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On the Space of Lebesgue-Bochner Summable Functions Generated by a Volume. (**)

In [1] is presented an approach to the theory of LEBESGUE-BOCHNER summable functions $L(v, Y)$, where v is the volume generating the integral and Y the BANACH space in which the functions have values. We show in this paper that the space $L(v, Y)$ is the closure, in a certain space, of the set of Y-valued simple functions over the prering on which the volume is defined.

We point out that M. H. STONE [7] used the notion of closure to define the space of real valued summable functions.

The terminology used in this paper is that of [5] and [6], where the LEBESGUE integral is defined axiomatically and representations are given for the integral and its completion by means of the volume generated by the integral.

We first state the following theorems of [5], as they will be frequently used in what follows:

Theorem A. Let $\int$ be a complete LEBESGUE integral and v the volume generated by $\int$. Then $D(\int) = L(v, R)$ and $\int f = \int f\, dv$ for all $f \in D(\int)$.

Theorem B. Let $\int$ be a LEBESGUE integral and v the volume generated by $\int$. Let m be the measure with smallest domain extending v. Then $D(\int) = L(m, R) = L(v, R) \cap M(m, R)$ and $\int f = \int f\, dv = \int f\, dm$ for all $f \in D(\int)$.

In [5] is also shown the following result.

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Theorem C. Let $\int$ be a complete LEBESGUE integral and v the volume generated by the integral. Then the family of null sets generated by $\int$ coincides with the family of null sets generated by v.

We also need the following result of [6].

Theorem D. Let $\int$ be a LEBESGUE integral with domain $D(\int)$ over X and $\int_c$ its completion with domain $D(\int_c)$ over X. Then a condition $C(x)$ holds $\int$-almost everywhere on X if and only if $C(x)$ holds $\int_c$-almost everywhere on X.

§ 1. - The space $S(\int, Y)$.

Let $\int$ be a LEBESGUE integral, with domain $D(\int)$, over an abstract set X; that is, $\int$ is a countably additive positive linear functional on the linear lattice $D(\int)$ of real valued functions satisfying the STONE condition over X (see [5]). We denote by $S(\int, Y)$ the space of all functions from X into a BANACH space $(Y, |\cdot|)$ such that for $f \in S(\int, Y)$ there exists a function $g \in D(\int)$ with the property that $|f(x)| \leqslant g(x)$ $\int$-a.e. On $S(\int, Y)$ we define the functional

$$\|f\|_{\int} = \inf\{\int g : |f(x)| \leqslant g(x) \ \int\text{-a.e.}, g \in (D)\} .$$

Proposition 1. Let $\int$ be a LEBESGUE integral and $\int_c$ its completion. Then the space $S(\int, Y)$ coincides with the space $S(\int_c, Y)$ and $\|f\|_{\int} = \|f\|_{\int_c}$ for all $f \in S(\int, Y)$.

Proof. Obviously, since $\int_c$ is an extension of $\int$, $S(\int, Y) \subset S(\int_c, Y)$. Now, take any $f \in S(\int_c, Y)$. Then there exists a function $g \in D(\int_c)$ such that $|f(x)| \leqslant g(x)$ $\int_c$-a.e. By definition of the completion, there exists $h \in D(\int)$ such that $g(x) = h(x)$ $\int$-a.e. This implies, by Theorem D, that $|f(x)| \leqslant h(x)$ $\int$-a.e. This proves $S(\int_c, Y) \subset S(\int, Y)$.

Since for any $g \in D(\int_c)$ such that $|f(x)| \leqslant g(x)$ $\int_c$-a.e. we have $h \in D(\int)$ such that $|f(x)| \leqslant h(x)$ $\int$-a.e. and $\int_c g = \int h$, it is clear that the two sets

$$\{\int_c g : |f(x)| \leqslant g(x) \ \int_c\text{-a.e.}, g \in D(\int_c)\}$$

and

$$\{\int h : |f(x)| \leqslant h(x) \ \int\text{-a.e.}, h \in D(\int)\}$$

coincide and therefore $\|f\|_{\int_c} = \|f\|_{\int}$ for $f \in S(\int_c, Y)$.

Proposition 2. The space $(S(\int, Y), \| \|_{\int})$ is a complete semi-normed space.

Proof. From the fact that $D(\int)$ is a linear space and the $\int$-null sets form a ring it is easy to see that $S(\int, Y)$ is a linear space.

Since $\int$ is a positive functional, $\|f\|_{\int} \geqslant 0$ for any $f \in S(\int, Y)$. If $f = 0$, then $|f(x)| \leqslant 0$ everywhere and therefore $\|f\|_{\int} = 0$. For $f_1, f_2 \in S(\int, Y)$, the triangle inequality follows from the fact that for $\varepsilon > 0$ there exist $g_i \in D(\int)$ such that $|f_i(x)| \leqslant g_i(x)$ $\int$-a.e. and $\int g_i \leqslant \|f_i\|_{\int} + \varepsilon/2$ for $i = 1, 2$.

To prove the inequality $|k| \, \|f\|_{\int} \leqslant \|kf\|_{\int}$ for $k \neq 0$, we notice that for $g \in D(\int)$ such that $|kf| \leqslant g$ $\int$-a.e. we have $|f| \leqslant (1/|k|) g$ $\int$-a.e. and therefore, $|k| \, \|f\|_{\int} \leqslant \int g$. The desired inequality is obtained by taking infemum over all such functions g. The inequality in the other direction can be proven similarly.

We shall now use Proposition 1 and establish the completeness of the semi-normed space $(S(\int_c, Y), \| \|_{\int_c})$. For the sake of convenience we shall denote the completion $\int_c$ of $\int$ by J. Let $f_n \in S(J, Y)$ such that the series $\sum_{n=1}^{\infty} \|f_n\|_J$ converges. We shall show that the series $\sum_{n=1}^{\infty} f_n$ converges in the space $S(J, Y)$.

There exist $g_n \in D(J)$ such that $|f_n(x)| \leqslant g_n(x)$ for $x \notin A_n$, where A_n is a J-null set, and $J g_n < \|f_n\|_J + 1/2^n$. Since J is complete, by Theorem A, $g_n \in L(v, R)$ and $\int g_n \, dv = J g_n$, where v is the volume generated by J.

The absolute convergence of the series $\sum_{n=1}^{\infty} g_n$ in the space $L(v, R)$ implies, by Lemma 1 of [**3**], that there exists $g \in L(v, R)$ such that $\sum_{n=1}^{\infty} \int g_n dv = \int g \, dv$. and $\sum_{n=1}^{\infty} g_n(x) = g(x)$ for $x \notin A_0$, where A_0 is a v-null set.

By Theorem C, A_0 is a J-null set and therefore for any x not in the J-null set $A = \bigcup_{n=0}^{\infty} A_n$ the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely and hence converges in the BANACH space Y.

Define a function f from X into Y by

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} f_n(x) & \text{for} \quad x \notin A\,, \\ 0 & \text{for} \quad x \in A\,. \end{cases}$$

For $x \notin A$, $|f(x)| \leqslant g(x)$, $g \in D(J)$ and therefore $f \in S(J, Y)$. Moreover, for $x \notin A$,

$$|f(x) - (f_1 + \ldots + f_k)(x)| \leqslant \sum_{n > k} g_n(x)$$

and

$$J\left(\sum_{n>k} g_n\right) = \int \sum_{n>k} g_n \, \mathrm{d}v = \int (g - g_1 - \ldots - g_k) \, \mathrm{d}v = \sum_{n>k} \int g_n \, \mathrm{d}v.$$

As $k \to \infty$, the right hand side of the equation goes to 0 and hence $\|f - (f_1 + \ldots + f_k)\|_J \to 0$.

Proposition 3. A function $f: X \to Y$ belongs to $S(\int, Y)$ and $\|f\|_\int = 0$ if and only if $f(x) = 0$ $\int$-a.e.

Proof. It is clear that if $f(x) = 0$ $\int$-a.e., then $f \in S(\int, Y)$ and $\|f\|_\int = 0$

Assume that $f \in S(\int, Y)$ and $\|f\|_\int = 0$. If J denotes the completion of $\int$, then $f \in S(J, Y)$ and $\|f\|_J = 0$. There exist $g_n \in D(J)$ such that $Jg_n \to 0$ and $|f(x)| \leqslant g_n(x)$ for $x \notin A_n$, where A_n is a J-null set. By Theorem A $Jg_n = \int g_n \, \mathrm{d}v$, where v is the volume generated by J, and therefore, by Theorem 2 of [1], there exists a subsequence g_{n_k} of g_n which converges to zero for $x \notin A_0$, where A_0 is a v-null set.

J being complete, by Theorem C, A_0 is a J-null set. For x not in the J-null set $A = \bigcup_{n=0}^{\infty} A_n$, $|f(x)| \leqslant g_{n_k}(x)$ and $g_{n_k}(x) \to 0$. This implies that $f(x) = 0$ for $x \notin A$. It follows from Theorem D that $f(x) = 0$ $\int$-a.e.

Proposition 4. Let f_n be a sequence of functions in $S(\int, Y)$ and f a function from X into Y. Then $f \in S(\int, Y)$ and $\|f_n - f\|_\int \to 0$ if and only if f_n is a Cauchy sequence in $S(\int, Y)$ and there exists a subsequence f_{n_k} of f_n such that $f_{n_k}(x)$ converges to $f(x)$ $\int$-a.e.

Proof. Let $f \in S(\int, Y)$ and $\|f_n - f\|_\int \to 0$. Since $\|\cdot\|_\int$ is a semi-norm, f_n is a Cauchy sequence in $S(\int, Y)$. If J is the completion of $\int$, then $\|f_n - f\|_J \to 0$. Further, there exist $g_n \in D(J)$ such that

$$|f_n(x) - f(x)| \leqslant g_n(x) \qquad \text{for } x \notin A_n,$$

where A_n is a J-null set, and

$$Jg_n < \|f_n - f\|_J + 1/n.$$

Since J is complete, by Theorem A, $\int g_n \, \mathrm{d}v \to 0$, where v is the volume generated by J. Then, by Theorem 2 of [1], there exists a subsequence g_{n_k} of g_n such that $g_{n_k}(x) \to 0$ for $x \notin A_0$, where A_0 is a v-null set and hence a

J-null set. This implies that $f_{n_k}(x)$ converges to $f(x)$ for any x not in the J-null set $\bigcup_{n=0}^{\infty} A_n$. By Theorem D, we see that $f_{n_k}(x)$ converges to $f(x)$ $\int$-a.e.

For the proof in the other direction, assume that f_n is a CAUCHY sequence in $S(\int, Y)$ and there exists a subsequence f_{n_k} such that $f_{n_k}(x)$ converges to $f(x)$ for $x \notin A$, where A is a $\int$-null set. By the completeness of $S(\int, Y)$, there exists $g \in S(\int, Y)$ such that f_n converges to g. By the first part of this proposition there exists a subsequence of f_{n_k} converging pointwise to g on $x \notin B$, where B is a $\int$-null set. This implies that $f(x) = g(x)$ for x not in the $\int$-null set $A \cap B$.

By the previous proposition $f - g \in S(\int, Y)$ and $\|f - g\|_f = 0$. This implies that $f \in S(\int, Y)$ and $\|f_n - f\|_f \to 0$.

§ 2. - The space of Lebesgue-Bochner summable functions.

Lemma. *Let $\int$ be a Lebesgue integral and w the volume generated by $\int$ on the ring W of summable sets. If Y is a Banach space, then the set $S(W, Y)$ of Y-valued simple functions is a subset of the space $S(\int, Y)$ and the semi-norms coincide on $S(W, Y)$, that is $\|s\|_w = \|s\|_f$ for $s \in S(W, Y)$.*

Proof. Let $s \in S(W, Y)$. Then,

$$s = y_1 c_{A_1} + \dots + y_n c_{A_n},$$

where $y_i \in Y$ and $A_i \in W$, A_i disjoint.

Let $\int$ be defined on $D(\int)$ over the set X.

For $x \in X$, $|s(x)| = |s|x|$, where

$$|s| = |y_1| c_{A_1} + \dots + |y_n| c_{A_n} \in D(\int).$$

Therefore $s \in S(\int, Y)$.

Since

$$|s| \in \{g\colon |s(x)| \leqslant g(x) \ \int\text{-a.e., } g \in D(\int)\},$$

we see that $\|s\|_f = \int |s|$ and, by Theorem B, $\int |s| = \int |s| \mathrm{d}w = \|s\|_w$.

Theorem 1. *Let Y be a Banach space and $L(v, Y)$ the space of Y-valued Lebesgue-Bochner summable functions generated by a volume space (X, V, v). Define a Lebesgue integral $\int$ by $D(\int) = L(v, R)$ and $\int f = \int f \,\mathrm{d}v$. Then $L(v, Y)$ is the closure in $S(\int, Y)$ of the set of Y-valued simple functions $S(V, Y)$ over the prering V.*

Proof. The functional $\int$ defined on $D(\int) = L(v, R)$ by $\int f = \int f\,dv$ for $f \in D(\int)$ is a complete LEBESGUE integral (see Example 5, [5]). Let w be the volume generated by the integral $\int$ over the ring W of summable sets (see Theorem 1, [5]).

According to Theorem 1 of [4], $L(v, Y) = L(w, Y)$ and $\|f\|_v = \|f\|_w$ for all $f \in L(v, Y)$ and the null sets generated by the two volumes v and w coincide. Notice that $V \subset W$, $S(V, Y) \subset S(W, Y) \subset L(v, Y)$ and $\|g\|_v = \|g\|_w$ for $g \in S(V, Y)$.

Let $f \in L(v, Y)$. Then by the definition of the space $L(v, Y)$, there exists a basic sequence $s_n \in S(V, Y)$ such that $s_n(x)$ converges to $f(x)$ v-a.e. By Lemma 4 of [1] $\|s_n - f\|_v$ converges to zero which implies, by the above lemma, that $\{s_n\}$ is a CAUCHY sequence in $S(\int, Y)$. Since $\int$ is a complete LEBESGUE integral and w the volume generated by it, by Theorem C, $s_n(x)$ converges to $f(x)$ $\int$-a.e. It follows from Proposition 4 that $f \in S(\int, Y)$ and $\|s_n - f\|_\int$ converges to zero. But

$$\|s_n\|_\int = \|s_n\|_v \qquad \text{and} \qquad \|s_n\|_v \to \|f\|_v\,.$$

This shows that $\|f\|_\int = \|f\|_v$.

Now take any $f \in S(\int, Y)$ such that there exists a sequence $s_n \in S(V, Y)$ with $\|s_n - f\|_\int$ converging to zero. By Proposition 4, $\{s_n\}$ is a CAUCHY sequence in $S(\int, Y)$ and there exists a subsequence s_{n_k} of s_n such that $s_{n_k}(x)$ converges to $f(x)$ $\int$-a.e.

Since $S(V, Y) \subset S(W, Y)$, the above lemma implies that s_n is a CAUCHY sequence in $S(W, Y)$ and, by Theorem C, $s_{n_k}(x)$ converges to $f(x)$ w-a.e. It follows from Theorem 2 of [1] that $f \in L(w, Y)$ and therefore $f \in L(v, Y)$.

The proof of the theorem is now complete.

We shall state below a theorem whose proof follows from the proof of the above theorem and from Theorem D above and Lemma 2 of [5].

Theorem 2. *Let $\int$ be a Lebesgue integral and w the volume generated by $\int$ on the ring W of summable sets. If Y is a Banach space, then $L(w, Y)$ is the closure in $S(\int, Y)$ of the set $S(W, Y)$ of simple functions.*

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