Riv. Mat. Univ. Parma (3) 1 (1972), 113-117

P. K. KAMTHAN and P. K. JAIN (*)

Remarks on the Geometric Means of Entire functions of Several Complex Variables. (**)

1. - Introduction.

Let $f(z_1, z_2)$ be an entire function of two complex variables (we restrict ourselves for two variables case for the sake of simplicity, otherwise our results are also valid for any finite number n of variables). The order ϱ of $f(z_1, z_2)$ is defined by

$$\varrho = \lim_{r_1, r_2 \to \infty} \sup \frac{\log \log M(r_1, r_2; f)}{\log (r_1 r_2)} \qquad 0 \leqslant \varrho \leqslant \infty ,$$

where

$$M(r_1, r_2; f) = \max_{|z_1| \leqslant r_1,\ |z_2| \leqslant r_2} |f(z_1, z_2)| .$$

Let us define the following geometric means of f:

$$G(r_1, r_2; f) = \exp \left\{ \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |f(r_1 \exp (i\theta_1), r_2 \exp (i\theta_2))| \, \mathrm{d}\theta_1 \mathrm{d}\theta_2 \right\} ;$$

$$g_{\lambda,\mu}(r_1, r_2; f) = \exp \left\{ \frac{(\mu + 1)(\lambda + 1)}{r_1^{\lambda+1} \ r_2^{\mu+1}} \int_0^{r_1} \int_0^{r_2} x_1^{\lambda} x_2^{\mu} \log G(x_1, x_2; f) \, \mathrm{d}x_1 \mathrm{d}x_2 \right\} .$$

(*) Indirizzo degli Autori: P. K. KAMTHAN, Department of Mathematics, I.I.T., Kanpur, India; P. K. JAIN, Department of Mathematics, Hans Raj College, Delhi University, Delhi-7, India.

(**) Ricevuto: 6-IX-1971.

When there is no confusion likely to arise, we will merely designate $G(r_1, r_2; f)$ and $g_{\lambda,\mu}(r_1, r_2; f)$ by $G(r_1, r_2)$ and $g_{\lambda,\mu}(r_1, r_2)$ respectively.

Recently AGARWAL has obtained a couple of results, expressing these mean values in terms of the order of the function f (see for instance [1], [2]). Unfortunately such of his results are in correct. For instance his proof of theorem 1 in ([2], p. 653) is incorrect, see: [5] consider for example $f(z_1, z_2) = \exp(z_1 z_2)$. Even the statement of this theorem is vague, in fact he mentions that this theorem is true for non-integral order functions but he nowhere makes use of this fact in the proof of this result. In this paper we bring forward a similar mistake which AGARWAL has committed in proving theorem 1 ([1], p. 177). We illustrate this claim of ours by an example. Moreover, we also give an improvement of AGARWAL's theorem 1 [1].

2. - Counter example for a result of Agarwal.

Before we mention this example, let us mention what AGARWAL has proved. In fact, his theorem 2 [1] runs as follows:

Theorem A. *If $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ are two entire functions of orders ϱ' and ϱ'' respectively, than*

$$\alpha \equiv \limsup_{r_1, r_2 \to \infty} \frac{\log\log G(r_1, r_2, f_1 f_2)}{\log(r_1 r_2)} = \limsup_{r_1, r_2 \to \infty} \frac{\log\log g_{\lambda,\lambda}(r_1, r_2, f_1 f_2)}{\log(r_1 r_2)}$$

$$\equiv \beta = \max(\varrho', \varrho'')$$

That $\alpha = \beta$ is obviously true. However, that $\alpha(\text{or } \beta) = \max(\varrho', \varrho'')$ is incorrect. Consider, for instance

$$f_1(z_1, z_2) = \exp(z_1 z_2)\ ; \qquad f_2(z_1, z_2) = \exp(z_1 + z_2)\ .$$

Than

$$\varrho' = 1 = \varrho''\ .$$

But

$$|f_1(r_1 \exp(i\theta_1), r_2 \exp(i\theta_2))\, f_2(r_1 \exp(i\theta_1), r_2 \exp(i\theta_2))| =$$

$$= \exp(r_1 r_2 \cos(\theta_1 + \theta_2) + r_1 \cos\theta_1 + r_1 \cos\theta_2)\ .$$

Therefore

$$\log G(r_1, r_2; f_1 f_2) = 0\ .$$

Hence $\alpha = \beta = -\infty$ which is certainly not equal to $1 = \max(\varrho', \varrho'')$.

3. – We now prove the following

Theorem. *For a class of entire function $f(z_1, z_2)$ for which*

$$\lim_{(r_1, r_2)\to\infty} \frac{\log\log g_{\lambda,\mu}(r_1, r_2)}{\log r_j} = \infty \qquad (j = 1, 2), \tag{3.1}$$

we have

$$\lim_{(r_1, r_2\to\infty)} \sup \frac{\log\log\log g_{\lambda,\mu}(r_1, r_2)}{\log (r_1 r_2)} = \log H_{\lambda,\mu}, \tag{3.2}$$

where

$$\lim_{(r_1, r_2)\to\infty} \begin{matrix}\text{sup.}\\ \text{inf.}\end{matrix} \left\{\frac{\log G(r_1, r_2)}{\log g_{\lambda,\mu}(r_1, r_2)}\right\}^{\frac{1}{\log (r_1 r_2)}} = \begin{matrix} H_{\lambda,\mu} \\ h_{\lambda,\mu}\end{matrix} \tag{3.3}$$

provided $h^2_{\lambda,\mu} < H_{\lambda,\mu}$.

Proof. We have

$$\frac{\partial^2}{\partial r_1 \partial r_2} \{\log [r_1^{\lambda+1} r_2^{\mu+1} \log g_{\lambda,\mu}(r_1, r_2)]\} =$$

$$= \frac{\partial}{\partial r_1}\left\{\frac{(\lambda+1)(\mu+1) r_2^{\mu} \int_0^{r_1} x_1^{\lambda} \log G(x_1, x_2)\, dx_1}{r_1^{\lambda+1} r_2^{\mu+1} \log g_{\lambda,\mu}(r_1, r_2)}\right\} =$$

$$= \{r_1^{\lambda+1} r_2^{\mu+1} \log g_{\lambda,\mu}(r_1, r_2)\}^{-2} [\{r_1^{\lambda+1} r_2^{\mu+1} \log g_{\mu,\lambda}(r_1, r_2)\} \cdot \tag{3.4}$$

$$\cdot \{(\lambda+1)(\mu+1) r_1^{\lambda} r_2^{\mu} \log G(r_1, r_2)\} -$$

$$-(\lambda+1)^2 (\mu+1)^2 \left(r_2^{\mu} \int_0^{r_1} x_1^{\lambda} \log G(x_1, r_2)\, dx_1\right) \left(r_1^{\lambda} \int_0^{r_2} x_2^{\mu} \log G(r_1, x_2)\, dx_2\right)]$$

$$\leqslant \frac{(\lambda+1)(\mu+1) \log G(r_1, r_2)}{r_1 r_2 \log g_{\lambda,\mu}(r_1, r_2)}. \tag{3.5}$$

Now, integrating both the sides of the inequality in (3,5), we get

$$\begin{cases} \log \{r_1^{\lambda+1} r_2^{\mu+1} \log g_{\lambda,\mu} g(r_1, r_2)\} \leqslant \\ \leqslant (\lambda+1)(\mu+1) \displaystyle\int_0^{r_1}\int_0^{r_2} \frac{\log G(r_1, r_2)}{\log g_{\lambda,\mu}(r_1, r_2)} \frac{dx_1}{x_1} \frac{dx_2}{x_2}. \end{cases} \tag{3.6}$$

Let $H_{\lambda,\mu} < \infty$. Then, for $\varepsilon > 0$ (3.6) gives

$$\log \{r_1^{\lambda+1} r_2^{\mu+1} \log g_{\lambda,\mu}(r_1, r_2)\} \leqslant$$

$$\leqslant (\lambda+1)(\mu+1) \int_{a_1}^{r_1}\int_{a_2}^{r_2} (H_{\lambda,\mu}+\varepsilon)^{\log (x_1, x_2)} (\mathrm{d}x_1/x_1)(\mathrm{d}x_2/x_2) <$$

$$< (\lambda+1)(\mu+1) \frac{(H_{\lambda,\mu}+\varepsilon)^{\log (r_1 r_2)}}{[\log (H_{\lambda,\mu}+\varepsilon)]^2} + O(1),$$

which implies

$$\lim_{r_1, r_2 \to \infty} \sup \frac{\log\log\log g_{\lambda,\mu}(r_1, r_2)}{\log (r_1 r_2)} \leqslant \log H_{\lambda,\mu},$$

in view (3.1).

Next, integrating (3.4), we get

$$(3.7) \quad \begin{cases} \log \{(2r_1)^{\lambda+1}(2r_2)^{\mu+1} \log g_{\lambda,\mu}(2r_1, 2r_2) > \\ > \displaystyle\int_{r_1}^{2r_1}\int_{r_2}^{2r_2} (\lambda+1)(\mu+1) \frac{\log G(x_1, x_2)}{\log g_{\lambda,\mu}(x_1, x_2)} \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2} - \\ - \displaystyle\int_{r_1}^{2r_1}\int_{r_2}^{2r_2} (\lambda+1)(\mu+1) \left\{\frac{\log G(x_1, x_2)}{\log g_{\lambda,\mu}(r_1, r_2)}\right\}^2 \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2} < \\ > (\lambda+1)(\mu+1)(\log 2)^2 \left\{\dfrac{\log G(r_1, r_2)}{\log g_{\lambda,\mu}(r_1, r_2)} - \left(\dfrac{\log G(2r_1, 2r_2)}{\log g_{\lambda,\mu}(2r_1, 2r_2)}\right)^2\right\}, \end{cases}$$

since $\{\log G(r_1, r_2)/\log g_{\lambda,\mu}(r_1, r_2)\}$ increases when of the variables (say) r_1 is fixed and the another variable r_2 increases, viceversa or both increase (see [4] p. 254).

Hence, using (3.3) in (3.7), we obtain for certain sequence of r_1 and r_2 (say) $\{r_{1,i}\}$ and $\{r_{2,j}\}$ $(r_{1,i} \to \infty$ with i and $r_{2,j} \to \infty$ with $j)$ the result

$$\log \{(2r_1)^{\lambda+1}(2r_2)^{\mu+1} \log g_{\lambda,\mu}(r_1, r_2)\} >$$

$$> (\lambda+1)(\mu+1)(\log g)^2 \{(H_{\lambda,\mu}-\varepsilon)^{\log (r_1 r_2)} - (h_{\lambda,\mu}+\varepsilon)^{\log (r_1 r_2)}\} >$$

$$> O(1)(H_{\lambda,\mu}-\varepsilon)^{\log (r_1 r_2)}, \; (h^2_{\lambda,\mu} < H_{\lambda,\mu}),$$

and the result follows.

Remark. If $\lambda = \mu$, this result reduces to a part of theorem 1 of AGARWAL [**1**]. Apart from this, we have also weakened the hypothesis of this theorem of AGARWAL referred to just now. Further, AGARWAL claims to have proved the second part of theorem 1 [**1**], namely the equality between the inferior limits, but gives nowhere any outline of the proof of this fact. We are of the view that to justify this claim is an extremely hard job. However, probably AGARWAL's claim is based on the ideas of a similar result proved earlier by one of us [**3**] for an entire function of one complex variable where such a claim is obviously justifiable (see also [**4**]).

References.

[**1**] A. K. AGARWAL, *On the geometric means of entire functions of several complex variables,* Riv. Mat. Unit. Parma (2) **9** (1968), 173-179.

[**2**] A. K. AGARWAL, *On the geometric means of entire functions of several complex variables,* Trans. Amer. Math. Soc. **151** (1970), 651-657.

[**3**] P. K. KAMTHAN, *On the mean values of an entire function,* Math. Student **32** (1964), 101-109.

[**4**] P. K. KAMTHAN and P. K. JAIN, *The geometric means of entire function,* Ann. Polon. Math. **21** (1968-69), 247-255.

[**5**] P. K. KAMTHAN, *A note on the geometric means of an entire functions of several complex variables,* Trans. Amer. Math. Soc. 169 (1972), 503-508.

* * *