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# Completion of a Lebesgue Integral. (**)

This paper is a continuation of [3]. In [3] we defined the LEBESGUE integral axiomatically as follows:

Definition. A LEBESGUE integral is a real valued functional $\int$ with domain $D(\int)$ of real valued functions on a set $X$ and satisfying the following conditions:

(1) $D(\int)$ is a real linear lattice and $\int$ a linear functional on $D(\int)$.

(2) $D(\int)$ satisfies the STONE condition; that is if $f \in D(\int)$ then $f \cap 1 \in D(\int)$.

(3) If $f$ is a non-negative function in $D(\int)$, then $\int f \geqslant 0$.

(4) $\int$ is a countably additive functional on $D(\int)$; that is if $f_n \in D(\int)$ is an increasing sequence convergent at every point of the set $X$ to a finite valued function $f$ and the sequence $\int f_n$ is bounded, then $f \in D(\int)$ and $\int f_n$ converges to $\int f$.

The LEBESGUE integral is complete if $D(\int)$ contains every non-negative function $g$ on $X$ to which there corresponds an $f \in D(\int)$ such that $0 \leqslant g(x) \leqslant f(x)$ for all $x \in X$ and $\int f = 0$.

In this paper we extend the given LEBESGUE integral to its completion, which is the smallest complete integral containing it, and then find a representation of the completion by means of the volume generated by the original integral.

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## § 1. - Definition of the completion.

We recall that for a given LEBESGUE integral $\int$ on $D(\int)$ over $X$, a set $A \subset X$ is a summable set generated by $\int$ ($\int$-summable set) if its characteristic function $c_A \in D(\int)$, and $A$ is a null set generated by $\int$ ($\int$-null set) if $c_A \in D(\int)$ and $\int c_A = 0$. The functional $v(A) = \int c(A)$, where $A$ is a $\int$-summable set, is an upper complete volume on the ring of summable sets.

Proposition 1. *Let $\int$ be a Lebesgue integral and $f \in D(\int)$. If $A$ is a null set generated by $\int$, then $c_A f \in D(\int)$ and $\int c_A f = 0$.*

Proof. It is sufficient to prove the proposition for the case when $f$ is non-negative. By Theorem 2 of [**3**], for any positive number $a$ the set

$$\{x \in X\colon\ f(x) > a\}$$

is in $V$, where $V$ is the family of summable sets generated by $\int$. Since $A$ is a member of the ring $V$ the set $A_a \in V$, where

$$A_a = \{x \in X\colon\ (c_A f)(x) > a\} = A \cap \{x \in X\colon\ f(x) > a\}\,.$$

For the positive integers $n$ and $j$, let

$$B_{nj} = \{x \in X\colon\ 2^{-n} j < (c_A f)(x) \leqslant 2^{-n}(j+1)\} = A_{2^{-n}j} \setminus A_{2^{-n}(j+1)}\,.$$

Then $B_{nj} \in V$ and since $v(B_{nj}) \leqslant v(A) = 0$, we get $v(B_{nj}) = 0$.

Consider now the sequence of functions

$$s_n = \sum_j 2^{-n} j\, c_{B_{nj}} \qquad (j = 1, \ldots, 4^{-n})\,.$$

The function $s_n$ is a simple function and if $v$ is the volume generated by $\int$,

$$\int s_n \, dv = \sum_{j=1}^{4^n} 2^{-n} j\, v(B_{nj}) = 0\,.$$

This implies that $s_n \in D(\int)$ and $\int s_n = 0$.

Since $s_n(x)$ converges increasingly to $c_A f(x)$ for all $x \in X$, it follows from the countable additivity of the integral that $c_A f \in D(\int)$ and $\int c_A f = 0$. This proves the proposition.

Proposition 2. *Let $\int$ be a Lebesgue integral and $f \in D(\int)$. If $f(x) = 0$ $\int$-almost everywhere, then $\int f = 0$.*

Proof. The proof follows from Proposition 1; indeed $f = c_A f$, where $A$ is the $\int$-null set outside of which $f(x) = 0$.

We shall now define the completion. Let $\int$ be a LEBESGUE integral with domain $D(\int)$ over a set $X$. Denote by $D(\int_c)$ the set of real valued functions on $X$ such that for every $f \in D(\int_c)$ there exists a $g \in D(\int)$ with $f(x) = g(x)$ $\int$-a.e., that is $f(x) = g(x)$ for $x \notin A$, where $A$ is a $\int$-null set, and define the functional $\int_c$ by $\int_c f = \int g$.

The fact that $\int_c$ is a well defined functional on $D(\int_c)$ follows from Proposition 2; for if $f(x) = g(x)$ $\int$-a.e. and $f(x) = h(x)$ $\int$-a.e., then $g(x) - h(x) = 0$ $\int$-a.e. and therefore $\int g = \int h$. The functional $\int_c$ will be called the completion of the integral $\int$. We shall prove that $\int_c$ is a complete integral.

Since the family of null sets generated by the integral $\int$ is closed with respect to countable unions, it is easy to see that $D(\int_c)$ is a linear lattice satisfying the STONE condition and $\int_c$ a linear functional on it. The following proposition shows that $\int_c$ is a positive functional.

Propostion 3. *Let $f \in D(\int_c)$ be such that $f(x) \geqslant 0$ for all $x \in X$. Then $\int_c f$ is non-negative.*

Proof. Let $g \in D(\int)$ be such that $f(x) = g(x)$ for $x \notin A$, where $A$ is a $\int$-null set. Then

$$g = c_{X \setminus A} g + c_A g \, .$$

By Proposition 1, $c_A g \in D(\int)$ and $\int c_A g = 0$. Therefore $c_{X \setminus A} \in D(\int)$ and $\int g = \int c_{X \setminus A} g \geqslant 0$.

We shall now proceed to establish the countable additivity of the functional $\int_c$.

For the given integral $\int$, the family of summable sets, the family of null sets and the volume generated by the integral will be denoted respectively by $V$, $V_0$ and $v$ in the remainder of this section.

Lemma 1. *Let $\int$ a Lebesgue integral over the space $X$. Let $g_n$ be a sequence of non-negative functions in $D(\int)$ such that $\int g_n < 4^{-n}$ for all $n$. Then there exist a set $A \in V_0$ and a function $g \in D(\int)$ such that*

$$g(x) = \sum_{n=1}^{\infty} g_n(x) \qquad \textit{for all } x \in X \setminus A$$

*and* $\int g = \sum_{n=1}^{\infty} \int g_n$.

Proof. Consider the sequence of sets

$$A_n = \{x \in X: \quad g_n(x) > 2^{-n}\}.$$

By Theorem 2 of [3] the set $A_n \in V$. Since $g_n \geqslant 2^{-n} c_{A_n}$, we have

$$2^{-n} v(A_n) \leqslant \int g_n < 4^{-n}$$

which gives $v(A_n) < 2^{-n}$.

Now

$$\sum_{j=n+1}^{m} v(A_j) < 2^{-n} \qquad \text{for all } m > n$$

and therefore, by the upper completeness of $v$, the set

$$B_n = \bigcup_{n+1}^{\infty} A_j \in V \qquad \text{and} \qquad v(B_n) < 2^{-n}.$$

From the upper completeness of $v$ on the ring $V$, we also get that the set $A = \bigcap_{n=1}^{\infty} B_n \in V$ and $v(A) \leqslant v(B_n) < 2^{-n}$ for all $n$. This implies that $v(A) = 0$, that is $A \in V_0$.

For any $x \notin A$ there exists a positive integer $n$ such that $0 \leqslant g_j(x) \leqslant 2^{-j}$ for all $j > n$ and therefore the series $\sum_{j=1}^{\infty} g_j(x)$ is convergent.

Now define a function $g$ by the formula $g(x) = \sum_{j=1}^{\infty} g_j(x)$ if $x \notin A$ and $g(x) = 0$ if $x \in A$. Consider $h_j = (1 - c_A) g_j$. By Proposition 1, $h_j \in D(\int)$ and $\int h_j = \int g_j$. Moreover, $g(x) = \sum_{j=1}^{\infty} h_j(x)$ for all $x \in X$.

The functions $s_n = \sum_{j=1}^{n} h_j \in D(\int)$ converge increasingly to $g$ and $\int s_n \leqslant \sum_{j=1}^{\infty} \int g_j < 1$. This implies by the countable additivity of the integral that $g \in D(\int)$ and

$$\int g = \sum_{j=1}^{\infty} \int h_j = \sum_{j=1}^{\infty} \int g_j.$$

The lemma is therefore proved.

Lemma 2. *Let $\int$ be a Lebesgue integral over $X$. If $f_n \in D(\int)$ is a sequence of functions monotone with respect to the relation less or equal $\int$-almost everywhere, such that the sequence $\int f_n$ is bounded, then there exists a function $f \in D(\int)$ such that $f_n(x)$ converges to $f(x)$ $\int$-almost everywhere and $\int f_n$ converges to $\int f$.*

Proof. We may assume that the functions are increasing $\int$-a.e. . Then there exist sets $A_n \in V_0$ such that $f_n(x) \leqslant f_{n+1}(x)$, if $x \notin A_n$.

If we let $B = \bigcup_{n=1}^{\infty} A_n$, then $B \in V_0$ and by Proposition 1 the function $h_n = (1 - c_B) f_n \in D(\int)$ and $\int h_n = \int f_n$. Moreover, $h_n(x) \leqslant h_{n+1}(x)$ for all $x \in X$ and all $n$.

The sequence $\int h_n$ being increasing and bounded is a CAUCHY sequence and therefore for every $n$ there exists $k_n$ such that

$$\int h_{k_{n+1}} - \int h_{k_n} < 4^{-n} .$$

By Lemma 1, there exist a set $C \in V_0$ and a function $h \in D(\int)$ such that

$$\sum_{j=1}^{\infty} (h_{k_{j+1}} - h_{k_j}) = h(x) \qquad \text{if } x \notin C$$

and

$$\sum_{j=1}^{\infty} \left(\int h_{k_{j+1}} - \int h_{k_j}\right) = \int h .$$

Let $f = h_{k_1} + h$. Then $\int h_{k_n}$ converges to $\int f$ and $h_{k_n}(x)$ converges to $f(x)$ for $x \notin C$. Since $\int f_{k_n} = \int h_{k_n}$ is an increasing sequence we see that $\int f_n$ converges to $\int f$, and since $h_n(x)$ is increasing at all $x \in X$ we see that $f_n(x)$ converges to $f(x)$ for any $x$ not in the $\int$-null set $B \cup C$.

Proposition 4. *Let $\int$ be a Lebesgue integral and $\int_c$ its completion. Let $f_n \in D(\int_c)$ be a sequence of functions converging increasingly at every point of the space $X$ to a finite valued function $f$, and let the sequence $\int_c f_n$ be bounded. Then $f \in D(\int_c)$ and $\int_c f_n$ converges to $\int_c f$.*

Proof. There exist functions $g_n \in D(\int)$ and sets $A_n \in V_0$ such that $f_n(x) = g_n(x)$ for $x \notin A_n$ and $\int_c f_n = \int g_n$. Therefore, by Lemma 2, there exists $g \in D(\int)$ such that $g_n(x)$ converges to $g(x)$ for $x \notin B \in V_0$ and $\int g_n$ converges to $\int g$.

The set $A = \bigcup_{n+1}^{\infty} A_n \in V_0$ and $f_n(x)$ converges to $g(x)$ for $x \notin A \cup B$. This implies that $f(x) = g(x)$ for $x \notin A \cup B \in V_0$. Therefore $f \in D(\int_c)$ and $\int_c f = \int g$.

Lemma 3. *Let $\int$ be a Lebesgue integral and $f \in D(\int)$ such that $\int |f| = 0$. Then $f(x) = 0$ $\int$-a.e..*

Proof. Let $A = \{x \in X\colon\ f(x) \neq 0\}$. Then $A = \bigcup_{n=1}^{\infty} A_n$, where

$$A_n = \{x \in X\colon\ |f(x)| > 1/n\}\,.$$

By Theorem 2 of [**3**], $c_{A_n} \in D(\int)$. Also, since $c_{A_n} \leqslant n|f|$ we get $v(A_n) = \int c_{A_n} \leqslant n\int|f| = 0$. This implies that $A_n \in V_0$ and since $V_0$ is closed under countable unions, $A \in V_0$. This proves the lemma.

Proposition 5. *Let $g$ be a non-negative function on $X$ such that there exists an $f \in D(\int_c)$ with $g(x) \leqslant f(x)$ for all $x \in X$ and $\int_c f = 0$. Then $g \in D\,(\int_c)$.*

Proof. Let $h \in D(\int)$ such that $f(x) = h(x)$ for $x \notin A$, where $A \in V_0$, and $\int_c f = \int h$. Then $h = |h| + (h - h)c_A$ and therefore, by Proposition 1, $\int|h| = \int h = 0$. Lemma 3 implies that $h(x) = 0$ if $x \notin B$, where $B \in V_0$.

Now, $g(x) \leqslant h(x) = 0$ if $x \notin A \cup B$. Since $A \cup B \in V_0$, we see that $g(x) = 0$ $\int$-a.e. . The conclusion now follows from the fact that $0 \in D(\int)$.

Theorem 1. *Let $\int$ be a Lebesgue integral on $D(\int)$ and $\int_c$ its completion on $D(\int_c)$. Then $\int_c$ is the smallest complete Lebesgue integral extending $\int$. In particular, if $\int$ is a complete Lebesgue integral, then its completion coincides with it.*

Proof. The fact that the completion is a complete LEBESGUE integral follows from the preceding propositions and discussions.

Since $0 \in D(\int)$, the empty set belongs to $V_0$ and therefore $D(\int) \subset D(\int_c)$ and $\int f = \int_c f$ for $f \in D(\int)$, that is $\int_c$ is an extension of $\int$.

Suppose, now that $J$ is a complete LEBESGUE integral such that $D(\int) \subset D(J)$ and $\int f = Jf$ for $f \in D(\int)$. We shall show that $D(\int_c) \subset D(J)$ and $\int_c f = Jf$ for $f \in D(\int_c)$.

Let $f \in D(\int_c)$. Then there exists $g \in D(\int)$ such that $f(x) = g(x)$ for $x \notin A$, where $A$ is a $\int$-null set, and $\int_c f = \int g$. Since $c_A \in D(\int) \subset D(\int)$, $Jc_A = \int c_A = 0$ and therefore $A$ is a $J$-null set. By Theorem 3 of [**3**], $f - g \in D(J)$ and $J(f-g) = 0$ which implies that $f \in D(J)$ and $Jf = Jg = \int_c f$.

## § 2. - Representation of the completion by means of volume.

Let $(X, V, v)$ be a volume space and $S(V, R)$ the family of real valued simple functions over $V$. (See [**1**].) Let $S_1(V, R)$ denote the set of all functions $f$ for which there exists a sequence $f_n \in S(V, R)$ such that $f_n(x)$ converges to $f(x)$ for all $x \in X$. The set $S_2(V, R)$ will consist of limits of sequences from the set $S_1(V, R)$ and $S_3(V, R)$ will consist of limits of sequences from $S_2(V, R)$. Denote by $L(v, R)$ the space of real valued LEBESGUE summable functions generated by volume $v$ and by $M(v, R)$ the space of real valued measurable functions generated by $v$. (See [**2**].)

For a measure $\mu$, denote by $M(\mu, R)$ the space of real valued measurable functions generated by $\mu$.

Lemma 4. *Let $(X, V, v)$ be a volume space and $f \in L(v, R)$. Let $\mu$ be a measure with domain $M$ extending $v$, that is $V \subset M$ and $\mu(A) = v(A)$ for $A \in V$. Then for any $f \in L(v, R)$ there exists a function $g \in M(\mu, R)$ such that $f(x) = g(x)$ $v$-a.e.* .

Proof. By Theorem 2 of [**2**], $L(v, R) \subset M(v, R)$ and, by Theorem 7 of the same paper, there exists a function $g \in S_3(V, R)$ such that $f(x) = g(x)$ $v$-a.e.. Since $M(\mu, R)$ is closed under convergence everywhere, $g \in M(\mu, R)$.

Lemma 5. *Let $\int$ be a Lebesgue integral and $\int_c$ its completion. Then a condition $C(x)$ holds $\int$-a.e. if and only if $C(x)$ holds $\int_c$-a.e.* .

Proof. If $C(x)$ holds $\int$-a.e., then obviously $C(x)$ holds $\int_c$-a.e.. Now, assume that $C(x)$ is true for all $x \notin A \subset X$, where $A$ is a $\int_c$-null set. We shall show that there exists a set $B \subset X$ such that $A$ is a $\int$-null set and $A \subset B$.

Since $c_A \in D(\int_c)$, there exists a function $g \in D(\int)$ such that $c_A(x) = g(x)$ for $x \notin E$, where $E$ is a $\int$-null set. This implies that $g(x) \geqslant 0$ for $x \notin E$ and, by Proposition 1, $\int |g| = \int g = 0$. Lemma 3 implies that $g(x) = 0$ for $x \notin F$, where $F$ is a $\int$-null set. Let $B = E \cup F$. Then $B$ is a $\int$-null set and we see that $c_A(x) = 0$ for $x \notin B$, that is $A \subset B$.

Theorem 2. *Let $\int$ be a Lebesgue integral with domain $D(\int)$ and $\int_c$ its completion with domain $D(\int_c)$. Let $v$ be the volume generated by $\int$ on the ring of summable sets. Then $D(\int_c) = L(v, R)$ and $\int_c f = \int f \, dv$ for $f \in D(\int_c)$.*

Proof. It was established in [**3**] that

$$D(\textstyle\int) = L(v, R) \cap M(\mu, R)$$

and $\int f = \int f dv$ for $f \in D(\int)$, where $\mu$ is the measure with smallest domain extending $v$.

Now, let $f \in D(\int_c)$. Then there exists a function $g \in D(\int) \subset L(v, R)$ and a $\int$-null set $A$ such that $f(x) = g(x)$ if $x \notin A$ and $\int_c f = \int g$. Since $v(A) = 0$,

$A$ is a $v$-null set and therefore $f(x) = g(x)$ v-a.e. . Since $g \in L(v, R)$, we get from Theorem 1, part. 4 of [**1**] that $f \in L(v, R)$ and $\int f dv = \int g dv = \int g = \int_c f$.

Now to prove that $L(v, R) \subset D(\int_c)$, take any $f \in L(v, R)$. By Lemma 4, there exists a function $g \in M(\mu, R)$ such that $f(x) = g(x)$ $v$–a.e. . Again, by Theorem 1, part 4, of [**1**], $g \in L(v, R)$ and therefore $g \in D(\int)$. By Lemma 2 of [**3**], $f(x) = g(x)$ $\int_c$-a.e. which implies by Lemma 5 that $f \in D(\int_c)$.

## References.

[1] W. M. BOGDANOWICZ, *A generalization of the Lebesgue-Bochner-Stieltjes integral and a new approach to the theory of integration*, Proc. Nat. Acad. Sci. U.S. **53** (1965), 492-498.

[2] W. M. BOGDANOWICZ, *An approach to the theory of Lebesgue-Bochner measurable functions and to the theory of measure*, Math. Ann. **164** (1966), 251-269.

[3] M. M. MATTAMAL, *Axiomatic characterization of the Lebesgue integral*, Riv. Mat. Univ. Parma (2) **12** (1971), 29-40.

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