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A Contour Integral Involving Modified Bessel Function and Fox's H - Function. (**)

1. - Introduction.

In this paper we have evaluated a contour integral involving modified BESSEL function and FOX's H-function. On specializing the parameters, the integral yields many results, some of which are recently given by MEIJER [8], MAC ROBERT [6] and BAJPAI [1].

The H-function introduced by FOX ([4], p. 408), will be represented and defined as follows:

$$
(1.1)\quad \begin{cases} H_{p,q}^{m,n}\left[\, x \left| \begin{matrix} (a_1, e_1), \dots, (a_p, e_p) \\ (b_1, f_1), \dots, (b_q, f_q) \end{matrix} \right.\right] = \\ \qquad = \dfrac{1}{2\pi i} \displaystyle\int_L \frac{\prod_{i=1}^{m} \Gamma(b_j - f_j s) \prod_{i=1}^{n} \Gamma(1 - a_j + e_j s)\, x^s}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + f_j s) \prod_{i=n+1}^{p} \Gamma(a_j - e_j s)}\, ds\,, \end{cases}
$$

where an empty product is interpreted as 1, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; $e's$ and $f's$ are all positive; L is a suitable contour of BARNES type such that the poles of $\Gamma(b_j - f_j s)$, $j = 1, \dots, m$ lie on the right hand side of the contour and those of $\Gamma(1 - a_j + e_j s)$, $j = 1, \dots, n$ lie on the left hand side of the contour.

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In what follows for sake of brevity (a_p, e_p) denotes $(a_1, e_1), \ldots, (a_p, e_p)$.

2. - The integral.

The integral to be evaluated is

$$(2.1)\quad \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} x^{-\lambda} \exp(xt - \alpha/x)\, I_\nu(\alpha/x)\, H_{p,q}^{m,n}\left[zx^h \,\middle|\, \begin{matrix}(a_p, e_p)\\(b_q, f_q)\end{matrix}\right] dx = \\ \qquad = \dfrac{2^{-\nu}\alpha^\nu t^{\lambda+\nu-1}}{\Gamma(\nu+1)} \displaystyle\sum_{r=0}^{\infty} \dfrac{(\nu+\frac{1}{2})_r (2\nu+1)_r}{r!} (-2\alpha t)^r H_{p+1,q}^{m,n} = \\ = \left[zE^h \,\middle|\, \begin{matrix}(a_p, e_p),\ (\lambda+\nu+r, h)\\(b_q, f_q)\end{matrix}\right], \end{cases}$$

where

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \leqslant 0\,, \qquad \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv k > 0\,,$$

$$|\arg z| < \tfrac{1}{2}k\pi\,, \qquad \mathrm{Re}\,(\lambda+\nu) > 0\,.$$

Proof. To establish (2.1), expressing the H-function as a MELLIN-BARNES type integral (1.1) and interchanging the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, we have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)\, z^s}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j - e_j s)} \cdot \frac{1}{2\pi i}\int_{c-i\infty} x^{-\lambda h s} \exp(xt - \alpha/x)\, I_\nu(\alpha/x)\, dx\, ds\,.$$

Evaluating the inner-integral with the help of the modified form of the formula ([3], p. 281, (18)), viz.

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-\lambda} \exp(xt - \alpha/x/) I_\nu(\alpha/x)dx =$$

$$= \frac{2^{-\nu}\alpha^\nu t^{\lambda+\nu-1}}{\Gamma(\nu+1)} \sum_{r=0}^{\infty} \frac{(\nu+\frac{1}{2})_r (2\nu+1)_r}{r!\,\Gamma(\lambda+\nu+r)} (-2\alpha t)^r\,, \qquad \mathrm{Re}\,(\lambda+\nu) > 0\,,$$

we get

$$\frac{(2^{-\nu}\alpha^{\nu}\, t^{\lambda+\nu-1})}{\Gamma(\nu+1)} \sum_{r=0}^{\infty} \frac{(\nu+\frac{1}{2})_r (2\nu+1)_r}{r!} (-2\alpha t)^r$$

$$x \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(1-b_j+f_j s) \prod_{j=1}^{n} \Gamma(1-a_j-e_j s)\, t^{-hs} z^s}{\prod_{j=m+1}^{q} \Gamma(1-b_j-f_j s) \prod_{j=n+1}^{p} \Gamma(a_j-e_j s)\, \Gamma(\lambda+\nu+r-hs)} \, ds .$$

On applying (1.1), the formula (2.1) is obtained.

3. - Particular cases.

In (2.1), putting $\nu = -\frac{1}{2}$ and using the formula

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \exp(x) ,$$

it reduces to the form

$$(3.1) \quad \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{c-i\infty}^{c+i\infty} x^{1/2-\lambda} \exp(xt)\, H_{p,q}^{m,n}\left[zx^h \middle| \begin{matrix} (a_p, e_p) \\ (b_q, f_q) \end{matrix} \right] dx = \\ \qquad = t^{\lambda-3/2}\, H_{p+1,q}^{m,n}\left[zt^{-h} \middle| \begin{matrix} (a_p, e_p), & (\lambda-\frac{1}{2}, h) \\ (b_q, f_q) & \end{matrix} \right] , \end{cases}$$

where

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \leqslant 0 , \qquad \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv k > 0 ,$$

$$|\arg z| > \tfrac{1}{2} k\pi , \qquad \operatorname{Re} \lambda > \tfrac{1}{2} .$$

The integral (3.1) is a generalization of many inverse LAPLACE transforms ([**3**], p. 227-301).

In (3.1), assuming h as a positive integer, putting $e_j = f_i = 1$ $(j = 1, ..., p;$ $i = 1, ..., q)$ using the formula

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_p, 1) \\ (b_q, 1) \end{matrix} \right] = G_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right],$$

and simplifying with the help of (1.1), ([2], p. 4, (11)) and ([2], p. 207, (1)), we get

$$(3.2)\quad \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} x^{1/2-\lambda} \exp(xt)\, G_{p,q}^{m,n}\left[zx^h \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right] dx = \\ \qquad = t^{\lambda-3/2}\, h^{1-\lambda}\, (2\pi)^{h/2-1/2}\, G_{p+h,q}^{m,n}\left[z\left(\dfrac{h}{t}\right)^h \,\middle|\, \begin{matrix} a_p, \Delta(h, \lambda-\frac{1}{2}) \\ b_q \end{matrix} \right], \end{cases}$$

where $2(m+n) > p+q$, $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $\operatorname{Re}\lambda > \frac{1}{2}$, and the symbol $\Delta(h, \lambda-\frac{1}{2})$ represents the set of parameters

$$\frac{\lambda-\frac{1}{2}}{h}, \frac{\lambda+\frac{1}{2}}{h}, \dots, \frac{\lambda+h-3/2}{h}.$$

In (3.2), putting $\lambda = \sigma - \mu$, we get a known result ([1], (2.2)).

In (3.2), putting $h = 1$, $t = -1$, $\lambda = \frac{3}{2} - \nu$, $m = 1$, $b_1 = 0$, $n = p$, replacing x by $-x$, q by $q+1$, b_k by b_{k-1} $(k = 2, 3, \dots, q+1)$, using ([7], p. 372, (20)), viz.

$$G_{p,q}^{1,n}\left[z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right] = \frac{\prod_{j=1}^{n} \Gamma(1+b_1-a_j) z^{b_1}}{\prod_{j=m+1}^{p} \Gamma(a_j - b_1)}\, {}_p\varphi_{q-1}\left[\begin{matrix} 1+b_1-a_1, \dots, 1+b_1-a_p\,; \\ 1+b_1-b_2, \dots, 1+b_1-b_q \end{matrix}\ (-1)^{p-n-1} \right],$$

and deforming the contour suitably it reduces to a result obtained by MEIJER ([8], p. 350, (98)).

In (3.1) assuming h as a positive integer, setting $m = q = p$, $n = 1$, $p = q+1$, $e_j = f_i = 1$ $(j = 1, \dots, p;\ i = 1, \dots, q)$, $a_1 = 1$, replacing a_{j+1} by a_j $(j = 1, \dots, q)$, using the formula

$$H_{q+1,p}^{p,1}\left[z \,\middle|\, \begin{matrix} (1, 1), (b_q, 1) \\ (a_p, 1) \end{matrix} \right] = E\left[\begin{pmatrix} a_p \\ b_q \end{pmatrix} : z \right],$$

and simplifying with the help of (1.1), ([2], p. 4, (11)) and ([5], p. 374, (36)), it reduces to the form

$$
(3.3)\quad \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+} x^{1/2=\lambda} \exp\ (xt)\, E\left[\begin{matrix}\alpha_p \\ \beta_q\end{matrix} : zx^h\right] \mathrm{d}x = \\ \qquad = t^{\lambda-3/2} - h^{1-\lambda}\,(2\pi)^{h/2-1/2}\, E\left[\begin{matrix}\alpha_p \\ \beta_q\end{matrix}, \Delta(h, \lambda-\tfrac{1}{2})^{:z\left(\frac{h}{t}\right)^h}\right] \end{cases}
$$

where $p \geqslant q+1$, $\operatorname{Re}\alpha_r > 0$, $(r = 1, \ldots, p-1)$, $\operatorname{Re}(\beta_s - \alpha_s) > 0$ $(s = 1, \ldots, q)$, $|\arg z| < \pi$, $\operatorname{Re}\lambda > \frac{1}{2}$.

In (3.3), taking $t = 1$, $\frac{1}{2} - \lambda = p$ and deforming the contour suitably, we obtain a known result given by MAC ROBERT ([**6**], p. 191, (8)).

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Abstract

In this paper a contour integral involving modified Bessel function and Fox's H-function has been evaluated. The integral is of general character and numerous particular cases of the integral are scattered throughout the literature.

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