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A Theorem on the Means of an Entire Function. (**)

1. - Introduction.

Let $f(z)$ be an entire function of order ϱ and lower order λ. Let us define the following means of $f(z)$:

$$I_\delta(r) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(r\exp(i\theta))|\,\mathrm{d}\theta\right\}^{1/\delta}, \qquad 0 < \delta < \infty ,$$

and

$$m_{\delta,h}(r) = \frac{1}{r^{h+1}}\int_0^r I_\delta(x)\,x^h\,\mathrm{d}x , \qquad -1 < h < \infty .$$

It is known that [1], [3]:

$$\overline{\underline{\lim}}_{r\to\infty} \left\{\frac{I_\delta(r)}{m_{\delta,h}(r)}\right\}^{1/\log r} = \frac{e^\varrho}{e^\lambda} \qquad (0 \leqslant \lambda \leqslant \varrho \leqslant \infty) .$$

My purpose in this note is to prove a result which gives a refinement of the above result when $f(z)$ is of non-zero finite order.

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2. – Theorem. *Let $f(z)$ be of order ϱ $(0<\varrho<\infty)$ and a proximate order* [1] *$\varrho(r)$. Let τ and t be the proximate type and the lower proximate type of $f(z)$ with respect to $\varrho(r)$ defined as:*

$$\genfrac{}{}{0pt}{}{\overline{\lim}}{\underline{\lim}}_{r\to\infty}\frac{\log M(r)}{r^{\varrho(r)}}=\genfrac{}{}{0pt}{}{\tau}{t}\qquad (0<t\leqslant\tau<\infty), \tag{2.1}$$

where $M(r)=\max\limits_{|z|=r}|f(z)|$. Then

$$\text{(i)}\qquad \varrho\tau\leqslant\overline{\lim_{r\to\infty}}\frac{I_\delta(r)}{m_{\delta,h}(r)\,r^{\varrho(r)}}\leqslant e\varrho\tau\,,$$

$$\text{(ii)}\qquad \underline{\lim}_{r\to\infty}\frac{I_\delta(r)}{m_{\delta,h}(r)\,r^{\varrho(r)}}\leqslant\varrho t\,.$$

Proof. It is seen from the definitions of $I_\delta(r)$ and $m_{\delta,h}(r)$, for $r\geqslant r_0$, that

$$\log m_{\delta,h}(r)=\log m_{\delta,h}(r_0)+\int_{r_0}^{r}\frac{\nu(x)}{x}\,\mathrm{d}x\,, \tag{2.2}$$

where

$$\nu(x)=\left\{\frac{I_\delta(x)}{m_{\delta,r}(x)}-(h+1)\right\}, \tag{2.3}$$

increases as x increases, since « $r^{h+1}I_\delta(x)$ is a convex function of $r^{h+1}m_{\delta,h}(x)$ » (see [**1**] lemma 3).

Also, from (2.1) and the fact that

$$\log M(r)\sim\log m_{\delta,h}(r)\,,$$

as $r\to\infty$, (see [**4**], theorem 2), we have

$$\overline{\lim_{r\to\infty}}\frac{\log m_{\delta,h}(r)}{r^{\varrho(r)}}=\tau\,.$$

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(1) For various conditions and properties of $\varrho(r)$ (see [**2**], p. 32).

Let

$$\overline{\lim_{r\to\infty}} \frac{\nu(r)}{r^{\varrho(r)}} = \gamma . \tag{2.4}$$

Then, from (2.2) and (2.4), we get

$$\log m_{\delta,h}(r) < \log m_{\delta,h}(r_0) + (\gamma + \varepsilon) \int_{r_0}^{r} x^{\varrho(x)-1}\, \mathrm{d}x$$

$$\sim \frac{\gamma + \varepsilon}{\varrho} r^{\varrho(r)} (1 + 0(1)) ,$$

for $r \geqslant r_0 = r_0(\varepsilon)$, $\varepsilon > 0$. This implies

$$\overline{\lim_{r\to\infty}} \frac{\nu(r)}{r^{\varrho(r)}} \geqslant \varrho\tau . \tag{2.5}$$

Further, for $\eta > 1$

$$\nu(r) \log \eta < \int_{r}^{\eta r} \frac{\nu(x)}{x}\, \mathrm{d}x < \log m_{\delta,h}(\eta r) ,$$

which gives

$$\overline{\lim_{r\to\infty}} \frac{\nu(r)}{r^{\varrho(r)}} \leqslant \frac{\eta^{\varrho}}{\log \eta} \tau .$$

Putting $\eta = e^{1/\varrho}$, which is the best value of η here, in the above inequality one gets

$$\overline{\lim_{r\to\infty}} \frac{\nu(r)}{r^{\varrho(r)}} \leqslant e\varrho\tau . \tag{2.6}$$

Hence (i) of the theorem follows from (2.5) and (2.6) through (2.3). Similary, (ii) of the theorem is proved.

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References.

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