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On Generalized Rice's Polynomial. (**)

1. - Introduction.

The generalized RICE's polynomial has been defined by KHANDEKAR ([1], p. 158, eqn. (2.3)):

$$H_n^{(\alpha,\beta)}(\xi, p, v) = \frac{(1+\alpha)_n}{n!}\, {}_3F_2\left[\begin{matrix} -n, & n+\alpha+\beta+1, & \xi \\ & 1+\alpha, & p \end{matrix}; v\right]. \tag{1.1}$$

When $\alpha = \beta = 0$ this reduces to RICE's polynomial ([2], p. 108):

$$H_n(\xi, p, v) = {}_3F_2(-n, n+1, \xi; 1, p; v), \tag{1.2}$$

where $n = 0, 1, 2, \ldots,$ and ξ, p, v are complex variables but $p \neq -n-1$, $-n-2$, $-n-3, \ldots$

With $\xi = p$ in (1.1), we obtain

$$P_n^{(\alpha,\beta)}(1-2v) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left[\begin{matrix} -n, & n+\alpha+\beta+1 \\ & 1+\alpha \end{matrix}; v\right], \tag{1.3}$$

a JACOBI polynomial which reduces to Ultra-spherical and LEGENDRE polynomials by setting $\beta = \alpha$ and $\alpha = \beta = 0$ respectively.

The generalized RICE's polynomial $H_n^{(\alpha,\beta)}(\xi, p, v)$ satisfies the following dif-

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ferential equation ([4], eqn. (1.2) p. 223):

$$(1.4)\quad \begin{cases} v^2(1-v)D^3H_n^{(\alpha,\beta)}(\xi, p, v) + \\ \qquad + [(p+\alpha+2)v - (4+\xi+\alpha+\beta)v^2]D^2 \\ H_n^{(\alpha,\beta)}(\xi, p, v) + \\ \qquad + [p(1+\alpha) + \{n(n+1) - (1+\xi)(\alpha+\beta+2) + n(\alpha+\beta)\}v] \\ DH_n^{(\alpha,\beta)}(\xi, p, v) + n\xi(n+\alpha+\beta+1)H_n^{(\alpha,\beta)}(\xi, p, v) = 0\,, \end{cases}$$

where $D = \mathrm{d}/\mathrm{d}v$.

We require the generating function of the generalized RICE's polynomial ([1], p. 159, eqn. (4.2)):

$$(1.5)\quad \begin{cases} (1-t)^{-1-\alpha-\beta}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}(1+\alpha+\beta),\ \frac{1}{2}(2+\alpha+\beta),\ \xi \\ 1+\alpha,\quad p\end{matrix}\ ;\ \dfrac{-4vt}{(1-t)^2}\right] = \\ = \sum_{n=0}^{\infty} \dfrac{(1+\alpha+\beta)_n}{(1+\alpha)_n} H_n^{(\alpha,\beta)}(\xi, p, v)t^n\,, \end{cases}$$

in the present work.

In this Note an interesting result involving a generalized RICE's polynomial has established with the help of the generating function. A number of known as well as new results are also obtained with proper choice of parameters. Therefore, the result derived in this Note is of general character.

2. – This section deals with the formula involving the generalized RICE's polynomial $H_n^{(\alpha,\beta)}(\xi, p, x)$.

The result to be established is

$$(2.1)\quad x^n = \frac{(1+\alpha)_n(p)_n}{(\xi)_n}\sum_{k=0}^{n}\frac{(-n)_k(1+\alpha+\beta)_k\,(1+\alpha+\beta+2k)}{(1+\alpha)_k(1+\alpha+\beta)_{n+k+1}}\,H_n^{(\alpha,\beta)}(\xi, p, x)\,.$$

Proof. To obtain (2.1), consider

$$(2.2)\quad \begin{cases} {}_3F_2\left[\begin{matrix}\frac{1}{2}(\alpha+\beta+1),\ \frac{1}{2}(\alpha+\beta+2),\ \xi \\ 1+\alpha,\ p\end{matrix}\ ;\ \dfrac{-4xt}{(1-t)^2}\right] = \\ = (1-t)^{1+\alpha+\beta}\sum_{k=0}^{\infty}\dfrac{(1+\alpha+\beta)_k}{(1+\alpha)_k}H_k^{(\alpha,\beta)}(\xi, p, x)t^k\,. \end{cases}$$

Next, in (2.2) put

$$\frac{-4t}{(1-t)^2} = v .$$

Then

$$t = 1 - \frac{2}{1+\sqrt{1-v}} = \frac{-v}{(1+\sqrt{1-v})^2}$$

and (2.2) becomes

$$(2.3) \quad \begin{cases} {}_3F_2\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1),\ \frac{1}{2}(\alpha+\beta+2), \xi \\ 1+\alpha,\ p \end{matrix} ; vx\right] = \\ = \sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)_k}{(1+\alpha)_k} \frac{(-1)^k v^k}{2^{2k}} \left(\frac{2}{1+\sqrt{1-v}}\right)^{1+\alpha+\beta+2k} H_k^{(\alpha,\beta)}(\xi, p, x) . \end{cases}$$

Now replacing

$$\left(\frac{2}{1+\sqrt{1-v}}\right)^{1+\alpha+\beta+2k} \quad \text{by} \quad {}_2F_1\left[\begin{matrix} \frac{1}{2}(2+\alpha+\beta+2k),\ \frac{1}{2}(1+\alpha+\beta+2k) \\ 2+\alpha+\beta+2k \end{matrix} ; v\right]$$

(which follows from example 10, p. 70 [**3**]), we have

$$(2.4) \quad {}_3F_2\left[\begin{matrix} \frac{1}{2}(\alpha+\beta+1),\ \frac{1}{2}(2+\alpha+\beta),\ \xi \\ 1+\alpha,\ p \end{matrix} ; vx\right] =$$

$$= \sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)_k}{(1+\alpha)_k} \frac{(-1)^k v^k H_k^{(\alpha,\beta)}(\xi, p, x)}{2^{2k}}$$

$${}_2F_1\left[\begin{matrix} \frac{1}{2}(1+\alpha+\beta+2k),\ \frac{1}{2}(2+\alpha+\beta+2k) \\ 2+\alpha+\beta+2k \end{matrix} ; v\right]$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)_k}{(1+\alpha)_k} \frac{(-1)^k v^k H_k^{(\alpha,\beta)}(\xi, p, x)}{2^{2k}} \cdot$$

$$\frac{(\frac{1}{2}[1+\alpha+\beta+2k])_n (\frac{1}{2}[2+\alpha+\beta+2k]) v^n}{n!\,(2+\alpha+\beta+2k)_n} .$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)_k}{(1+\alpha)_k} \frac{(-1)^k H_k^{(\alpha,\beta)}(\xi, p, x)}{2^{2n+2k} n!} \frac{(1+\alpha+\beta+2k)_{2n}}{(2+\alpha+\beta+2k)_n} v^{n+k}$$

$\left[\text{obtained on using } (\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n\right]$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)_k}{(1+\alpha)_k} \frac{(-1)^k H_k^{(\alpha,\beta)}(\xi, p, x)}{2^{2n+2k} n!} \frac{(1+\alpha+\beta)_{2k+2n}(1+\alpha+\beta+2k)}{(1+\alpha+\beta)_{n+2k+1}} v^{n+k}$$

$\left[\text{obtained with the help of } \frac{(1+\alpha+\beta+2k)_{2n}}{(2+\alpha+\beta+2k)_n} = \frac{(1+\alpha+\beta)_{2k+2n}(1+\alpha+\beta+2k)}{(1+\alpha+\beta)_{n+2k+1}}\right]$

Therefore

$$(2.5)\quad \left\{\begin{aligned} &\sum_{n=0}^{\infty} \frac{\{\frac{1}{2}(\alpha+\beta+1)\}_n\{\frac{1}{2}(\alpha+\beta+2\}_n(\xi)_n}{(1+\alpha)_n\,(p)_n\,n!} v^n x^n = \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(1+\alpha+\beta)_k}{(1+\alpha)_k} \frac{(-1)^k(1+\alpha+\beta)_{2n}(1+\alpha+\beta+2k)H_k^{(\alpha,\beta)}(\xi, p, x)}{2^{2n}(n-k)!(1+\alpha+\beta)_{n+k+1}} v^n \end{aligned}\right.$$

which yields equation (2.1) on equating coefficients of v^n and using

$$(\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k} \qquad \text{for} \qquad 0 \leqslant k \leqslant n\,.$$

3. - Corollaires of (2.1).

a) Substituting $\alpha = \beta = 0$ and $x = v$, etc., we obtain a known result ([**3**], p. 288, eqn. (8)):

$$(3.1)\qquad v^n = \frac{(p)_n(n!)^2}{(\xi)_n} \sum_{k=0}^{n} \frac{(-1)^k(1+2k)H_k(\xi, p, v)}{(n-k)!\,(n+k+1)!}\,.$$

b) Taking $\xi = p$ and replacing x by $\{(1-x)/2\}$, we have

$$(3.2)\qquad (1-x)^n = 2^n(1+\alpha)_n \sum_{k=0}^{n} \frac{(-n)_k(1+\alpha+\beta+2k)(1+\alpha+\beta)_k P_k^{(\alpha,\beta)}(x)}{(1+\alpha+\beta)_{n+1+k}(1+\alpha)_k}\,.$$

This is a known result ([**3**], p. 262, eqn. (2)).

References.

[1] P. R. KHANDEKAR, *On a generalization of Rice's polynomial* - I, Proc. Nat. Acad. Sci. India, Sect. A, part. II **34** (1964), 257-262.

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[3] E. D. RAINVILLE, *Special Functions,* Macmillan Company, New York 1960.

[4] MANILAL SHAH, *On applications of Hermite polinomials,* Metu J. of pure and applied Sciences (4) **3** (1971), 221-235.

Abstract

The object of this Note is to obtain a relation involving a generalized Rice's polynomial with the use of the generating function. Particular interesting cases are also obtained on specializing the parameters.

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