Riv. Mat. Univ. Parma (4) 2 (1976), 83-94

V. D. CHUGH and P. SINGH (*)

On Nevanlinna's proximity function. (**)

1. - Introduction.

Let $f(z)$ be a meromorphic function in the entire complex plane C. Familiarity with the definitions of basic quantities of Nevalinna's theory of meromorphic functions: $m(r, f)$, $m(r, 1/f)$, $n(r, f)$, $n(r, 1/f)$, $N(r, f)$, $N(r, 1/f)$, $T(r, f)$ etc. is assumed (see [2], [5]). We also write

$$m(r) = m(r, f) + m(r, 1/f)$$

and

$$N_p(r) = \int_0^r \frac{n(t)}{t^{p+1}} \, dt \qquad (p = 0, 1, 2, 3, \ldots),$$

where

$$n(r) = n(r, f) + n(r, 1/f)$$

and it is assumed, without any loss of generality, that $n(r) = 0$, for $r \leqslant 1$.

Our aim in this paper is to investigate the growth of $m(r)$ with regard to $n(r)$, $N(r)$, $N_p(r)$ and $r^{\varrho(r)}$ where $\varrho(r)$ is a proximate order of $f(z)$, ϱ being the usual order of $f(z)$ in terms of $T(r, f)$. Section **2** deals with the statement and discussion of the main results, whereas the remaining sections have been devoted to the proofs of the main results.

(*) Indirizzo degli Autori: Faculty of Mathematics, University of Delhi, Delhi-7, India.

(**) Ricevuto: 6-VI-1972.

2. - Statement and discussion of results.

Recently in [**3**], Jain and Kamthan have proved, for a meromorphic function of order ϱ $(0<\varrho<1)$, that

$$\lim_{r\to\infty}\inf \frac{m(r)}{n(r)} \leqslant A(\varrho)\,, \tag{2.1}$$

where

$$A(\varrho) = [\varrho(1-\varrho)]^{-1} + 2(1-\varrho)^{-1}\cot\tfrac{1}{2}\pi\varrho\,. \tag{2.2}$$

We prove here the following

Theorem 1. *Let $f(z)$ be a meromorphic function of non-integral order ϱ, $(0<\varrho<\infty)$. Then*

$$\lim_{r\to\infty}\inf \frac{m(r)}{n(r)} < \infty\,. \tag{2.3}$$

The above result is not necessarily true for functions of integral order. Consider for example (see [**2**], p. 7) the function $f(z)=\exp[z]$. Then $m(r)=(2r/\pi)$ for all $r>0$ and $n(r)=0$ for all r. Hence the left-hand expression in (2.3) is infinite.

Further, let us consider an example of a function which possesses poles. Take for instance $f(z)=\Gamma(z)$. It is of order 1 and $n(r)\sim N(r)\sim r$; $m(r)\sim \sim r\log r$ and so Theorem 1 is not true in this case also.

Further, we have

Theorem 2. *Let $f(z)$ be a meromorphic function of non-integral order ϱ $(p<\varrho<p+1$, p being any integer) and proximate order $\varrho(r)$. Then*

$$\lim_{r\to\infty}\operatorname{Sup} \frac{m(r)}{r^{\varrho(r)}} \leqslant \beta[p(\varrho-p)^{-1} + (p+1-\varrho)^{-1}\{p+1+2\varrho\cot\tfrac{1}{2}\pi(\varrho-p)\}]\,, \tag{2.4}$$

where β is defined as

$$\lim_{r\to\infty} \frac{N(r)}{r^{\varrho(r)}} = \beta < \infty.$$

An immediate corollary of this theorem is the following

Corollary 1. *For* $p < \varrho < p+1$,

$$\operatorname*{Lim\,Sup}_{n\to\infty} \frac{m(r)}{N(r)} \leqslant p(\varrho-p)^{-1} + (p+1-\varrho)^{-1}[p+1+2\varrho \cot \tfrac{1}{2}\pi(\varrho-p)]\,. \tag{2.5}$$

The corollary follows immediately, by using Theorem 2, since

$$\operatorname*{Lim\,Sup}_{r\to\infty} \frac{m(r)}{r^p N_p(r)} = \operatorname*{Lim\,Sup}_{r\to\infty} \frac{m(r) r^{\varrho(r)}}{r^p N_p(r)/r^{\varrho(r)}} = \beta^{-1} \operatorname*{Lim\,Sup}_{r\to\infty} \frac{m(r)}{r^{\varrho(r)}}\,.$$

However, if we replace $N(r)$ by $r^p N_p(r)$ in Theorem 2, we have

Theorem 3. *Under the hypothesis of Theorem 2, we have*

$$\operatorname*{Lim\,Sup}_{r\to\infty} \frac{m(r)}{r^{\varrho(r)}} \leqslant \beta^*(p+1-\varrho)^{-1}[1+2(\varrho-p)\cot \tfrac{1}{2}\pi(\varrho-p)]\,, \tag{2.6}$$

where β^* *is defined as*

$$\operatorname*{Lim}_{r\to\infty} \frac{r^p N_p(r)}{r^{\varrho(r)}} = \beta^* < \infty\,.$$

Corollary 2. *For* $p < \varrho < p+1$,

$$\operatorname*{Lim\,Sup}_{r\to\infty} \frac{m(r)}{r^p N_p(r)} \leqslant (p+1-\varrho)^{-1}[1+2(\varrho-p)\cot \tfrac{1}{2}\pi(\varrho-p)]\,. \tag{2.7}$$

Remark 1. For functions of order ϱ $(0 < \varrho < 1)$ i.e. for which $p = 0$, the results in Theorems 2 and 3, and consequently their respective corollaries are identical.

Next, we have a striking deduction from Corollary 1 as follows

Corollary 3. *For* $p < \varrho < p+1$

$$\operatorname*{Lim\,Sup}_{r\to\infty} \frac{m(r)}{n(r)\log r} \leqslant \varrho A(\varrho)\,, \tag{2.8}$$

where $A(\varrho)$ *is given by* (2.2).

Remark 2. The inequalities in (2.5), (2.7) and (2.8) cannot be removed.

Consider, for instance, (see [2], p. 100) the function

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^2}\right).$$

Then $m(r) = 2r^{\frac{1}{2}}$, $n(r) \sim r^{\frac{1}{2}}$, $N(r) \sim 2r^{\frac{1}{2}}$, $\varrho = 1/2$.

Finally, we separate meromorphic functions of non-integral order into two classes. According to Nevanlinna [5], a meromorphic function $f(z)$ of positive non-integral order ϱ is of convergence or divergence class according as

$$\int^{\infty} r^{-\varrho-1} n(r)\,\mathrm{d}r \tag{2.9}$$

converges or diverges.

Now, we have

Theorem 4. *The meromorphic function of non-integral order ϱ is of convergence or divergence class according as*

$$\int^{\infty} r^{-\varrho-1} m(r)\,\mathrm{d}r \tag{2.10}$$

converges or diverges.

In the proofs of our theorems we shall very much rely on the following result due to Kamthan ([4], p. 40), which is in fact based on an estimate of $m(r)$ obtained earlier by Edrei and Fuchs ([1], p. 300).

Lemma A [4]. *Let $f(z)$ be a meromorphic function of non-integral order ϱ and let p be an integer, $p < \varrho < p+1$. Then*

$$\begin{aligned} m(r) \leqslant {} & r^{p}\int_0^r x^{-p-1} n(x)\,\mathrm{d}x + r^{p+1}\int_r^{\infty} x^{-p-2} n(x)\,\mathrm{d}x + \\ & + \frac{4}{\pi}\sum_{m=0}^{\infty}(2m+1)^{-1}\left[r^{p-2m}\int_0^r x^{-p+2m-1} n(x)\,\mathrm{d}x \right. \\ & \left. + r^{p+2m+2}\int_r^{\infty} x^{-p-2m-3} n(x)\,\mathrm{d}x\right] + O(r^p)\,. \end{aligned} \tag{2.11}$$

3. - Proof of Theorem 1.

We have to consider two cases according as $f(z)$ is of convergence class or not. In either case $p < \varrho < p+1$ and $\int^{\infty} x^{-\alpha} n(x)\,dx$ converges for $\alpha > \varrho + 1$ and diverges for $\alpha < \varrho + 1$.

It is sufficient to prove that it is impossible to have

$$n(r) \leqslant \varepsilon m(r)\,, \qquad r > r_0(\varepsilon)\,, \qquad \varepsilon > 0\,. \tag{3.1}$$

Choose α with $\varrho + 1 \leqslant \alpha < p + 2$ and such that $\int^{\infty} x^{-\alpha} n(x)\,dx$ converges (it converges for $\alpha > \varrho + 1$ in both cases).

Multiplying (3.1) by $r^{-\alpha}$, integrating over (R, ∞) and changing the order of integration in the resulting integrated integrals, we get

$$\begin{aligned}\int_R^{\infty} r^{-\alpha} n(r)\,dr \leqslant{} & \varepsilon \int_R^{\infty} r^{p-\alpha}\,dr \int_0^r x^{-p-1} n(x)\,dx + \\ & + \varepsilon \int_R^{\infty} r^{p-\alpha+1}\,dr \int_r^{\infty} x^{-p-2} n(x)\,dx + \\ & + \frac{4}{\pi}\,\varepsilon \sum_{m=0}^{\infty} (2m+1)^{-1} \left[\int_R^{\infty} r^{p-2m-\alpha}\,dr \int_0^r x^{-p+2m-1} n(x)\,dx + \right. \\ & \left. + \int_R^{\infty} r^{p+2m+2-\alpha}\,dr \int_r^{\infty} x^{-p-2m-3} n(x)\,dx \right] + O\Big(\int_R^{\infty} r^{p-\alpha}\,dr\Big),\end{aligned} \tag{3.2}$$

$$\int_R^{\infty} r^{-\alpha} n(r)\,dr \leqslant \varepsilon I_1 + \varepsilon I_2 + \frac{4\varepsilon}{\pi} \sum_{m=0}^{\infty} (2m+1)^{-1}(I_3 + I_4) + I_5 \qquad \text{(say)}\,. \tag{3.3}$$

Now by changing the order of integration in (3.2), we notice that

$$\begin{aligned} I_1 &= \int_0^R x^{-p-1} n(x)\,dx \int_R^{\infty} r^{p-\alpha}\,dr + \int_R^{\infty} x^{-p-1} n(x)\,dx \int_x^{\infty} r^{p-\alpha}\,dr \\ &= (\alpha - p - 1)^{-1} R^{p+1-\alpha} \int_0^R x^{-p-1} n(x)\,dx + (\alpha - p - 1)^{-1} \int_R^{\infty} x^{-\alpha} n(x)\,dx\,; \end{aligned}$$

$$I_2 = \int_R^{\infty} x^{-p-2} n(x)\,dx \int_R^{\infty} r^{p+1-\alpha}\,dr \leqslant (p + 2 - \alpha)^{-1} \int_R^{\infty} x^{-\alpha} n(x)\,dx\,;$$

$$I_3 = \int_0^R x^{-p+2m-1} n(x)\,dx \int_R^\infty r^{p-2m-\alpha}\,dr + \int_R^\infty x^{-p+2m-1} n(x)\,dx \int_R^\infty r^{p-2m-\alpha}\,dr$$

$$\leqslant (2m-p-1+\alpha)^{-1} R^{p-\alpha+1} \int_0^R x^{-p-1} n(x)\,dx + (2m-p-1+\alpha)^{-1} \int_R^\infty x^{-\alpha} n(x)\,dx;$$

$$I_4 = \int_R^\infty x^{-p-2m-3} n(x)\,dx \int_R^\infty r^{p+2m+2-\alpha}\,dr \leqslant (2m+p+3-\alpha)^{-1} \int_R^\infty x^{-\alpha} n(x)\,dx;$$

$$I_5 \leqslant O(R^{p+1-\alpha}) .$$

Therefore, (3.3) implies that

$$\int_R^\infty r^{-\alpha} n(r)\,dr \leqslant \varepsilon A(\alpha, p) \int_R^\infty x^{-\alpha} n(x)\,dx + \tag{3.4}$$

$$+ \varepsilon B(\alpha, p) R^{p+1-\alpha} \int_0^R x^{-p-1} n(x)\,dx + O(R^{p+1-\alpha}) ,$$

where

$$A(\alpha, p) = (\alpha-p-1)^{-1} + (p+2-\alpha)^{-1} + \frac{4}{\pi} \sum_{m=0}^{\infty} (2m+1)^{-1} \times$$

$$\times \left[\frac{1}{2m-p-1+\alpha} + \frac{1}{2m+p-\alpha+3}\right],$$

and

$$B(\alpha, p) = (\alpha-p-1)^{-1} + \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)(2m-p-1+\alpha)} .$$

It is easily seen that $A(\alpha, p) < \infty$ and $B(\alpha, p) < \infty$. Therefore, by choosing $\varepsilon < 1/2A(\alpha, p)$ and collecting the terms in (3.4), we get

$$\tfrac{1}{2} \int_R^\infty r^{-\alpha} n(r)\,dr \leqslant \varepsilon B(\alpha, p) R^{p+1-\alpha} \int_1^R x^{-p-1} n(x)\,dx + O(R^{p+1-\alpha}) . \tag{3.5}$$

In case $f(z)$ is of divergence class (holding R fixed), let $\alpha \to \varrho + 1$, the left-hand side of (3.5) becomes infinite, while its right hand side approaches a finite limit. So (3.1) leads to a contradiction.

In case $f(z)$ is of convergence class, we may take $\alpha = \varrho + 1$, since $n(r)$ increases, (3.5) implies

$$\tfrac{1}{2} n(R)\varrho^{-1}R^{-\varrho} \leqslant \varepsilon R^{p-\varrho}\left[\left\{(\varrho - p)^{-1} + \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)(2m-p+\varrho)}\right\} \times \right.$$
$$\left. \times \int_1^R x^{-p-1}n(x)\,\mathrm{d}x\right] + O(R^{p-\varrho}),$$

and since this holds for large enough R, for every positive ε, we have

$$n(r) = o[r^p \int_1^r x^{-p-1}n(x)\,\mathrm{d}x].$$

Since $\int^{\infty} r^{-p-\alpha}n(r)\,\mathrm{d}r$ diverges for $1 < \alpha < \varrho - p + 1$, for such α, we have as $R \to \infty$

$$\int_1^R r^{-p-\alpha}n(r)\,\mathrm{d}r = O[\int_1^R r^{-\alpha}\,\mathrm{d}r \int_1^r x^{-p-1}n(x)\,\mathrm{d}x]$$
$$= O[\int_1^R x^{-p-1}n(x)\,\mathrm{d}x \int_x^R r^{-\alpha}\,\mathrm{d}r] = O[\int_1^R x^{-p-\alpha}n(x)\,\mathrm{d}x],$$

a contradiction. Hence (2.3) follows.

4. - Proof of Theorem 2.

Since $n(x)\,\mathrm{d}x = x\,\mathrm{d}N(x)$ almost everywhere in $[0, x]$, the result in the Lemma A may be rewritten as

$$m(r) \leqslant pr^p \int_0^r x^{-p-1}N(x)\,\mathrm{d}x + (p+1)r^{p+1}\int_0^{\infty} x^{-p-2}N(x)\,\mathrm{d}x \tag{4.1}$$
$$+ \frac{4}{\pi}\sum_{m=0}^{\infty}(2m+1)^{-1}\left[(p-2m)r^{p-2m}\int_0^r x^{2m-p-1}N(x)\,\mathrm{d}x\right.$$
$$\left. + (2m+p+2)r^{2m+p+2}\int_r^{\infty} x^{-2m-p-3}N(x)\,\mathrm{d}x\right] + O(r^p)$$
$$= I_6 + I_7 + \frac{4}{\pi}\sum_{m=0}^{\infty}(2m+1)^{-1}[I_8 + I_9] + O(r^p) \qquad \text{(say)}.$$

From the hypothesis, we have $\beta-\varepsilon<N(r)/r^{\varrho(r)}<\beta+\varepsilon$ for $r>r_0=r_0(\varepsilon)$, $\varepsilon>0$. Therefore

$$I_6\leqslant(\beta+\varepsilon)pr^p\int_{r_0}^{r}x^{\varrho(x)-p-1}\,dx+O(r^p)\sim(\beta+\varepsilon)p(\varrho-p)^{-1}r^{\varrho(r)}+O(r^p)\,,$$

and

$$I_7\leqslant(\beta+\varepsilon)(p+1)r^{p+1}\int_{r}^{\infty}x^{\varrho(x)-p-2}\,dx\sim(\beta+\varepsilon)(p+1)(p+1-\varrho)^{-1}r^{\varrho(r)}\,.$$

Also for $m<p/2$

$$I_8\leqslant(\beta+\varepsilon)(p-2m)r^{p-2m}\int_{r_0}^{r}x^{\varrho(x)-p+2m-1}\,dx+O(r^{p-2m})\sim$$

$$\sim(\beta+\varepsilon)(p-2m)(\varrho-p+2m)^{-1}r^{\varrho(r)}+O(r^{p-2m}),$$

and for $m\geqslant p/2$

$$I_8\leqslant-(\beta-\varepsilon)(2m-p)r^{p-2m}\int_{r_0}^{r}x^{\varrho(x)-p+2m-1}\,dx+O(r^{p-2m})\sim$$

$$\sim-(\beta-\varepsilon)(2m-p)(\varrho-p+2m)^{-1}r^{\varrho(r)}+O(r^{p-2m})\,.$$

Furthermore for all $m\geqslant 0$

$$I_9\leqslant(\beta+\varepsilon)(p+2m+2)r^{p+2m+2}\int_{r}^{\infty}x^{\varrho(x)-p-2m-3}\,dx\sim$$

$$\sim(\beta+\varepsilon)(p+2m+2)(p+2m+2-\varrho)^{-1}r^{\varrho(r)}\,.$$

Hence for $m<p/2$

$$I_8+I_9\leqslant(\beta+\varepsilon)[(p-2m)(\varrho-p+2m)^{-1}+$$

$$+(p+2m+2)(p+2m+2-\varrho)^{-1}]r^{\varrho(r)}+O(r^{p-2m}),$$

whereas for $m\geqslant p/2$

$$I_8+I_9\leqslant-(\beta-\varepsilon)(2m-p)(\varrho-p+2m)^{-1}r^{\varrho(r)}+$$

$$+(\beta+\varepsilon)(p+2m+2)(p+2m-2\varrho)^{-1}r^{\varrho(r)}+O(r^{p-2m}).$$

Using the above estimates (4.1) gives

$$\frac{m(r)}{r^{\varrho(r)}} \leqslant (\beta+\varepsilon)p(\varrho-p)^{-1}+(\beta+\varepsilon)(p+1)(p+1-\varrho)^{-1}+$$

$$+\frac{4}{\pi}\sum_{m<p/2}(2m+1)^{-1}[(\beta+\varepsilon)(p-2m)(\varrho-p+2m)^{-1}+$$

$$+(\beta+\varepsilon)(p+2m+2)(p+2m+2-\varrho)^{-1}+O(r^{p-2m})]+$$

$$+\frac{4}{\pi}\sum_{m\geqslant p/2}(2m+1)^{-1}[(\beta+\varepsilon)(p+2m+2)(p+2m+2-\varrho^{-1})-$$

$$-(\beta-\varepsilon)(2m-p)(\varrho-p+2m)^{-1}+O(r^{p-2m})]+O(r^{p-\varrho(r)}) \qquad (r\geqslant r_0),$$

which implies

$$\operatorname*{Lim\,Sup}_{r\to\infty}\frac{m(r)}{r^{\varrho(r)}} \leqslant \beta\left[p(\varrho-p)^{-1}+(p+1)(p+1-\varrho)^{-1}+\right.$$

$$\left.+\frac{8\varrho}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+\varrho-p)(2m+p+2-\varrho)}\right]$$

$$=\beta\left[p(\varrho-p)^{-1}+(p+1)(p+1-\varrho)^{-1}+\frac{2\varrho}{\pi}(p+1-\varrho)^{-1}\right.$$

$$\left.\times\sum_{m=0}^{\infty}\left\{\frac{1}{m+(\varrho-p)/2}\;\frac{1}{(m+1)-(\varrho-p)/2}\right\}\right]$$

$$=\beta[p(\varrho-p)^{-1}+(p+1-\varrho)^{-1}\{p+1+2\varrho\cot\tfrac{1}{2}\pi(\varrho-p)\}].$$

Hence the theorem is established.

5. - Proof of Theorem 3.

From the definition of $N_p(r)$, we note that $n(x)\,dx = x^{p+1}\,dN_p(x)$, almost everywhere in $[0, x]$. Now using the arguments of Kamthan (see [4], p. 38) the Lemma A gives

$$m(r) \leqslant r^{p+1} \int_r^\infty x^{-2} N_p(x)\,\mathrm{d}x + \frac{4}{\pi} \sum_{m=0}^{\infty} (2m+1)^{-1} \times$$

$$\times \Big[-2mr^{p-2m} \int_0^r x^{2m-1} N_p(x)\,\mathrm{d}x +$$

$$+ (2m+2) r^{p+2m+2} \int_r^\infty x^{-2m-3} N_p(x)\,\mathrm{d}x\Big] + O(r^p)\,,$$

$$m(r) = I_{10} + \frac{4}{\pi} \sum_{m=0}^{\infty} (2m+1)^{-1} [I_{11} + I_{12}] + O(r^p) \qquad \text{(say)}\,; \tag{5.1}$$

From the hypothesis, we have

$$\beta^* - \varepsilon < \frac{r^p N_p(r)}{r^{\varrho(r)}} < \beta^* + \varepsilon \qquad (r > r_0(\varepsilon),\ \varepsilon > 0)\,.$$

Now computing along the lines of Theorem 2, we have

$$I_{10} \leqslant (\beta^* + \varepsilon)(p + 1 - \varrho)^{-1} r^{\varrho(r)}\,,$$

$$I_{11} \leqslant O(r^{p-2m}) - 2m(\beta^* - \varepsilon)(\varrho - p + 2m)^{-1} r^{\varrho(r)},$$

$$I_{12} \leqslant (\beta^* + \varepsilon)(2m+2)(p + 2m + 2 - \varrho)^{-1} r^{\varrho(r)}\,,$$

and hence the theorem follows.

6. - Proof of Theorem 4.

To prove this, it is enough to show that convergence of

$$\int^\infty r^{-\varrho-1} n(r)\,\mathrm{d}r \Leftrightarrow \text{convergence of } \int^\infty r^{-\varrho-1} J_k(r)\,\mathrm{d}r\,,$$

where $J_k(r)$, $(k = 1, 2, 3, 4, 5)$ stands for the kth term in the right hand expression in (2.11). Clearly

$$\int_0^\infty r^{-\varrho-1} J_1(r)\,\mathrm{d}r = \int_0^\infty x^{-p-1} n(x)\,\mathrm{d}x \int_x^\infty r^{-\varrho+p-1}\,\mathrm{d}r = (\varrho - p)^{-1} \int_0^\infty x^{-\varrho-1} n(x)\,\mathrm{d}x\,,$$

and

$$\int_R^\infty r^{-\varrho-1}J_2(r)\,\mathrm{d}r=\int_R^\infty x^{-p-2}n(x)\,\mathrm{d}x\int_R^\infty r^{p-\varrho}\,\mathrm{d}r$$

$$=(p+1-\varrho)^{-1}\Big[\int_R^\infty x^{-\varrho-1}n(x)\,dx-R^{p+1-\varrho}\int_R^\infty x^{-p-2}n(x)\,\mathrm{d}x\Big]\,.$$

Therefore, (6.1) implies

$$(1-R^{p+1-\varrho})(p+1-\varrho)^{-1}\int_R^\infty x^{-\varrho-1}n(x)\,\mathrm{d}x<$$

$$<\int_R^\infty r^{-\varrho-1}J_2(r)\,\mathrm{d}r<(p+1-\varrho)^{-1}\int_R^\infty x^{-\varrho-1}n(x)\,\mathrm{d}x\,,$$

since $\varrho+1<p+2$. Also, we have

$$\int_0^\infty r^{-\varrho-1}J_3(r)\,\mathrm{d}r=\frac{4}{\pi}\sum_{m=0}^\infty(2m+1)^{-1}\int_0^\infty x^{-p+2m-1}n(x)\,\mathrm{d}x\int_x^\infty r^{p-\varrho-2m-1}\,\mathrm{d}r$$

$$=S_1\int_0^\infty x^{-\varrho-1}n[x)\,\mathrm{d}x$$

and

$$\int_R^\infty r^{-\varrho-1}J_4(r)\,\mathrm{d}r=\frac{4}{\pi}\sum_{m=0}^\infty(2m+1)^{-1}\int_R^\infty x^{-p-2m-3}n(x)\,\mathrm{d}x\int_R^x r^{p-\varrho+2m+1}\,\mathrm{d}r,$$

$$\int_R^\infty r^{-\varrho-1}J_4(r)\,\mathrm{d}r=\frac{4}{\pi}\sum_{m=0}^\infty(2m+1)^{-1}(2m+p-\varrho+2)^{-1}\times \tag{6.2}$$

$$\times\Big[\int_R^\infty x^{-\varrho-1}n(x)\,\mathrm{d}x-R^{2m+p-\varrho+2}\int_R^\infty x^{-2m-p-3}n(x)\,\mathrm{d}x\Big],$$

where

$$S_1=\frac{4}{\pi}\sum_{m=0}^\infty\frac{1}{(2m+1)(2m+\varrho-p)}\,.$$

It is easily seen, from (6.2), that

$$S_2\int_R^\infty x^{-\varrho-1}n(x)\,\mathrm{d}x<\int_R^\infty r^{-\varrho-1}J_4(r)\,\mathrm{d}r<S_3\int_R^\infty x^{-\varrho-1}n(x)\,\mathrm{d}x\,,$$

where

$$S_2 = \sum_{m=0}^{\infty} \frac{1 - R^{2m+p-\varrho+2}}{(2m+1)(2m+p-\varrho+2)}, \qquad S_3 = \sum_{m=0}^{\infty} \frac{1}{(2m+1)(2m+p-\varrho+2)}.$$

Also,

$$\int_R^{\infty} r^{-\varrho-1} J_5(r)\, \mathrm{d}t = O(R^{p-\varrho}).$$

This completes the proof of Theorem 4, since $p < \varrho < p+1$, and S_1, S_2, S_3 are convergent series.

Finally, we take this opportunity to thank Dr. P. K. Jain (University of Delhi) for his valuable suggestions in the preparation of this paper. We are also grateful to Professor U. N. Singh (Pro-Vice Chancellor, University of Delhi) for his keen interest and encouragement in our research work.

References.

[1] A. Edrei and W. H. J. Fuchs, *On the growth of meromorphic functions with several deficient values*, Trans. Amer. Math. Soc. (2) **93** (1959), 292-328.

[2] W. K. Hayman, *Meromorphic functions*, Oxford Univ. Press, Oxford 1964.

[3] P. K. Jain and P. K. Kamthan, *On Nevalinna's proximity functions of a meromorphic functions*, Annales Polonici Mathematici **26** (1972), 109-113.

[4] P. K. Kamthan, *Growth of a meromorphic function*, Collectanea Matematica **19** (1968), 33-49.

[5] R. Nevalinna, *Analytic functions*, Springer-Verlag, Berlin 1971.

Abstract.

Certain growth relations of the Nevanlinna proximity function $m(r)$ of a meromorphic function $f(z)$ of non-integral order ϱ, relative to $N(r)$, $N_p(r)$, $n(r)$ and $r^{\varrho(r)}$ ($\varrho(r)$ being a proximate order of $f(z)$), have been considered. It has been observed, by means of examples, that these results are not necessarily true for meromorphic functions of integral order. Also, it has been shown that a meromorphic function of non-integral order ϱ is of convergence or divergence class according as

$$\int^{\infty} r^{-\varrho-1} m(r)\, \mathrm{d}r$$

converges or diverges.

* * *