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Absolute Nörlund Summability of a Factored Fourier Series. (**)

1.1. - Definitions [5].

Let $\sum a_n$ be a given infinite series and $\{s_n\}$ is the sequence of its partial sums. The sequence to sequence transformation

$$t_n = (1/P_n) \sum_{\nu=0}^{n} p_{n-\nu} s_\nu , \qquad P_n = \sum_{\nu=0}^{n} p_\nu \neq 0 , \tag{1.1.1}$$

where p_n is a sequence of constants, real or complex, defines the (N, p_n) mean of the sequence $\{s_n\}$. If is of bounded variation, i.e.

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty , \tag{1.1.2}$$

then we say that $\sum a_n$, or $\{s_n\}$, is summable $|\mathrm{N}, p_n|$.

In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\,\Gamma(\alpha)} \qquad (\alpha > 0) \tag{1.1.3}$$

the NÖRLUND mean reduces to the familiar (C, α) mean, and when

$$p_n = \frac{1}{n+1} \tag{1.1.4}$$

the (N, p_n) mean reduces to the harmonic mean.

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1.2. – Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$, without loss of generality, we may assume that the constant term in the FOURIER series of $f(t)$ is zero, so that

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{1}^{\infty} A_n(t)$$

and

$$\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = 0 ,$$

we use the following notations throughout:

$$\varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\} ,$$

$$\Phi_0(t) = \varphi(t) ,$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \varphi(u)\,\mathrm{d}u \qquad (\alpha > 0) ,$$

$$\varphi_\alpha(t) = \Gamma(\alpha-1)\, t^{-\alpha}\, \Phi_\alpha(t) \qquad (\alpha \geqslant 0) ,$$

$$\Phi_\alpha(t) = \frac{\mathrm{d}}{\mathrm{d}t} \Phi_{\alpha+1}(t) \qquad (-1 < \alpha \leqslant 0) ,$$

$$\varphi_{-1}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \{t\varphi(t)\} ,$$

$$g(n, t) = (2/\pi) \sum_{k=1}^{n} p_{n-k}\, \lambda_k\, k^{-\alpha} \sin kt ,$$

$$J(n, u) = \frac{1}{\Gamma(1-\alpha)} \int_u^\pi (t-u)^{-\alpha} \left\{\frac{\mathrm{d}}{\mathrm{d}t} g(n, t)\right\} \mathrm{d}t \qquad (0 < \alpha < 1,\ u \geqslant 0) ,$$

$$V(n, u) = \int_0^u v^\alpha \left\{\frac{\mathrm{d}}{\mathrm{d}v} J(n, v)\right\} \mathrm{d}v ,$$

$$P_n = \sum_{\nu=0}^{\nu} p_\nu , \qquad \tau = [1/t] ,$$

$$Q_n \quad = \sum_{\nu=0}^{n} q_\nu = \sum_{\nu=1}^{n} \lambda_\nu \nu^{-\alpha},$$

$$\mu_n \quad = (1/p_n) \sum_{k=1}^{n} p_{n-k} \lambda_k ,$$

$$m \quad = [n/2] , \qquad \text{where } [x] = (\text{the integral part of } x) ,$$

$$\Delta\lambda_n \quad = \lambda_n - \lambda_{n+1} .$$

2.1. – BOSANQUET [**2**] established:

Theorem A. [**2**] If $\varphi_\alpha(t)$ is of bounded variation in $(0, \pi)$, then the FOURIER series of $f(t)$ is summable $|C, \beta|$ at the point $t = x$, where $\beta > \alpha \geqslant 0$.

Extending the above theorem, PRASAD and BHATT [**7**] proved:

Theorem B [**7**]. If $\{\lambda_n\}$ is a convex sequence such that $\sum \lambda_n/n < \infty$, and for $0 < \alpha \leqslant 1$,

$$\int_0^t u\,|\mathrm{d}\varphi_\alpha(u)| = O(t) \qquad (0 \leqslant t \leqslant \pi) ,$$

then the series $\sum \lambda_n A_n(t)/\log(n+1)$, at $t = x$ is summable $|C, \alpha|$.

BHATT [**1**] further extended BOSANQUET's result for $|H, 1|$ summability.

Theorem C [**1**]. If $\{\lambda_n\}$ be a convex sequence such that $\sum \lambda_n/n < \infty$. If $\varphi_\alpha(t) \in B.V.$ in $(0, \pi)$, then the factored FOURIER series $\sum A_n(t) n^{-\alpha} \log(n+1) \lambda_n$ is summable $|H, 1|$.

Recently NANDKISHORE [**6**] has extended this further for absolute NÖRLUND summability.

The aim of this paper is to prove:

Theorem. *If*

$$(2.1.1) \qquad \int_0^t u\,|\mathrm{d}\varphi_\alpha(u)| = O(t) \qquad (0 \leqslant t \leqslant \pi) ,$$

then the factored Fourier series $\sum A_n(t) n^{-\alpha} \lambda_n$ is $|N, p_n|$ summable, at $t = x$, provided $\{\lambda_n\}$ is a convex sequence such that $\sum \lambda_n/n < \infty$, and $\{p_n\}$ is a monotonic non-increasing sequence of non-negative numbers, such that

(i) *the sequence $\{(n+1) p_n/P_n\}$ is of bounded variation, and*

(ii) *the sequence $\{Q_n n^{-\alpha}\}$ is monotonic.*

Our Theorem generalises the theorem of NANDKISHORE. This also extends Theorem B for absolute NÖRLUND summability.

3.1. – We require the following Lemmas for proof of our Theorem.

Lemma 1. *If $\{\lambda_n\}$ be a convex sequence such that $\sum \lambda_n/n < \infty$. If $\sum a_n$ is bounded, then a necessary and sufficient condition for $|N, p_n|$ summability of the series $\sum_{n=1}^{\infty} \lambda_n n^{-\alpha} a^n$ is that the series*

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \sum_{k=1}^{n} \frac{p_{n-k} k \lambda_k a_k}{k^{\alpha}} \right| < \infty .$$

Proof. Following BHATT [1] we have,

$$t_n - t_{n-1} = \frac{1}{(n+1)p_{n-1}} \sum_{k=0}^{n-1} p_k(n-k)\, a_{n-k}\, q_{n-k} +$$

$$+ \frac{1}{(n+1)P_{n-1}} \sum_{k=0}^{n=1} p_k\left(\frac{1}{P_k} - \frac{1}{P_n}\right) a_{n-k}\, q_{n-k} .$$

Hence to prove the Lemma it is sufficient to show that

$$\sum \frac{1}{(n+1)P_{n-1}} |M_n| < \infty , \tag{3.1.1}$$

where

$$M_n = (1/P_n) \sum_{k=1}^{n} (P_n - P_{n-k})\, a_k\, q_k$$

$$= (1/P_n)\left\{ \sum_{k=1}^{m} + \sum_{k=m+1}^{n} \right\} (P_n - P_{n-k})\, a_k\, q_k$$

$$= M_{n1} + M_{n2} , \qquad \text{say} .$$

By ABEL's transformation, we have

$$M_{n1} = (1/P_n) \sum_{k=1}^{m-1} \Delta(P_n - P_{n-k}) \sum_{\mu=1}^{k} a_\mu\, q_\mu + \frac{P_n - P_{n-m}}{P_n} \sum_{\mu=1}^{m} a_\mu\, q_\mu = O\left(\frac{1}{P_n}\right)$$

hence

$$M_{n1} = O(1) , \tag{3.1.2}$$

since,

$$P_n - P_{n-m} = O(1)$$

and

$$\sum_{\mu=1}^{k} a_\mu q_\mu = \sum_{\mu=1}^{k} a_\mu \lambda_\mu / \mu^\alpha = O(1) .$$

Similarly,

$$M_{n2} = O(1) . \tag{3.1.3}$$

Lemma 2 [3]. (i) *If* $\beta > \alpha > -1$, *and* $\Phi_{\alpha+1}(t)$ *is of bounded variation in* $(0, \eta)$, (ii) $\Phi_{\alpha+1}(+0) = 0$, *then* $\Phi_\beta(t)$ *exists for almost all* t *in* $(0, \eta)$ *and satisfies the relation*

$$\Phi_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^t (t-u)^{\beta-\alpha-1} \, \mathrm{d}\Phi_{\alpha+1}(u) .$$

Lemma 3 [4]. *If* $\{p_n\}$ *is non-negative, non-increasing sequence, then for* $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant t$, *and for any* n

$$\left| \sum_{k=a}^{b} p_k \exp\left(i(n-k)t\right) \right| \leqslant CP\tau .$$

Lemma 4. *Let* $\{p_n\}$ *be monotonic non-increasing sequence of real non-negative numbers and* $\{\lambda_n\}$ *be a convex sequence such that* $\sum \lambda_n n^{-1} < \infty$, *then*

$$|g(n,t)| = \begin{cases} O(P_n \mu_n) \\ O(p_m Q_\tau) + O(\lambda_m P_m P_\tau m^{-\alpha}) , \end{cases} \tag{3.1.4}$$

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} g(n,t) \right| = \begin{cases} O(nP_n \mu_n) \\ O(m\, p_m Q_\tau) + O(m \lambda_m P_m P_\tau m^{-\alpha}) . \end{cases} \tag{3.1.5}$$

Proof. We have

$$|g(n,t)| = \left| \frac{2}{\pi} \sum_{k=1}^{n} p_{n-k} \lambda_k k^{-\alpha} \sin kt \right| \leqslant C \left| \frac{2}{\pi} \sum_{k=1}^{n} p_{n-k} \lambda_k \right|$$

$$= O(P_n M_n) .$$

Also

$$|g(n,t)| \leqslant |2/\pi[\sum_{k=1}^{m} + \sum_{k=m+1}^{n}]p_{n-k}\lambda_k\, k^{-\alpha}\sin kt| = g_1 + g_2\,, \qquad \text{say}\,.$$

$$|g_1| = 2/\pi[\sum_{k=1}^{m-1}\Delta p_{n-k}\sum_{\mu=1}^{k}\lambda_\mu\,\mu^{-\alpha}\sin\mu t + p_{n-m}\sum_{\mu=1}^{m}\lambda_\mu\,\mu^{-\alpha}\sin\mu t]$$

$$\leqslant C\,Q_\tau[\sum_{k=1}^{m-1}|\Delta p_{n-k}| + p_m] = O(p_m\,Q_\tau)\,,$$

and clearly

$$g_2 = O(\lambda_m\,P_m\,p_\tau\,P^{-\alpha})\,.$$

This completes the proof of (3.1.4). Similarly we can obtain (3.1.5).

Lemma 5. *Under the conditions of Lemma* 4

$$|J(n,u)| = \begin{cases} O(n^\alpha P_n\mu_n) & \text{for all } u\,, \\ O(n^\alpha p_m Q[1/u]) + O(\lambda_m P_m p[1/u]) & \text{for } u \geqslant n^{-1}. \end{cases}$$

Proof. We shall prove the second inequality; the first can be similarly obtained.

$$J(n,u) = \int_u^{u+n^{-1}} \dots + \int_{u+n^{-1}}^{\pi} \dots = J_1 + J_2\,, \qquad \text{say}\,.$$

Now for $u \geqslant 1/n$

$$|J_1| \leqslant \frac{1}{\Gamma(1-\alpha)}\int_u^{u+n^{-1}}(t-u)^{-\alpha}\{O(mp_m Q_\tau) + O(n\lambda_m P_m P_\tau m^{-\alpha})\}\,\mathrm{d}t$$

$$= O\{n^\alpha p_m Q[1/u]\} + O\{\lambda_m P_m P_{[1/u]}\}$$

and using the second mean-value theorem,

$$|J_2| \leqslant C\,n^\alpha\int_{u+n^{-1}}^{u}\frac{\mathrm{d}}{\mathrm{d}t}\,g(n,t)\,\mathrm{d}t$$

$$= C\,n^\alpha\{|g(n,u)|\} = O(n^\alpha p_m Q[1/u]) + O(\lambda_m P_m P_{[1/u]})\,.$$

3.2. - Proof of the Theorem.

We consider the case $0<\alpha<1$. We first observe that $\Phi_\alpha(t)$ is of bounded variation in $(0,\pi)$, and $\Phi_\alpha(+0)=0$. For, if $0<t<\pi$, we have by (2.1.1.), writing $\Phi^*(t)=\int\limits_0^t u\,|\mathrm{d}\varphi_\alpha(u)|$,

$$\int\limits_t^\pi |\mathrm{d}\varphi_\alpha(u)| = \int\limits_t^\pi 1/u\,|u\ \mathrm{d}\varphi_\alpha(u)| = [1/u\ \Phi^*(u)]_t^\pi + \int\limits_t^\pi \Phi^*(u)\,\frac{\mathrm{d}u}{u^2}$$

$$= O(1) + O(\log \pi/t)\,.$$

It follows that

$$\varphi_\alpha(t) = O\big(\log(2\pi/t)\big)$$

and hence $\Phi_\alpha(+0)=0$. Hence also, for $0<t<\pi$,

$$\Gamma(\alpha+1)\int\limits_t^\pi |\mathrm{d}\Phi_\alpha(u)| = \int\limits_t^\pi |\mathrm{d}\{u^\alpha\varphi_\alpha(u)\}|$$

$$\leqslant \int\limits_t^\pi u^\alpha|\mathrm{d}\varphi_\alpha(u)| + \alpha\int\limits_t^\pi u^{\alpha-1}|\varphi_\alpha(u)|\,\mathrm{d}u$$

$$\leqslant \int\limits_t^\pi u^\alpha|\mathrm{d}\varphi_\alpha(u)| + \alpha\int\limits_t^\pi u^{\alpha-1}|\varphi_\alpha(u)|\,\mathrm{d}u$$

$$= O(1) + O(t^\alpha \log 2\pi/t) = O(1)\,.$$

since

$$nA_n = 2/\pi\int\limits_0^\pi \varphi(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\sin nt)\,\mathrm{d}t\,.$$

We now have, by Lemma 2, with O in place of β and α in place of $\alpha+1$, and proceeding as BOSANQUET [2] or more particularly as BHATT [1], we have

$$\sigma_n = \sum_{k=1}^n p_{n-k}\lambda_k\, k^{-\alpha}\, 2/\pi\int\limits_0^\pi \varphi(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\sin kt)\,\mathrm{d}t$$

$$= O\left(\int\limits_0^\pi \frac{\mathrm{d}}{\mathrm{d}t}\,g(n,t)\,\mathrm{d}t\int\limits_0^t (t-u)^{-\alpha}\,\mathrm{d}\Phi_\alpha(u)\right)$$

$$= O\left\{\int_0^\pi \mathrm{d}\Phi_\alpha(u) \int_u^\pi (t-u)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} g(n, t)\,\mathrm{d}t\right\}$$

$$= O\left\{[\Phi_\alpha(u)\, J(n, u)]_0^\pi + \int_0^\pi \Phi_\alpha(u) \frac{\mathrm{d}}{\mathrm{d}u} J(n, u)\,\mathrm{d}u\right\}$$

$$= O\left\{\Phi_\alpha(\pi)\, J(n, \pi) + \int_0^\pi u^\alpha \varphi_\alpha(u) \frac{\mathrm{d}}{\mathrm{d}u} J(n, u)\,\mathrm{d}u\right\}$$

$$= O\,\{\Phi_\alpha(\pi)\; J(n, \pi)\} + O\left\{\left[\varphi_\alpha(u) \int_0^u v^\alpha \frac{\mathrm{d}}{\mathrm{d}v} J(n,\; v)\,\mathrm{d}v\right]_0^\pi + \right.$$

$$\left. + \int_0^\pi \mathrm{d}\varphi_\alpha(u) \int_0^u v^\alpha \frac{\mathrm{d}}{\mathrm{d}v} \cdot J(n, v)\,\mathrm{d}v\right\}$$

$$= O\,\{\Phi_\alpha(\pi)\, J(n, \pi) - \varphi_\alpha(\pi)\, V(n, \pi) + \int_0^\pi V(n, u)\,\mathrm{d}\varphi_\alpha(u)\,\mathrm{d}u\}$$

$$= O(n^\alpha p_m) + O(P_m \lambda_m) - \varphi_\alpha(\pi)\, V(n, \pi) + \int_0^\pi V(n, u)\,\mathrm{d}\varphi_\alpha(u)\,\mathrm{d}u\}\,.$$

If in particular we suppose that $\varphi(t) = 1$, for all t, in which case $\varphi_\alpha(t) = 1$ for all t and $\sigma_n = 0$ for every n, we get

$$V(n, \pi) = O(n^\alpha p_m) + O(P_m \lambda_m)\,.$$

Thus

$$\sigma_n = O(n^\alpha p_m) + O(P_m \lambda_m) + \int_0^\pi V(n, u)\,\mathrm{d}\varphi_\alpha(u)\,.$$

We can now prove

$$(3.2.1) \qquad |V(n, u)| = \begin{cases} O(n^\alpha u^\alpha P_n \mu_n) & \text{for all } u\,, \\ O(n^\alpha p_m) + O(P_m \lambda_m) + O(n^\alpha u^\alpha p_m Q[1/u]) + & \\ \qquad + O(u^\alpha \lambda_m P_m P_{[1/u]}) & \text{for } u \geqslant n^{-1}. \end{cases}$$

For that, we have

$$V(n, u) = 1/\Gamma(1+\alpha) \int_0^u v^\alpha \frac{\mathrm{d}}{\mathrm{d}v} J(n, v)\,\mathrm{d}v = O(n^\alpha u^\alpha p_n \mu_n)\,,$$

from the first inequality of Lemma 5.

Again, for $u \geqslant n^{-1}$

$$V(n, \pi) - V(n, u) = \left[\frac{u^\alpha J(n, u)}{\Gamma(1+\alpha)}\right]_u^\pi - \frac{1}{\Gamma(\alpha)} \int_u^\pi v^{\alpha-1} J(n, v)\,\mathrm{d}v\,.$$

Thus,

$$V(n, u) = V(n, \pi) - \left[\frac{u^\alpha J(n, u)}{\Gamma(1+\alpha)}\right]_u^\pi + \frac{1}{\Gamma(\alpha)} \int_u^\pi v^{\alpha-1} J(n, v)\,\mathrm{d}v$$

$$= O(n^\alpha p_m) + O(P_m \lambda_m) + O\big(n^\alpha u^\alpha p_m Q[1/u]\big) + O\big(u^\alpha \lambda_m P_m P_{[1/u]}\big) + |I|\,,$$

where

$$|I| = \left|\frac{1}{\Gamma(\alpha)} \int_u^\pi v^{\alpha-1} \left\{\frac{1}{\Gamma(1+\alpha)} \int_v^\pi (t-v)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} g(n, t)\,\mathrm{d}t\right\} \mathrm{d}v\right.$$

$$= \mathrm{C}\left|\int_u^\pi \frac{\mathrm{d}}{\mathrm{d}t} g(n, t)\,\mathrm{d}t \int_{v/t}^1 (1-v)^{-\alpha} v^{\alpha-1}\,\mathrm{d}v\right| = O\Big(\int_u^\pi \frac{\mathrm{d}}{\mathrm{d}t} g(n, t)\,\mathrm{d}t\Big)$$

$$= O\big(p_m Q[1/u]\big) + O\big(\lambda_m P_m P_{[1/u]} m^{-\alpha}\big) \qquad \text{by Lemma 4}\,,$$

$$= O[n^\alpha u^\alpha p_m Q[1/u]] + O[u^\alpha P_m \lambda_m P_{[1/u]}]\,.$$

Consequently, for $0 < \alpha < 1$,

$$\sum_{n=1}^\infty \frac{|\sigma_n|}{(n+1)p_{n-1}} \leqslant \left[\sum_{n=1}^\infty \frac{1}{(n+1)P_{n-1}} \left\{O(n^\alpha p_m) + O(P_m \lambda_m) + \left|\int_0^\pi V(n, u)\,\mathrm{d}\varphi_\alpha(u)\right|\right\}\right]$$

hence

$$\sum_{n=1}^\infty \frac{|\sigma_n|}{(n+1)P_{n-1}} = O(1) + \sum_{n=1}^\infty \frac{1}{(n+1)P_{n-1}} \left|\int_0^\pi V(n, u)\,\mathrm{d}\varphi_\alpha(u)\right|. \tag{3.2.2}$$

Now

$$\sum_{n=1}^\infty \frac{1}{(n+1)P_{n-1}} \left|\left[\int_0^{1/n} + \int_{1/n}^\pi\right] V(n, u)\,\mathrm{d}\varphi_\alpha(u)\right| = K_1 + K_2\,, \qquad \text{say.} \tag{3.2.3}$$

$$|K_1| \leqslant \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \int_0^{1/n} O(n^{\alpha} u^{\alpha-1} P_n \mu_n) \, |u \; \mathrm{d}\varphi_{\alpha}(u)|$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left(\sum_{k=1}^{m} + \sum_{k=m+1}^{n} \right) p_{n-k} \lambda_k$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \{p_m \sum_{k=1}^{m} \lambda_k + \lambda_m \sum_{k=m+1}^{n} p_{n-k}\} = \sum_{k=1}^{\infty} \lambda_k \sum_{n=k}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{\lambda_m}{m}$$

hence

$$|K_1| = O(1) \tag{3.2.4}$$

and

$$|K_2| \leqslant \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left[\int_{1/n}^{\pi} \{O(n^{\alpha} p_m u^{-1}) + O(u^{-1} P_m \lambda_m\} \right] +$$

$$+ O\big(n^{\alpha} u^{\alpha-1} p_m Q[1/u]\big) + O\big(u^{\alpha-1} \lambda_m P_m P_{[1/u]}\big) \, |u \, \mathrm{d}\varphi_{\alpha}(u)|$$

hence

$$|K_2| = K_{2,1} + K_{2,2} + K_{2,3} + K_{2,4}, \qquad \text{say.} \tag{3.2.5}$$

$$\begin{cases} K_{2,1} = O(1) \\ K_{2,2} = O(1). \end{cases} \tag{3.2.6}$$

And since $u^{\alpha} Q[1/u]$ is monotonic, we have

$$|K_{2,3}| \leqslant \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left[\int_{1/n}^{\pi} O\big(n^{\alpha} u^{\alpha-1} p_m Q[1/u]\big) \, |u \, \mathrm{d}\varphi_{\alpha}(u)| \right.$$

$$= \sum_{n=1}^{\infty} \frac{n^{\alpha} p_m}{(n+1)P_{n-1}} \left\{ \frac{Q_n}{n^{\alpha-1}} \int_{1/n}^{\xi} |u \, \mathrm{d}\varphi_{\alpha}(u)| + \pi^{\alpha} Q[1/u] \int_{\xi}^{\pi} u \, |\mathrm{d}\varphi_{\alpha}(u)| \right\} \qquad (1/n \leqslant \xi \leqslant \pi)$$

$$= O\left[\sum_{\nu=1}^{\infty} \frac{\lambda_{\nu}}{\nu^{\alpha}} \sum_{n=\nu}^{\infty} \frac{n \, p_m}{(n+1) \, P_{n-1}} \right] + O(1) = O\left(\sum_{\nu=1}^{\infty} \frac{\lambda_{\nu}}{\nu} \right) + O(1)$$

hence

$$|K_{2,3}| = O(1) \tag{3.2.7}$$

since $u^{\alpha} P_{[1/u]}$ is non-increasing, we have

$$|K_{2,4}| = \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} O(P_m \lambda_m) \int_{1/n}^{\xi'} |u \, d\varphi_\alpha(u)| = O\left(\sum_{n=1}^{\infty} \frac{\lambda_m}{m}\right) \qquad (1/n < \xi' \leqslant \pi),$$

$$|K_{2,4}| = O(1) . \tag{3.28}$$

Collecting (3.2.4), (3.2.6), (3.2.7), and (3.2.8) the Proof of the Theorem is complete.

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References.

[1] S. N. BHATT, *On the absolute summability of factored Fourier series*, Ann. Mat. Pura Appl. **72** (1966), 253-265.

[2] L. S. BOSANQUET, *The absolute Cesàro summability a Fourier series*, Proc. London Math. Soc. (2) **41** (1936), 517-528.

[3] L. S. BOSANQUET, *Some arithmetic means connected with Fourier series*, Trans. Amer. Math. Soc. **39** (1936), 189-204.

[4] L. MCFADDEN, *Absolute Nörlund summability*, Duke Math. J. **9** (1942), 168-207.

[5] F. M. MEARS, *Some multiplication theorems for the Nörlund mean*, Bull. Amer. Math. Soc. **41** (1935), 875-880.

[6] N. KISHORE, *Absolute Nörlund summability of a factored Fourier series*, Indian J. Math. **9** (1967), 123-136.

[7] B. N. PRASAD and S. N. BHATT, *The summability factors of a Fourier series*, Duke Math. J. **24** (1957), 103-117.

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