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On common fixed points through rational expression. (**)

1. - Introduction.

Let (X, d) be a complete metric space. A mapping $f: X \to X$ is called contraction if there exists a real number k, $0 < k < 1$, such that

$$d(f(x), f(y)) \leqslant k d(x, y) , \qquad x, y \in X .$$

The celebrated Banach's contraction principle [1] states that every such mapping admits of a unique fixed point. The condition that the mapping is a contraction is a very severe restriction as also such maps are uniformly continuous Kannan [2] established the same theory for a mapping f which satisfies

$$d(f(x), f(y)) \leqslant \alpha [d(x, f(x)) + d(y, f(y))] , \qquad 0 < \alpha < \tfrac{1}{2} ,$$

and showed that f need not be even continuous. Later on Reich [4] and Wong [5] established the same results through generalized contractions.

In this paper we have made an attempt to establish a fixed point theorem through rational expression for product of two self-mappings defined on a metric space (not necessarily complete). Also the mappings under study need not be continuous. Some results which follow as its consequences are also derived.

We have

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Theorem 1. *Let T_1 and T_2 be two self-mappings defined on a metric space (X, d) such that*

$$\text{(i)}\quad d(T_1T_2(x), T_2T_1(y)) \leqslant \frac{\beta d(x, T_1T_2(x))[\alpha + d(y, T_2T_1(y))]}{\alpha + \beta d(x, T_1T_2(x))},$$

for all $x, y \in X$, $\alpha > 0$, $0 < \beta < 1$,

(ii) *for some $x \in X$, the sequence $\{x_n\}$ defined as $x_1 = T_1T_2(x)$ and for $n > 1$,*

$$x_n = \begin{cases} T_1T_2(x_{n-1}) & \text{if } n \text{ is odd} \\ T_2T_1(x_{n-1}) & \text{if } n \text{ is even} \end{cases}$$

has a subsequence $\{x_{n_k}\}$ with $x_0 = \lim_{n\to\infty} x_{n_k}$. Then x_0 is the unique common fixed point of T_1 and T_2.

Proof. For the sequence $\{x_n\}$ as defined in (ii) we have

$$d(x_1, x_2) = d(T_1T_2(x), T_2T_1(x_1))$$
$$\leqslant \frac{\beta d(x, T_1T_2(x))[\alpha + d(x_1, T_2T_1(x_1))]}{\alpha + \beta d(x, T_1T_2(x))},$$

i. e.

$$\left\{1 - \frac{\beta d(x, x_1)}{\alpha + \beta d(x, x_1)}\right\} d(x_1, x_2) \leqslant \frac{\alpha\beta d(x, x_1)}{\alpha + \beta d(x, x_1)},$$

which gives $d(x_1, x_2) \leqslant \beta d(x, x_1)$.

Also

$$d(x_2, x_3) = d(T_2T_1(x_1), T_1T_2(x_2))$$
$$\leqslant \frac{\beta d(x_2, T_1T_2(x_2))[\alpha + d(x_1, T_2T_1(x_1))]}{\alpha + \beta d(x_2, T_1T_2(x_2))},$$

i.e., $\alpha + \beta d(x_2, x_3) \leqslant \alpha\beta + \beta d(x_1, x_2)$ which implies

$$d(x_2, x_3) \leqslant \frac{\alpha}{\beta}(\beta - 1) + \beta d(x, x_1).$$

Again

$$d(x_3, x_4) = d\big(T_1T_2(x_2),\, T_2T_1(x_3)\big)$$

$$\leqslant \frac{\beta d(x_2, T_1T_2(x_2))[\alpha + d(x_3, T_2T_1(x_3))]}{\alpha + \beta d(x_2, T_1T_2(x_2))}$$

which implies

$$d(x_3, x_4) \leqslant \beta d(x_2, x_3) \leqslant \alpha(\beta - 1) + \beta^2 d(x, x_1)\,,$$

and

$$d(x_4, x_5) = d\big(T_2T_1(x_3),\, T_1T_2(x_4)\big)$$

$$\leqslant \frac{\beta d(x_4, T_1T_2(x_4))[\alpha + d(x_3, T_2T_1(x_3))]}{\alpha + \beta d(x_4, T_1T_2(x_4))}$$

i.e., $\alpha + \beta d(x_4, x_5) \leqslant \alpha\beta + \beta d(x_3, x_4)$ which implies

$$d(x_4, x_5) \leqslant \frac{\alpha}{\beta}(\beta - 1) + \alpha(\beta - 1) + \beta^2 d(x, x_1) = \frac{\alpha}{\beta}(\beta^2 - 1) + \beta^2 d(x, x_1)\,.$$

In general,

$$d(x_{2n}, x_{2n+1}) \leqslant \frac{\alpha}{\beta}(\beta^n - 1) + \beta^n d(x, x_1)$$

and

$$d(x_{2n+1}, x_{2(n+1)}) \leqslant \alpha(\beta^n - 1) + \beta^{n+1} d(x, x_1)\,.$$

Hence for $m = 2r$, we have

$$d(x_m, x_{m+n}) \leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{m+n-1}, x_{m+n}) \leqslant$$

$$\leqslant \frac{\alpha}{\beta}(\beta^r - 1) + \beta^r d(x, x_1) + \alpha(\beta^r - 1) + \beta^{r+1} d(x, x_1) + \ldots \quad (2n \text{ terms})$$

$$= \frac{\alpha}{\beta}(\beta^r - 1) + \alpha(\beta^r - 1) + \frac{\alpha}{\beta}(\beta^{r+1} - 1) + \alpha(\beta^{r+1} - 1) + \ldots (n \text{ terms}) +$$

$$+ \big(\beta^r + 2\beta^{r+1} + 2\beta^{r+2} + \ldots,\ (n \text{ terms})\big)\, d(x, x_1)$$

$$< \alpha\beta^{r-1}(1 + \beta + \beta^2 + \ldots) + \alpha\beta^r(1 + \beta + \beta^2 + \ldots) +$$

$$+ 2\beta^r(1 + \beta + \beta^2 + \ldots)\, d(x, x_1)$$

$$= \frac{\alpha\beta^{r-1}(1 + \beta)}{1 - \beta} + \frac{2\beta^\nu}{1 - \beta} d(x, x_1) \to 0 \text{ as } r \to \infty\,, \quad \text{i.e.,} \quad m \to \infty\,.$$

Similarly, for $m = 2r + 1$, we have

$$d(x_m, x_{m+n}) \to 0 \quad \text{as } m \to \infty ,$$

showing thereby that $\{x_n\}$ is a Cauchy sequence. As a consequence of (ii), there exists $x_0 \in X$ such that

$$\lim_{n \to \infty} x_n = x_0.$$

We shall show that $T_1 T_2(x_0) = T_2 T_1(x_0) = x_0$. Now

$$\begin{aligned} d(T_1 T_2(x_0), x_0) &\leqslant d(T_1 T_2(x_0), x_t) + d(x_t, x_0) \qquad \text{(where } t \text{ is even)} \\ &= d(T_1 T_2(x_0), T_2 T_1(x_{t-1})) + d(x_t, x_0) \\ &\leqslant \frac{\beta d(x_0, T_1 T_2(x_0))[\alpha + d(x_{t-1}, T_2 T_1(x_{t-1}))]}{\alpha + \beta d(x_0, T_1 T_2(x_0))} + d(x_t, x_0) , \end{aligned}$$

i. e.,

$$d(T_1 T_2(x_0), x_0) \leqslant d(x_t, x_0) \left\{ 1 - \frac{\alpha\beta + \beta d(x_{t-1}, x_t)}{\alpha + \beta d(x_0, T_1 T_2(x_0))} \right\}^{-1} \to 0$$

for sufficiently large t. Thus $T_1 T_2(x_0) = x_0$. Similarly $T_2 T_1(x_0) = x_0$.

Next, we show that x_0 is the unique fixed point of $T_1 T_2$ and $T_2 T_1$. Let y_0 also satisfy $T_1 T_2(y_0) = T_2 T_1(y_0) = y_0$. Then

$$d(x_0, y_0) = d(T_1 T_2(x_0), T_2 T_1(y_0)) \leqslant 0.$$

Therefore $x_0 = y_0$. It can be easily seen that x_0 is the only fixed point of $T_1 T_2$ and $T_2 T_1$ which is common. Next, we show that $T_1(x_0) = x_0 = T_2(x_0)$.

We have $T_1 T_2(x_0) = x_0 = T_2 T_1(x_0)$ which implies $T_1(T_2 T_1(x_0)) = T_1(x_0)$ i.e., $T_1 T_2(T_1(x_0)) = T_1(x_0)$. Therefore $T_1(x_0) = x_0$, since x_0 is the unique fixed point of $T_1 T_2$. Similarly $T_2(x_0) = x_0$. Lastly, we show that x_0 is the only common fixed point of T_1 and T_2. Let $z_0 \neq x_0$ be an element of X enjoying the property $T_1(z_0) = T_2(z_0) = z_0$ which implies $T_1 T_2(z_0) = T_2 T_1(z_0) = z_0$ i.e., $z_0 = x_0$, since x_0 is the unique common fixed point of $T_1 T_2$ and $T_2 T_1$. This completes the proof of the theorem.

Remark. The result by setting $T_2 x = x$ for all $x \in X$ in the above theorem, has been proved independently in [**3**].

2. – In the following theorem we show that if T_1 and T_2 are continuous, then it is sufficient for the validity of Theorem 1 that T_1, T_2 satisfy condition (i) only on a dense subset of X.

Theorem 2. *Let T_1 and T_2 be two continuous mappings of X into itself such that for any pair of points x, y belonging to an everywhere dense subset M of X*

$$(*) \qquad d(T_1T_2(x), T_2T_1(y)) \leqslant \frac{\beta d(x, T_1T_2(x))[\alpha + d(y, T_2T_1(y))]}{\alpha + \beta d(x, T_1T_2(x))}$$

$\alpha > 0$, $0 < \beta < 1$, and (ii) *holds, then T_1 and T_2 have a unique common fixed point.*

Proof. The proof will follow from Theorem 1 if we show that the expression in (*) holds for any pair of points $x, y \in X$.

Let $x, y \in X$. If $x \in X$, $y \in X \sim M$, then a sequence $\{y_n\}$ in M converges to y and we have

$$\begin{aligned} d(T_1T_2(x), T_2T_1(y)) &\leqslant d(T_1T_2(x), T_2T_1(y_n)) + d(T_2T_1(y_n), T_2T_1(y)) \\ &\leqslant \frac{\beta d(x, T_1T_2(x))[\alpha + d(y_n, T_2T_1(y_n))]}{\alpha + \beta d(x, T_1T_2(x))} + d(T_2T_1(y_n), T_2T_1(y)) \\ &= \frac{\beta d(x, T_1T_2(x))[\alpha + d(y, T_2T_1(y))]}{\alpha + \beta d(x, T_1T_2(x))}, \end{aligned}$$

since T_1 and T_2 are continuous.

Now we consider the case when $x, y \in X \sim M$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in M converging to x and y respectively. Again we shall have condition (i) using the continuity of T_1 and T_2 and the inequality

$$\begin{aligned} d(T_1T_2(x), T_2T_1(y)) \leqslant d(T_1T_2(x), T_1T_2(x_n)) &+ \\ + d(T_1T_2(x_n), T_2T_1(y_n)) &+ d(T_2T_1(y_n), T_2T_1(y)) . \end{aligned}$$

3. – A much more interesting situation when a function which is the limit of a convergent sequence of functions is shown to have unique fixed point which is the limit of the fixed points of the sequence of functions is handled in the next theorem.

Theorem 3. *Let $\{(T_1T_2)_i\}$ and $\{(T_2T_1)_i\}$ be two sequences of mappings of X into itself converging pointwise to T_1T_2 and T_2T_1 respectively such that*

$$d((T_1T_2)_i(x), (T_2T_1)_i(y)) \leqslant \frac{\beta d(x, (T_1T_2)_i(x))[\alpha + d(y, (T_2T_1)_2(y))]}{\alpha + \beta d(x, (T_1T_2)_i(x))}$$

for all $x, y \in X$, $\alpha > 0$, $0 < \beta < 1$, $i = 1, 2, \ldots$.

If $(T_1T_2)_i$ and $(T_2T_1)_i$ have common fixed point ξ_i and ξ is the common fixed point of T_1T_2 and T_2T_1, then the sequence ξ_n converges to ξ.

Proof. We have

$$d(\xi, \xi_n) = d(T_1T_2(\xi), (T_2T_1)_n(\xi_n))$$

$$\leqslant d(T_1T_2(\xi), (T_1T_2)_n(\xi)) + d((T_1T_2)_n(\xi), (T_2T_1)_n(\xi_n))$$

$$\leqslant d(T_1T_2(\xi), (T_1T_2)_n(\xi)) + \frac{\beta d(\xi, (T_1T_2)_n(\xi))[\alpha + d(\xi_n, (T_2T_1)_n(\xi_n))]}{\alpha + \beta d(\xi, (T_1T_2)_n(\xi))}$$

$$= \left\{1 + \frac{\alpha\beta}{\alpha + \beta d(T_1T_2(\xi), (T_1T_2)_n(\xi))}\right\} d(T_1T_2(\xi), (T_1T_2)_n(\xi)) \to 0$$

as $(T_1T_2)_n(\xi) \to T_1T_2(\xi)$. Hence ξ_n converges to ξ.

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References.

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Abstract.

The well known Banach's contraction mapping theorem has been generalized by various authors through generalized contractions. In this paper, an attempt has been made to derive certain results pertaining to fixed point theorems in metric spaces through a rational expression.

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