A) + (C, A, B) - (B, A, C) - (A, C, B) - (C, B, A) \qquad (A, B, C \equiv \boldsymbol{R});$$

consequently for the elements of $\boldsymbol{P}\{\boldsymbol{R}\}$ we have two similar relationships obtained by replacing A by $A(z)$, B by $B(z)$, and so on and holding for $A(z), \ldots, D(z) \equiv \boldsymbol{P}\{\boldsymbol{R}\}$ in the first case and for $A(z), B(z), C(z) \equiv \boldsymbol{P}\{\boldsymbol{R}\}$ in the second. The latter relationships may be proved either by manipulating the elements of $\boldsymbol{P}\{\boldsymbol{R}\}$ as a distributive ring or, from first principles, using products of power series expansions involving the elements of $\boldsymbol{R}$.

The defining relationships lead to special systems of derived identities in each of the cases adumbrated in Definition 3. For example, the identity $AB + BA = 0$ follows from the relationship $A^2 = 0$ in the case of a Lie ring, whilst the formula $(A, B, C) - (C, B, A) = 0$ follows from the relationship $(A, B, A) = 0$ in the case of a flexible ring. If A, B, C are elements of an alternative ring, we have Zorn's identities [**21**]$_1$ $(A, B, C) = -(B, A, C) = -(C, B, A) = -(A, C, B)$, Moufang's identities [**12**] $(AB)(CA) = \{A(BC)\}A = A\{(BC)A\}$, $(A, B, C)A = (AB, C, A)$, $(A, B, CA) = A(C, B, A)$ and, setting $A \cdot B = AB + BA$, the identities of Bruck and Kleinfeld [**4**] $(A^2, B, C) = A \cdot (A, B, C)$, $(C, B, A^2) = A \cdot (C, B, A)$. All of these derived identities are satisfied by the corresponding power series of Theorem 1. If $\boldsymbol{R}$ is a Lie ring, we also have the relationship $A(z)B(z) + B(z)A(z) = O(z)$ $(A(z), B(z) \equiv \boldsymbol{P}\{\boldsymbol{R}\})$, and so on.

Rings belonging to classes similar to those considered in clause (v) of Theorem 1 (for example, those studied by Kosier [**10**]$_2$ for which only the right alternative identity $(B, A, A) = 0$ holds) can also be treated by use of the same methods.

Once it has been shown that results concerning homogeneous linear expressions of the third degree can be extended to the theory of formal power series with coefficients over a strong distributive ring, such power series fall within the domain of application of a number of theories. For example, Zorn [**21**]$_2$

bases a classification of distributive rings upon the use of four rules:

$$\text{I:}\quad (A, A, A) = 0\ ,$$

$$\text{II:}\quad [[A, B], C] + [[B, C], A] + [[C, A], B] = 0\ ,$$

$$\text{III:}\quad (B, A, A) - (A, A, B) = 0\ ,$$

$$\text{IV:}\quad (B, A, A) - 2(A, B, A) + (A, A, B) = 0\ ,$$

showing that an alternative ring is defined by the scheme I + II + IV, that a flexible ring is defined by the scheme I + IV, and that Jordan's cyclic system for every triad of elements A, B, C of which $(A, B, C) = (B, C, A)$ is defined by the scheme III + IV. Outcalt [**15**] has shown that if $(A, B, C) = (B, C, A)$ $(A, B, C \equiv \boldsymbol{R})$, where $\boldsymbol{R}$ is a strong distributive ring, then $\boldsymbol{R}$ is strong and alternative if and only if $(A, A, A) = 0$ $(A \equiv \boldsymbol{R})$; Kosier [**10**]$_1$ has given a theory of distributive rings whose elements satisfy these two relationships. Again, Osborne [**14**] defines a partial ordering on the set of all homogeneous identities satisfied by the elements of distributive ring, and finds necessary and sufficient conditions that an identity does not imply an identity lower than it in the ordering. All of these theories now have application to the power series under discussion: the power series with coefficients over a strong distributive ring which constitute an alternative ring both satisfy and are defined by formulae analogous to I + II + IV above, and so on.

Knopfmacher [**11**] has given a theory of distributive rings whose elements satisfy a system of identities $\boldsymbol{S}$, and shown that special choices of $\boldsymbol{S}$ lead to the definitions of Lie, Jordan, alternative, associative and commutative distributive rings; with Theorem 1 in hand, this theory can be applied to formal power series with coefficients over a distributive ring.

It should perhaps be pointed out that the elements of many of the rings considered in Definition 2 may be represented as elements of a linear algebra over a field. For example, square, upper triangular and lower triangular matrices of finite dimension, and quaternions are elements of a linear associative algebra; the numbers of a hierarchy of systems of 2^n units $(n = 0, 1, \ldots)$ due to Albert [**1**]$_1$, which places the real numbers, complex numbers, quaternions, Cayley numbers and extended Cayley numbers in a unified setting, are elements of a flexible algebra [**17**]; Cayley numbers [**5**] (see also [**7**]$_1$ [**7**]$_3$ and ch. 7 of [**3**]) are elements of an alternative algebra with eight units; various representations of Jordan algebras are known [**3**]. We remark, however, that formal power series with coefficients over a linear algebra are not themselves elements of a linear algebra.

Theorem 2. *If a distributive ring* $\boldsymbol{R}$ *possesses a unit element* I, *then* $\boldsymbol{P}\{\boldsymbol{R}\}$ *also possesses a unit element.*

Proof. The unit element in question is $\mathfrak{p}\{0; A_\nu|z\}$, where $A_0 = I$, $A_\nu = 0$ $(\nu = 1, 2, \ldots)$.

Definition 4. *The set* $\boldsymbol{C}\{\boldsymbol{R}\}$ *of all elements* ξ *of a distributive ring* $\boldsymbol{R}$ *for which* $[A, \xi] = 0$ $(A \equiv \boldsymbol{R})$, $(A, B, \xi) = (A, \xi, B) = (\xi, A, B) = 0$ $(A, B \equiv \boldsymbol{R})$ *is the centre of* $\boldsymbol{R}$.

As is easily shown, the elements of the centre of a distributive ring form a commutative ring.

Theorem 3. *If a distributive ring* $\boldsymbol{R}$ *possesses a centre, so does* $\boldsymbol{P}\{\boldsymbol{R}\}$.

Proof. It is a simple matter to verify that the set of formal power series with coefficients over $\boldsymbol{C}\{\boldsymbol{R}\}$ also satisfy relationships analogous to those of Definition 4.

Albert [**1**]$_2$ and Kosier [**10**]$_3$ have constructed a theory of strong rings $\boldsymbol{R}$ (a which possess a centre b) whose elements satisfy a nontrivial identity of the form

$$\alpha_1(CA)B + \alpha_2(CB)A + \alpha_3B(CA) + \alpha_4A(CB) + \\ + \alpha_5(AC)B + \alpha_6(BC)A + \alpha_7B(AC) + \alpha_8A(BC) = 0$$

$(A, B, C \equiv \boldsymbol{R},\ \alpha_1, \ldots, \alpha_8 \in \boldsymbol{C}\{\boldsymbol{R}\})$ and c) possess a nonflexible subring of elements and d) possess a unit element. This theory can be extended to the formal power series of Theorems 2 and 3.

Definition 5. *A distributive ring* $\boldsymbol{R}$ *for which, when* $A, B \in \boldsymbol{R}$, $AB = 0$ *if and only if either* $A = 0$ *or* $B = 0$ *is said to be without divisors of zero.*

Theorem 4. *If the distributive ring* $\boldsymbol{R}$ *is without divisors of zero, then so is* $\boldsymbol{P}\{\boldsymbol{R}\}$.

Proof. If $A(z)B(z) = O(z)$ then, equating coefficients of z^{a+b} upon both sides of this equation, $A_aB_b = 0$, i.e. either A_a or $B_b = 0$. Assume that it has been shown that $A_\nu = 0$ $(\nu = a, a+1, \ldots, a+a'-1)$ $B_\nu = 0$ $(\nu = b, b+1, \ldots, b+b'-1)$ where $a' + b' = r$. Equating the coefficient of z^{a+b+r} upon the left hand side of the equation $A(z)B(z) = O(z)$ to zero, we find that $A_{a+a'}B_{b+b'} = 0$, i.e. a' or b' can be increased by unity. In this way we show that all coefficients of at least one of the series $A(z)$ and $B(z)$ can be reduced to zero.

As is easily verified, if $\boldsymbol{R}$ is without divisors of zero and the formal power series $A(z)$ satisfies either of the equations $A(z)B(z) = C(z)$, $B(z)A(z) = C(z)$ $\big(A(z), B(z), C(z) \in \boldsymbol{P}\{\boldsymbol{R}\}, B(z) \neq O(z)\big)$, then it is the only one to do so.

4. - Transformations of formal power series.

Definition 6. *A transformation* $\mathfrak{D}$ *operating upon every element of a distributive ring* $\boldsymbol{R}$ *and such that* $\mathfrak{D}(A+B) = \mathfrak{D}A + \mathfrak{D}B$, $\mathfrak{D}(AB) = A(\mathfrak{D}B) + (\mathfrak{D}A)B$ $(A, B \equiv \boldsymbol{R})$ *is called a derivation.*

Definition 7. *The transformation* $A(z) \to \mathfrak{D}A(z) = \mathfrak{p}\{a; \mathfrak{D}A_\nu | z\}$ *where* $\boldsymbol{R}$ *is a distributive ring permitting a derivation* $\mathfrak{D}$ *and* $A_\nu \in \boldsymbol{R}$ $(\nu = a, a+1, \ldots)$ *is called the coefficient derivation* $\mathfrak{C}\{\mathfrak{D}\}$ *of the formal power series* $A(z)$.

Theorem 5. *Let* $\boldsymbol{R}$ *be a distributive ring permitting a derivation* $\mathfrak{D}$; *then the coefficient derivation* $\mathfrak{C}\{\mathfrak{D}\}$ *over* $\boldsymbol{P}\{\boldsymbol{R}\}$ *is a derivation*

Proof. From Theorem 1, $\boldsymbol{P}\{\boldsymbol{R}\}$ is a distributive ring.

We have

$$\mathfrak{D}\{A(z)B(z)\} = \sum_{j=0}^{\infty} \mathfrak{D}\{\sum_{i=0}^{j} A_{a+j-i} B_{b+i}\} z^{a+b+j}$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{j} \{\mathfrak{D}(A_{a+j-i} B_{b+i})\} z^{a+b+j}$$

and two similar series for $A(z)\{\mathfrak{D}B(z)\}$ and $\{\mathfrak{D}A(z)\}B(z)$ in which the expression $\mathfrak{D}(A_{a+j-i}B_{b+i})$ is replaced by $A_{a+j-i}(\mathfrak{D}B_{b+i})$ and $(\mathfrak{D}A_{a+j-i})B_{b+i}$ respectively. Hence

$$\mathfrak{D}\{A(z)B(z)\} = A(z)\{\mathfrak{D}B(z)\} + \{\mathfrak{D}A(z)\}B(z) \cdot \big(A(z), B(z) \equiv \boldsymbol{P}\{\boldsymbol{R}\}\big) .$$

Definition 8. *A distributive ring* $\boldsymbol{R}$ *is said to permit a distributive involution* $\mathfrak{J}$ *if associated with every element* $A \in \boldsymbol{R}$ *there exists a transformation* $A \to \hat{A}$ $(\hat{A} \in \boldsymbol{R})$ *such that* $\hat{\hat{A}} = A$ $(A \equiv \boldsymbol{R})$ *and* $\widehat{A+B} = \hat{A} + \hat{B}$, $\widehat{AB} = \hat{B}\hat{A}$ $(A, B \equiv \boldsymbol{R})$; *the sum* $\mathfrak{t}(A) = A + \hat{A}$ *is called the trace of* A; *the product* $\mathfrak{n}(A) = A\hat{A}$ *is called the norm of* A; $\mathfrak{J}$ *is said to have central trace if* $\mathfrak{t}(A) \in \boldsymbol{C}\{\boldsymbol{R}\}$ $(A \equiv \boldsymbol{R})$, *and to have a central norm if* $\mathfrak{n}(A) \in \boldsymbol{C}\{\boldsymbol{R}\}$ $(A \equiv \boldsymbol{R})$.

Definition 9. *The transformation* $A(z) \to \hat{A}(z) = \mathfrak{p}\{a; \hat{A}_\nu | z\}$ *where* $\boldsymbol{R}$ *is a distributive ring permitting a distributive involution* $\hat{\mathfrak{J}}$ *and* $A_\nu \in \boldsymbol{R}$ $(\nu = a, a+1, \ldots)$ *is called a coefficient involution* $\mathfrak{C}\{\hat{\mathfrak{J}}\}$ *of the formal power series* $A(z)$; *the formal power series* $\mathfrak{t}\{A(z)\} = A(z) + \hat{A}(z)$ *and* $\mathfrak{n}\{A(z)\} = A(z)\hat{A}(z)$ *are called the trace and norm respectively of the formal power series* $A(z)$.

Theorem 6. *The coefficient involution* $\mathfrak{C}\{\hat{\mathfrak{J}}\}$ *associated with each series of the set* $\boldsymbol{P}\{\boldsymbol{R}\}$, *where* $\boldsymbol{R}$ *permits the distributive involution* $\hat{\mathfrak{J}}$, *is a distributive involution over* $\boldsymbol{P}\{\boldsymbol{R}\}$; *if* $\hat{\mathfrak{J}}$ *has a central trace (norm) over* $\boldsymbol{R}$, *then* $\mathfrak{C}\{\mathfrak{J}\}$ *has a central trace (norm) over* $\boldsymbol{P}\{\boldsymbol{R}\}$; *if* $\mathfrak{T}$ *has a central trace over* $\boldsymbol{R}$, *then* $\mathfrak{n}\{\hat{A}(z)\} = \mathfrak{n}\{A(z)\}$ $(A(z) \equiv \boldsymbol{P}\{\boldsymbol{R}\})$.

Proof. In the notation of Definition 9, we have $\hat{A}(z) \in \boldsymbol{P}\{\boldsymbol{R}\}$, and it is also clear that $\hat{\hat{A}}(z) = A(z)$ $(A(z) \equiv \boldsymbol{P}\{\boldsymbol{R}\})$, $\widehat{A(z) + B(z)} = \hat{A}(z) + \hat{B}(z)$ $(A(z), B(z) \equiv \boldsymbol{P}\{\boldsymbol{R}\})$. To prove the first part of the theorem it remains to show that $\widehat{A(z)B(z)} = \hat{B}(z)\hat{A}(z)$. We have

$$\widehat{A(z)B(z)} = \sum_{j=0}^{\infty} \sum_{i=0}^{j} \widehat{A_{a+j-i} B_{b+i}}\, z^{a+b+j} =$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{j} \hat{B}_{b+i}\hat{A}_{a+j-i} z^{a+b+j} = \hat{B}(z)\hat{A}(z) \cdot (A(z), B(z) \equiv \boldsymbol{P}\{\boldsymbol{R}\}) .$$

Turning to the second part of the theorem, we remark that the coefficients of the series $\mathfrak{t}\{A(z)\}$ are $\mathfrak{t}(A_\nu)$ $(\nu = a, a+1, \ldots)$ each of which belongs to the centre of $\boldsymbol{R}$; hence $\mathfrak{t}\{A(z)\} \in \boldsymbol{C}\{\boldsymbol{P}\{\boldsymbol{R}\}\}$. We also have

$$\mathfrak{n}\{A(z)\} = \sum_{j=0}^{\infty} \{ \sum_{i=0}^{j} A_{a+j-i} \hat{A}_{a+i}\} z^{2a+j}.$$

If $\mathfrak{n}(A) = A\hat{A} \in \boldsymbol{C}\{\boldsymbol{R}\}$ $(A \equiv \boldsymbol{R})$, then $(A + B)(\hat{A} + \hat{B}) \in \boldsymbol{C}\{\boldsymbol{R}\}$ $(A, B \equiv \boldsymbol{R})$ and, since the elements of $\boldsymbol{C}\{\boldsymbol{R}\}$ form a distributive ring $B\hat{A} + A\hat{B} \in \boldsymbol{C}\{\boldsymbol{R}\}$ $(A, B \equiv \boldsymbol{R})$. Hence

$$A_{a+j-i}\hat{A}_{a+i} + A_{a+i}\hat{A}_{a+j-i} \in \boldsymbol{C}\{\boldsymbol{R}\} \quad (i = 0, 1, \ldots, [(j-1)/2]; j = 0, 1, \ldots)$$

$$A_{a+j-i}\hat{A}_{a+i} \in \boldsymbol{C}\{\boldsymbol{R}\} \quad \text{if} \quad j = 2i \quad \text{when} \quad i = [j/2] \quad (j = 0, 1, \ldots) .$$

Hence the coefficients of the series $\mathfrak{n}\{A(z)\}$ belong to $\boldsymbol{C}\{\boldsymbol{R}\}$, i.e. $\mathfrak{n}\{A(z)\} \in \boldsymbol{C}\{\boldsymbol{P}\{\boldsymbol{R}\}\}$.

With regard to the last part of the theorem we have, subject to the stated conditions,

$$\mathfrak{n}\{\hat{A}(z)\} = \hat{A}(z)A(z) = [\mathfrak{t}\{A(z)\} - A(z)]A(z)$$
$$= A(z)\mathfrak{t}\{A(z)\} - A(z)^2 = A(z)[\mathfrak{t}\{A(z)\} - A(z)] = \mathfrak{n}\{A(z)\}.$$

The above theorem can be applied to the cases in which the elements of $\boldsymbol{R}$ are quaternions over a field, Cayley numbers over a field, the numbers of Albert's hierarchy and those of the author's generalisation of it [**19**]$_1$.

We proved the result $\mathfrak{n}\{\hat{A}(z)\} = \mathfrak{n}\{A(z)\}$ when $\hat{\mathfrak{S}}$ has a central trace only because we make subsequent use of it. Once it has been shown that $\mathfrak{C}\{\hat{\mathfrak{S}}\}$ is an involution, the general theory of involutions can be applied to the power series in question; we mention, in particular, a number of results concerning involutions over distributive rings given in [**19**]$_1$, and the special result due to Zorn [**21**]$_1$ that $\mathfrak{n}(AB) = \mathfrak{n}(A)\mathfrak{n}(B)$ if A, B are elements of an alternative ring permitting involution.

5. - Properties of the inverse.

We saw in § 3 that the multiplicative properties of the distributive rings $\boldsymbol{R}$ are largely carried over to the rings $\boldsymbol{P}\{\boldsymbol{R}\}$ considered in that section. The same is not true with regard to the properties of the inverse of the elements of $\boldsymbol{R}$.

Definition 10. *If to the element A of a distributive ring $\boldsymbol{R}$ posssessing a unit element I there corresponds an element $A^{-1(L)} \in \boldsymbol{R}$ such that $A^{-1(L)}A = I$, then $A^{-1(L)}$ is called a left inverse of A. A right inverse $A^{-1(R)}$ of A is similarly defined. If both inverses of A exist , are unique, and are equal, A is said to possess a two sided inverse.*

Notation 5. *The set of elements A of a distributive ring $\boldsymbol{R}$ possessing a left (right) inverse is denoted by $\boldsymbol{R}_{LI}(\boldsymbol{R}_{RI})$. $\boldsymbol{R}_I$ denotes the intersection of $\boldsymbol{R}_{LI}$ and $\boldsymbol{R}_{RI}$.*

The set of equations determining the left inverse $\mathfrak{p}\{-a, A'_\nu|z\}$ of the power series $A(z)$ with coefficients over a distributive ring $\boldsymbol{R}$ is

$$A'_{-a}A_a = I, \quad \sum_{\nu=0}^{r} A'_{-a+\nu}\, A_{a+r-\nu} = 0 \qquad (r = 1, 2, \ldots). \tag{7}$$

If $\boldsymbol{R}_{LI}$ exists and $A_a \in \boldsymbol{R}_{LI}$, then we may take $A'_\nu = A^{-1(L)}$ to ensure that the first of these equations is satisfied. However, without assuming something further concerning the multiplicative properties of $\boldsymbol{R}$ it is in general impossible to determine the further coefficients $A'_{-a+\nu}$ $(\nu = 1, 2, \ldots)$: in this case a left inverse is defined for the elements of $\boldsymbol{R}$ but not for those of $\boldsymbol{P}\{\boldsymbol{R}\}$.

Of the further assumptions concerning the elements of $\boldsymbol{R}$ that can be made, $\boldsymbol{R}$ can be taken to be a field, and we have the well known result (Ch. 18 of [**18**]) that $\boldsymbol{P}\{\boldsymbol{R}\}$ is also a field. We shall derive some further special results.

Theorem 7. *Let* $\boldsymbol{R}$ *be a ring containing a non-empty set* $\boldsymbol{R}_I$ *of elements with left and right inverse; then* $\boldsymbol{P}\{\boldsymbol{R}\}$ *is a ring with a non-empty set of two sided inverses.*

Proof. As is well known (see Ch. 3 of [**18**]) if $\boldsymbol{R}$ is a ring, $A \in \boldsymbol{R}$, and $A^{-1(L)}$ and $A^{-1(R)}$ exist, then these inverses are unique and identical; we denote them both by A^{-1}. If $A_a \in \boldsymbol{R}_I$, then equations (7) may be solved and we derive

$$A'_{-a} = A_a^{-1}, \quad A'_{-a+r} = - \sum_{\nu=0}^{r-1} A'_{-a+\nu} A_{a+r-\nu} A_a^{-1} \qquad (r = 1, 2, \ldots). \tag{8}$$

The equations to determine the right inverse $\mathfrak{p}\{-a; A''_\nu | z\}$ of $A(z)$ are

$$A_a A''_{-a} = I, \quad \sum_{\nu=0}^{r} A_{a+r-\nu} A''_{-a+\nu} = 0 \qquad (r = 1, 2, \ldots); \tag{9}$$

they may be solved, and we have

$$A''_{-a} = A_a^{-1}, \quad A''_{-a+r} = - \sum_{\nu=0}^{r-1} A_a^{-1} A_{a+r-\nu} A''_{-a+\nu} \quad (r = 1, 2, \ldots). \tag{10}$$

Since the left and right inverse of $A(z)$ exist and $\boldsymbol{P}\{\boldsymbol{R}\}$ is a ring (clause (ii) of Theorem 1) it follows without further deliberation that these inverses are unique and identical, and that the same holds for all series $\mathfrak{p}\{a; A_\nu | z\}$ for which $A_a \in \boldsymbol{R}_I$.

It is, however, instructive to prove from first principles that the series $\mathfrak{p}\{-a; A'_\nu | z\}$ and $\mathfrak{p}\{-a; A''_\nu | z\}$ are identical. We have $A'_{-a} = A''_{-a}$. Assume that $A'_{-a+\nu} = A''_{-a+\nu}$ $(\nu = 0, 1, \ldots, r-1)$. Replacing $A'_{-a+\nu}$ by $A''_{-a+\nu}$ in the second of formulae (8) and using the appropriate equations of the set (10), we have

$$A'_{-a+r} = - A_a^{-1} A_{a+r} A_a^{-1} + \sum_{\nu=1}^{r-1} A_a^{-1} \sum_{\nu'=0}^{\nu-1} A_{a+\nu-\nu'} A''_{-a+\nu'} A_{a+r-\nu} A_a^{-1}$$

and, by rearrangement,

$$(11)\qquad A'_{-a+r} = -A_a^{-1} A_{a+r} A_a^{-1} + \sum_{\nu=0}^{r-2} \sum_{\nu'=1}^{r-\nu-1} A_a^{-1} A_{a+\nu'} A''_{-a+\nu} A_{a+r-\nu-\nu'} A_a^{-1} .$$

In the same way we derive an identical formula for A''_{-a+r} in terms of the coefficients $A'_{-a+\nu}$ ($\nu = 0, 1, \ldots, r-2$). Since $A'_{-a+\nu} = A''_{-a+\nu}$ ($\nu = 0, 1, \ldots, r-2$), we have $A'_{-a+r} = A''_{-a+r}$. It follows by induction that $A'_{-a+r} = A''_{-a+r}$ ($r = 0, 1, \ldots$).

The above theorem has applications to quaternions, to square matrices, lower triangular matrices and upper triangular matrices of finite dimension, and to certain classes of infinite matrices. We give a further result concerning the nature of the inverse series below (see Theorem 11).

Definition 11. *A distributive ring with centre $\boldsymbol{C}\{\boldsymbol{R}\}$ is said to possess an invertible subcentre $\boldsymbol{C}_I\{\boldsymbol{R}\}$ if $\boldsymbol{R}$ possesses a unit element I and $\boldsymbol{C}\{\boldsymbol{R}\}$ contains a nonempty subset $\boldsymbol{C}_I\{\boldsymbol{R}\}$ of elements such that to any $\xi \in \boldsymbol{C}_I\{\boldsymbol{R}\}$ there corresponds an element $\xi^{-1} \in \boldsymbol{C}\{\boldsymbol{R}\}$ for which $\xi^{-1}\xi = I$.*

Since the elements of $\boldsymbol{C}\{\boldsymbol{R}\}$ form a commutative ring it follows that the number ξ^{-1} of the above definition is also a right inverse and is unique.

Theorem 8. *Let the coefficients of the formal power series $\mathfrak{p}\{a; A_\nu | z\}$ be elements of a flexible ring $\boldsymbol{R}$ with invertible subcentre $\boldsymbol{C}_I\{\boldsymbol{R}\}$, and let $A_a \in \boldsymbol{C}_I\{\boldsymbol{R}\}$. Then this series has both a left and a right inverse, and these series are identical.*

Proof. As in the proof of Theorem 7, we find that equations of the form (8) and (10) can be solved. Assuming that $A'_{-a+\nu} = A''_{-a+\nu}$ ($\nu = 0, 1, \ldots, r-1$), we derive in analogy with equation (11)

$$(12)\quad A'_{-a+r} = -(A_a^{-1} A_{a+r}) A_a^{-1} + \sum_{\nu=0}^{r-2} \sum_{\nu'=0}^{r-\nu-1} [\{A_a^{-1} (A_{a+\nu'} A'_{-a+\nu})\} A_{a+r-\nu-\nu'}]\, A_a^{-1},$$

$$(13)\quad A''_{-a+r} = -A_a^{-1} (A_{a+r} A_a^{-1}) + \sum_{\nu=0}^{r-2} \sum_{\nu'=1}^{r-\nu-1} A_a^{-1} [A_{a+\nu'}\, \{(A'_{-a+\nu} A_{a+r-\nu-\nu'}) A_a^{-1}\}] .$$

The first terms on the right hand sides of equations (12) and (13) are, of course, equal. The second term on the right hand side of equation (12) is a sum of terms of the form $[\{\xi(AB)\}C]\xi + [\{\xi(CB)\}A]\xi$ (setting $\xi = A_a^{-1}$, $A = A_{a+\nu'}$, $B = A'_{-a+\nu}$, $C = A_{a+r-\nu-\nu'}$) and (when $r - \nu = 2\nu'$) of the form $[\{\xi(AB)\}A]\xi$. The corresponding terms upon the right hand side of equation (13) are $\xi[A\{(BC)\xi\}] + \xi[C\{(BA)\xi\}]$ and $\xi[A\{(BA)\xi\}]$. As in the proof of clause (iv) of Theorem 1, these two pairs of terms are shown to be identical.

Theorem 8 has applications to Cayley numbers over a field, to the numbers of Albert's hierarchy and to the flexible rings of a system constructed by the author [**19**]$_1$.

Theorem 9. *If $\boldsymbol{R}$ possesses an invertible subcentre, so does $\boldsymbol{P}\{\boldsymbol{R}\}$.*

Proof. If $A_{a+\nu} \in \boldsymbol{C}\{\boldsymbol{R}\}$ $(\nu = 0, 1, \ldots)$ and $A_a \in \boldsymbol{C}_I\{\boldsymbol{R}\}$, both the left and right inverses of $\mathfrak{p}\{a; A_\nu | z\}$ can be constructed; these inverses belong to $\boldsymbol{C}\{\boldsymbol{P}\{\boldsymbol{R}\}\}$, they are identical and unique.

The presence of an invertible subcentre of a distributive ring of formal power series allows such series to be brought within the domain of application of further theories. As an example of such a theory, we mention that Rodabaugh [**16**] considers a ring $\boldsymbol{R}$ for which $(A, A, A) = 0$, $(A, B, C) = \varepsilon(\pi(A), \pi(B), \pi(C))$ where $\varepsilon \in \boldsymbol{C}_I\{\boldsymbol{R}\}$ and π is in the symmetric group on three letters.

Definition 12. *A distributive ring $\boldsymbol{R}$ with unit element I to each nonzero element A of which there corresponds a unique element $A^{-1} \in \boldsymbol{R}$ such that $A^{-1}A = AA^{-1} = I$ is called a division ring.*

Theorem 10. *Let $\boldsymbol{R}$ be a distributive ring possessing an invertible subcentre $\boldsymbol{C}_I\{\boldsymbol{R}\}$ and permitting a distributive involution $\mathfrak{S}$ with central trace and norm, and let $A(z) = \mathfrak{p}\{a; A_\nu | z\}$ be a fixed formal power series with coefficients over $\boldsymbol{R}$ and such that $\mathfrak{n}(A_a) \in \boldsymbol{C}_I\{\boldsymbol{R}\}$; then there exists a formal power series $A^{-1}(z)$ satisfying the relationships*

$$A(z)^{-1}A(z) = I(z)\,, \qquad A(z)A(z)^{-1} = I(z) \tag{14}$$

and furthermore

$$A(z)^{-1} = \hat{A}(z)\mathfrak{n}\{A(z)\}^{-1} = \mathfrak{n}\{A(z)\}^{-1}\hat{A}(z)\,. \tag{15}$$

If $\boldsymbol{R}$ has no divisors of zero, the series $A^{-1}(z)$ is uniquely determined by either of relationships (14) *and if, in addition, $\mathfrak{n}(A) \in C_I\{\boldsymbol{R}\}$ for every nonzero element A of $\boldsymbol{R}$, then $\boldsymbol{R}$ and $\boldsymbol{P}\{\boldsymbol{R}\}$ are division rings.*

Proof. We set $\mathfrak{n}\{A(z)\} = \mathfrak{p}\{2a; \eta_\nu | z\}$ where, in particular, $\eta_{2a} = \mathfrak{n}(A_a)$ and $\eta_\nu \in \boldsymbol{C}\{\boldsymbol{R}\}$ $(\nu = 2a+1, 2a+2, \ldots)$. From Theorem 9 it follows that the inverse series $\mathfrak{n}\{A(z)\}^{-1}$ exists and belongs to $\boldsymbol{C}\{\boldsymbol{P}\{\boldsymbol{R}\}\}$. Using the first of formulae (15) we then have

$$A(z)A^{-1}(z) = A(z)[\hat{A}(z)\mathfrak{n}\{A(z)\}^{-1}] = \{A(z)\hat{A}(z)\}n\{A(z)\}^{-1} = I(z)\,.$$

Again, since $\mathfrak{n}\{A(z)\} = \mathfrak{n}\{\hat{A}(z)\}$ when $\mathfrak{F}$ has a central trace (Theorem 6), we have

$$A(z)^{-1}A(z) = \mathfrak{n}\{A(z)\}^{-1}\{\hat{A}(z)A(z)\} = \mathfrak{n}\{A(z)\}^{-1}\mathfrak{n}\{A(z)\} = I(z) .$$

That $A(z)^{-1}$ is the only series to satisfy either of relationships (14) if $\boldsymbol{R}$ is without divisors of zero is a trivial consequence of Theorem 4. Lastly, if $\mathfrak{n}(A) \in \boldsymbol{C}_I\{\boldsymbol{R}\}$ for every nonzero element of $\boldsymbol{R}$, the inverse $A^{-1} = \hat{A}\mathfrak{n}(A)^{-1}$ of every nonzero element A of $\boldsymbol{R}$ exists, and the inverse of every series distinct from $O(z)$ belonging to $\boldsymbol{P}\{\boldsymbol{R}\}$ can be constructed; thus, if $\boldsymbol{R}$ is without divisors of zero, $\boldsymbol{R}$ and $\boldsymbol{P}\{\boldsymbol{R}\}$ are division rings.

Theorem 10 in its entirety can be applied to the cases in which the elements of $\boldsymbol{R}$ are quaternions over a formally real field, Cayley numbers over a formally real field, elements of any one of the recursively constructed algebra's of Albert's hierarchy or of the author's recursively constructed system of rings.

Theorem 11. *Let $\boldsymbol{R}$ be a distributive ring permitting a distributive involution $A \to \hat{A}$ $(A \equiv \boldsymbol{R})$; let $A(z)$ be a formal power series with coefficients over $\boldsymbol{R}$ which possesses a two-sided inverse $A(z)^{-1}$ and for which $\hat{A}(z) = A(z)$. Then $\widehat{A(z)^{-1}} = A(z)^{-1}$.*

Proof. The equations determing the coefficients of the left inverse series $\mathfrak{p}\{-a; A'_\nu | z\}$ are (7). Applying the involution to both sides of these equations, we have

$$\hat{A}_a\hat{A}'_{-a} = \hat{I} , \quad \sum_{\nu=0}^{r} \hat{A}_{a+r-\nu}\hat{A}'_{-a+\nu} = \hat{0} \qquad (r = 1, 2, \ldots),$$

or

$$A_a\hat{A}'_{-a} = I , \quad \sum_{\nu=0}^{r} A_{a+r-\nu}\hat{A}'_{-a+\nu} = 0 \qquad (r = 1, 2, \ldots)$$

These are, however, the equations which determine the coefficients of the right inverse series; since this is identical with the left inverse series, the numbers $\{A'_\nu\}$ and $\{\hat{A}'_\nu\}$ are equal.

The above theorem may be applied to the case in which the $\{A_\nu\}$ are square matrices, and the involution $A \to \hat{A}$ is either $A \to A^T$ or $A \to \overline{A}^T$, where A^T is the transpose of A, and the bar denotes a complex conjugate.

We conclude this section with the remark that the generalised inverse of a nonzero formal power series with vector valued coefficients can also be defined [**19**]$_2$. However, vectors are not closed with respect to binary multiplication (in the sense of matrix multiplication) and hence such inverses are not subsumed within the theory of this section.

6. - Recursive application of the theory.

It is clearly possible to apply the theory of this paper recursively, considering power series in one variable whose coefficients are power series in another, and to do so a denumerably infinite number of times.

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Sommario.

Si dimostra che il sistema $\boldsymbol{P}\{\boldsymbol{R}\}$ *di serie di potenze con coefficienti definiti su un anello distributivo* $\boldsymbol{R}$ *condivide in gran parte le proprietà moltiplicative dei coefficienti; in particolare che se gli elementi di* $\boldsymbol{R}$ *si moltiplicano in modo commutativo, così pure si moltiplicano quelli di* $\boldsymbol{P}\{\boldsymbol{R}\}$, *che se* $\boldsymbol{R}$ *è un anello, pure* $\boldsymbol{P}\{\boldsymbol{R}\}$ *è un anello e che lo stesso vale per anelli di Lie, anelli flessibili, alternativi e per anelli di Jordan commutativi e non. Si dimostra anche che se* $\boldsymbol{R}$ *ha un elemento unità, anche* $\boldsymbol{P}\{\boldsymbol{R}\}$ *ha un elemento unità; che* $\boldsymbol{P}\{\boldsymbol{R}\}$ *ha un centro se* $\boldsymbol{R}$ *ne ha uno; che se* $\boldsymbol{R}$ *è senza divisori dello zero, lo stesso è vero per* $\boldsymbol{P}\{\boldsymbol{R}\}$; *che se esiste un'algebra derivativa definita sugli elementi di* $\boldsymbol{R}$ *lo stesso è vero per* $\boldsymbol{P}\{\boldsymbol{R}\}$; *che se* $\boldsymbol{R}$ *ammette un'involuzione distributiva, anche* $\boldsymbol{P}\{\boldsymbol{R}\}$ *la ammette e che se l'involuzione su* $\boldsymbol{R}$ *ha traccia centrale e norma, lo stesso è vero per quella di* $\boldsymbol{P}\{\boldsymbol{R}\}$. *Si dimostra che una serie di potenze con coefficienti definiti su un anello* $\boldsymbol{R}$, *avente un sottoinsieme nonnullo* $\boldsymbol{R}_I$ *di elementi invertibili e il cui coefficiente della potenza più elevata appartiene a* $\boldsymbol{R}_I$, *possiede inversa biunivoca. Si analizzano anche altri casi in cui una serie di potenze possiede inversa biunivoca, in particolare si studiano quelle serie in cui la norma del coefficiente potenza più elevata appartiene al centro di un anello distributivo ed è invertibile. Si dimostra che se una serie di potenze ha coefficienti autoinvolutivi ed ha inversa biunivoca, allora tale inversa è anche autoinvolutiva.*

Abstract.

It is shown that the system $\mathbf{P}\{\mathbf{R}\}$ *of formal power series with coefficients over a distributive ring* $\mathbf{R}$ *shares to a large extent the multiplicative properties of the coefficients; in particular that if the elements of* $\mathbf{R}$ *multiply commutatively, so do those of* $\mathbf{P}\{\mathbf{R}\}$*, that if* $\mathbf{R}$ *is a ring,* $\mathbf{P}\{\mathbf{R}\}$ *is also a ring, and that the same holds with regard to Lie, flexible, alternative, commutative and noncommutative Jordan rings. It is also shown that if* $\mathbf{R}$ *possesses a unit element,* $\mathbf{P}\{\mathbf{R}\}$ *also possesses a unit element, that* $\mathbf{P}\{\mathbf{R}\}$ *has a centre if* $\mathbf{R}$ *has one, that if* $\mathbf{R}$ *is without divisors of zero, the same is true of* $\mathbf{P}\{\mathbf{R}\}$*, that if there exists a derivative algebra over the elements of* $\mathbf{R}$*, the same is true of* $\mathbf{P}\{\mathbf{R}\}$*, that if* $\mathbf{R}$ *permits a distributive involution,* $\mathbf{P}\{\mathbf{R}\}$ *does the same and that if the involution over* $\mathbf{R}$ *has a central trace and norm, the same is true of that over* $\mathbf{P}\{\mathbf{R}\}$*. It is proved that a formal power series with coefficients over a ring* $\mathbf{R}$ *possessing a nonempty subset* $\mathbf{R}_I$ *of invertible elements, and whose leading coefficient belongs to* $\mathbf{R}_I$ *possesses a two sided inverse. Further cases in which a formal power series possesses a two side inverse are investigated, in particular that in which the norm of the leading coefficient belongs to the centre of a distributive ring and is invertible. It is shown that if a formal power series has self-involutive coefficients and has a two sided inverse, then this inverse is self-involutive.*

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