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HARBANS LAL (*)

Semi-primary Ideals of Commutative Rings. (**)

1. - Introduction.

GILMER in [3] (p. 738) has characterised the rings in which each ideal is primary. In this paper, we characterize the rings in which each ideal is semi-primary and Noetherian rings in which each non-zero ideal is semi-primary.

2. - Preliminaries.

In the following, a ring will always mean a commutative ring. We call a ring R to be a generalised semi-primary (notation: G.S.P.) ring if all the ideals of R are semi-primary. Also we call a ring R to be a restricted generalised semi-primary ring if all the non-zero ideals of R are semi-primary. An ideal A of a ring R is said to be simple if there is no ideal properly between A and A^2 ([**6**], p. 842). An ideal A of a ring R such that $A \neq R$ is called a genuine ideal ([**6**], p. 838). A ring R is said to have KRULL dimension (notation: $\dim R$) equal to n if there exists a chain $P_1 \subset P_2 \subset \ldots \subset P_{n+1}$ of $n+1$ genuine prime ideals of R and there is no such longer chain. Noetherian rings are not assumed to have unity always. The notations and terminology, in general, are of [**4**].

3. - Rings in which all ideals are semi-primary.

Let R be a G.S.P. ring. Rings in which every ideal is primary are such rings. Converse is false as is illustrated by Example 3.7.

(*) Indirizzo: Department of Mathematics, Hans Raj College, Delhi-7, India.
(**) Ricevuto: 1-IX-1971.

3.1. – Theorem. *Let R be a commutative ring (not necessarily with unity); then the following are equivalent:*

(1) *R is a G.S.P. ring.*
(2) *Every principal ideal of R is semi-primary.*
(3) *Prime ideals of R are totally ordered under set inclusion.*

Proof. (1) $\Rightarrow$ (2) is immediate. Let (2) hold and P and Q be two prime ideals of R such that neither of $P \subseteq Q$ or $Q \subseteq P$ holds. Choose $x \in P - Q$ and $y \in Q - P$. There exist prime ideals P_1 and Q_1 with $\sqrt{(x)} = P_1$ and $\sqrt{(y)} = Q_1$. Also $\sqrt{(xy)} = P'$ for some prime ideal P'. Then $P' = P_1 \cap Q_1$. This gives $P_1 \subseteq P'$ or $Q_1 \subseteq P'$. Both are impossible and we arrive at the required contradiction which proves (2) $\Rightarrow$ (3). Let (3) hold now. For any ideal A of R, $\sqrt{A} = \cap P_\alpha$, where the intersection is over all prime ideals $P_\alpha \supseteq A$. The totally ordered nature of prime ideals yields that $\sqrt{A} = P$, for some prime P, so that A is semi-primary.

3.2. – Corollary. *A G.S.P. ring with unity is quasi-local.*

3.3. – Lemma. *Let R be a Noetherian G.S.P. ring with unity; then* $\dim R \leqslant 1$ *and R can have at most two genuine prime ideals.*

Proof. Let M be the unique maximal ideal of R. If $\sqrt{(0)} = M$, the Lemma follows. So Let $\sqrt{(0)} \neq M$ and choose $x \in M - \sqrt{(0)}$. Further if we suppose $\sqrt{(x)} \neq M$, take $y \in M - \sqrt{(x)}$. Now $\sqrt{(0)} \subset \sqrt{(x)} \subset \sqrt{(y)} \subseteq M$. By [**5**] (theorem 3, p. 84), we get a prime ideal P such that $\sqrt{(y)} \subseteq P \subseteq M$ and P is a minimal prime ideal of $\sqrt{(y)}$. Thus $\sqrt{(0)} \subset \sqrt{(x)} \subset P$, which is impossible in view of [**5**] (theorem 21, p. 217). Hence $\sqrt{(x)} = M$ and $\sqrt{(0)}$, M are the only two genuine prime ideals.

We will denote the prime ideal $\sqrt{(0)}$ by Q in the remainder of this section.

3.4. – Lemma. *Let R be a G.S.P. ring with unity such that its unique maximal ideal M is principal. Now if $P \subset M$, for a prime ideal P of R such that there is no prime ideal properly between P and M; then M is simple, $P = \bigcap_{n=1}^{\infty} M^n$, every prime ideal properly contained in M is contained in P and each genuine ideal not contained in P is a power of M.*

Proof. That M is simple and $P \subseteq \bigcap_{n=1}^{\infty} M^n$ are immediate. Consider the domain R/P which has just three prime ideals, each of which being principal;

so by [1] (theorem 2) R/P is a Noetherian domain and thus by [7] (p. 216) $\bigcap_{n=1}^{\infty} [M/P]^n = (\bar{0})$ which in turn implies $\bigcap_{n=1}^{\infty} M^n \subseteq P$. The third assertion is obvious. As for the last, let A be a genuine ideal of R such that $A \not\subseteq P$. As R is a G.S.P. ring, $\sqrt{A} = M$ which being principal, $M^k \subseteq A \subseteq M$ for some positive integer k. Lemma 3 of [2] proves this assertion, since M is simple.

3.5. – Theorem. *Let R be a commutative Noetherian ring with unity. Then R is a G.S.P. ring if and only if R is one of the following types:*

(1) *Every ideal of R is a primary ideal.*

(2) *R is a one dimensional local ring with only two genuine prime ideals Q and M (the unique maximal ideal of R).*

Proof. If R is any one of these types, the R is clearly a G.S.P. ring. Let, now, R be a G.S.P. ring. By Lemma **3.3**, $\dim R \leqslant 1$. If $\dim R = 0$, then R is of type (1) and if $\dim R = 1$, then R is of type (2).

3.6. – Corollary. *If in type (2) above, M is simple, then R is a discrete valuation ring of rank one.*

Proof. First we prove that M is principal. Let $x \in M - Q$, then $\sqrt{(x)} = M$, R being Noetherian, there exists a positive integer n such that $M^n \subseteq (x) \subseteq M$. Thus by [2] (lemma 3) $(x) = M^k$ for some k. If $k = 1$, M is principal. Suppose $k > 1$. For a y in $M - (x)$, and a repetition of the above argument gives that for some k_1, $(y) = M^{k_1}$ and $k_1 < k$. This way we reach at an element m in R, in a finite number of steps, such that $M = (m)$. Revoking to Lemma **3.4**, we get $Q = \bigcap_{n=1}^{\infty} M^n$ and as R is a local ring, by [7] (p. 217) $Q = (0)$. Thus we have a local one dimensional domain, with its unique maximal ideal to be simple. By [1] (theorem 8) R is a DEDEKIND domain and hence a discrete valuation ring of rank one.

The following example is of a G.S.P. ring in which every ideal is not primary.

3.7. – Example. Let $R = \{a_0 + a_1 x + \sum_{n=1}^{\infty} b_n y^n :\ a_0, a_1, b_i \in F\}$, F any field, $x^2 = xy = yx = 0$, $fx = xf$ and $fy = yf$ for every f in F. R is a ring with the usual addition, multiplication and equality as in power series rings. Further this is a local ring, with its unique maximal ideal $M = (x, y)$ and contains one more genuine prime ideal (x). This is a G.S.P. ring by Theorem **3.1** and is of type (2) by Theorem **3.5**. But in this ring, all ideals except (0) are primary, whereas (0) is only a semi-primary ideal.

4. - Rings in which all non-zero ideals are semi-primary.

The ring of integers modulo pq where p, q are different primes and $F_1 \oplus F_2$, F_i fields, are the examples of restricted G.S.P. rings which are not G.S.P. rings. We start with

4.1. – Lemma. *Let R be a restricted G.S.P. ring (not necessarily with unity) such that $\sqrt{(0)} \neq (0)$; then R is a G.S.P. ring.*

Proof. Obvious.

4.2. – Remark. For a ring R to be restricted G.S.P. but not G.S.P., we have necessarily $\sqrt{(0)} = (0)$ and R is not an integral domain. Clearly such a ring is always a semi-prime ring.

4.3. – Lemma. *Let R be a Noetherian restricted G.S.P. ring which is not a G.S.P. ring (not necessarily with unity). Then it contains only two minimal prime ideals P and Q, such that $P \cap Q = (0)$, $P \nsubseteq Q$, $Q \nsubseteq P$ and every prime ideal of R contains either P or Q.*

Proof. In view of the above remark, there exist x and y in R such that $xy=0$, $x \neq 0$, $y \neq 0$. Then $\sqrt{(x)}=P$ and $\sqrt{(y)}=Q$ where P and Q are suitable non-zero prime ideals. R being Noetherian, there exist integers m and n such that $P^m \subseteq (x)$ and $Q^n \subseteq (y)$. Thus $P^m Q^n = (0)$. The last assertion follows from this. Now $(0) = \sqrt{(0)} = \sqrt{P^m Q^n} = P \cap Q$ and since (0) is not a prime ideal, therefore, $P \nsubseteq Q$ and $Q \nsubseteq P$. These P and Q are minimal primes.

4.4. – Lemma. *Let R be a Noetherian restricted G.S.P. ring (not necessarily with unity). Then for any prime ideal T and R, the family of all those prime ideals which contain T is totally ordered.*

Proof. We can assume R to be a restricted G.S.P. ring which is not a G.S.P. ring, in view of Theorem **3.1**. Then by Lemma **4.3**, R has only two minimal prime ideals P and Q such that any prime ideal of R contains either P or Q. It clearly suffices to prove the result for $T=P$ and $T=Q$. Let $T=P$ and P_1, P_2 be any two prime ideals each containing P. As $P \neq 0$, $P_1 \cap P_2 \neq 0$. But then $P_1 \cap P_2$ is a prime ideal and therefore $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. Thus the prime ideals containing P are totally ordered. Similarly when $T=Q$.

4.5. – Theorem. *Let R be a commutative Noetherian ring with unity. Then R is a restricted G.S.P. ring which is not a G.S.P. ring if and only if R is one of the following types*:

(1) *R is a local ring of dimension one having only two minimal prime ideals P and Q such that $P \cap Q = (0)$.*

(2) *R is a direct sum of two fields.*

Proof. Let R be a restricted G.S.P. ring but not a G.S.P. ring. By Lemma 4.3, it contain only two minimal prime ideals P and Q such that $P \not\subseteq Q$, $Q \not\subseteq P$ and $P \cap Q = (0)$ and every prime ideal of R contains either P or Q. Now two cases arise:

Case I. When $P + Q = R$. There P and Q are easily seen to be maximal ideals and therefore, $R \approx R/P \oplus R/Q$. Thus R is of type (2).

Case II. When $P + Q \neq R$. These there exists a maximal ideal M containing $P + Q$. Let M' be any other maximal ideal of R. Then either $P \subseteq M'$ or $Q \subseteq M'$. Suppose $P \subseteq M'$. Passing onto R/P, which is a G.S.P. ring and hence quasi-local, we get $M' = M$. Also from this, we have $\dim R = 1$, in view of Lemma 3.3. Thus R is a local ring of dimension 1, with only two minimal primes P and Q such that $P \cap Q = (0)$. Thus R is of type (1). Converse is easily proved.

Combining Theorems **3.5** and **4.5** we obtain the following

4.6. – Theorem. *Let R be a commutative Noetherian ring with unity; then R is a restricted G.S.P. ring if and only if R is one of the following types*:

(1) *Every ideal of R is a primary ideal.*

(2) *R is a one dimensional local ring with only twog enuine prime ideals Q and M (the unique maximal ideal of R).*

(3) *R is a local ring of dimension one having only two minimal prime ideals P and Q such that $P \cap Q = (0)$.*

(4) *R is a direct sum of two fields.*

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