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Linear Operators Defined with Poisson Type Distribution. (**)

1. - Introduction.

Recently CHENEY and SHARMA [2] have considered a generalization of BERNSTEIN polynomials based upon the generalized binomial theorem due to JENSEN [3]. CIMOCA and LUPAS [1] have generalized the MEYER-KÖNIG and ZELLER operator [5] by using a generalized BERNSTEIN power series given by POLYA [8]. The purpose of this paper is to consider, in the same way a generalization of the SZASZ-MIRAKYAN operator [6] by using another interesting result of JENSEN [3]. The SZASZ-MIRAKYAN operator is defined on $[a, b]$, as

$$(B_n f)(x) = \sum_{k=0}^{\infty} m_{nk}(x)\, f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$m_{nk}(x) = \exp(-nx).\,(nx)^k/k! \qquad (k = 0, 1, 2, \ldots), \tag{1.2}$$

is a POISSON distribution.

JENSEN's result is given by

$$\exp(\alpha z)/(1-\beta z) = \sum_{k=0}^{\infty} (\alpha + k\beta)^k\, \big(z\exp(-\beta z)\big)^k/k!, \qquad |\beta z| < 1. \tag{1.3}$$

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Starting with the LAGRANGE's formula

$$\frac{\varphi(z)}{1-\{zf'(z)/f(z)\}}=\sum_{k=0}^{\infty}\frac{1}{k!}\left\{\frac{\mathrm{d}^k}{\mathrm{d}z^k}[\{f(z)\}^k\varphi(z)]\right\}_{z=0}\left[\frac{z}{f(z)}\right]^k$$

and setting

$$\varphi(z)=\exp(\alpha z) \qquad \text{and} \qquad f(z)=\exp(\beta z)$$

the expression (1.3) can easily be derived.

We substitute $z=1$ and $\alpha=nx$ in (1.3) to obtain an extension of (1.2) as

$$(1.4) \qquad W_k(nx,\beta)=(1-\beta)(nx+k\beta)^k\,\{\exp(-(nx+k\beta))\}/k! \qquad (k=0,1,2,\ldots),$$

such that

$$\sum_{k=0}^{\infty}W_k(nx,\beta)=1\,.$$

We may now define a new operator with the help of a POISSON type distribution (1.4), consider its convergence properties and give its degree of approximation. It will be seen in the sequal that our operator has approximation properties similar to those of BERNSTEIN polynomials. Further the results for SZASZ-MIRAKYAN operator can easily be obtained from our operator as a particular case when $\beta=0$.

2. - The operators and their convergence.

In analogy with (1.1), the generalized SZASZ-MIRAKYAN operator may be defined for $0\leqslant\beta<1$ as

$$(2.1) \qquad (P_n f)(x)=(1-\beta)\sum_{k=0}^{\infty}(nx+\beta k)^k\,\exp(-(nx+\beta k))\cdot f(k/n)/k!\,.$$

The parameter β may depend only on the natural number n.

It is clear that the SZASZ-MIRAKYAN operator is a special of our operator (2.1) and is obtained when $\beta=0$.

The convergence property of the operator $(P_n f)(x)$ may be proved in the following

Theorem (2.1). *If $0\leqslant\beta=\beta(n)\to 0$ as $n\to\infty$, then $P_n f\to f$ (uniformly) for all $f\in C[0,\lambda]$.*

The proof can be made to depend on the following lemma.

Lemma. *Let*

$$S(r, nx, \beta) = \sum_{k=0}^{\infty} (nx + k\beta)^{k+r-1} \{\exp(-(nx + k\beta))\}/k! \tag{2.2}$$

such that

$$S(1, nx, \beta) = 1/(1-\beta) . \tag{2.3}$$

Then

$$S(r, nx, \beta) = \sum_{k=0}^{\infty} \beta^k (nx + \beta k) S(r-1, nx + \beta k, \beta) . \tag{2.4}$$

Proof. It can be seen from (2.2) that the functions $S(r, nx, \beta)$ satisfy the recurrence relation

$$S(r, nx, \beta) = nx S(r-1, nx, \beta) + \beta S(r, nx + \beta, \beta) . \tag{2.5}$$

A repeated use of (2.5) proves the Lemma.

Particular values. By applying (2.3) and (2.4) it further follows that

$$S(2, nx, \beta) = \sum_{k=0}^{\infty} \beta^k \frac{nx + \beta k}{1-\beta} = \frac{nx}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^3} , \tag{2.6}$$

$$\begin{cases} S(3, nx, \beta) = \sum_{k=0}^{\infty} \beta^k (nx + \beta k) \left[\frac{nx + \beta k}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^3} \right] = \\ \quad = \frac{n^2 x^2}{(1-\beta)^3} + \frac{3nx\beta^2}{(1-\beta)^4} + \frac{\beta^3}{(1-\beta)^4} + \frac{3\beta^4}{(1-\beta)^5} . \end{cases} \tag{2.7}$$

These formulae will be used in proving the Theorem (2.1).

Since $(P_n f)(x)$ is +ve linear operator for $1 > \beta \geqslant 0$, it is sufficient, by KOROVKIN's theorem [4], to verify the uniform convergence for test functions $f(t) = 1$, t and t^2. It is obvious from (1.4) that

$$P_n 1 = 1 . \tag{2.8}$$

Now putting $f(t) = t$ in (2.1) and using (2.2) and (2.6) it can be easily shown that

$$(2.9)\qquad \begin{cases} (P_n t)(x) = (1-\beta)\displaystyle\sum_{k=0}^{\infty} \frac{(nx+k\beta)^k \exp(-(nx+k\beta))}{k!}\,\frac{k}{n} \\ \quad = \dfrac{(1-\beta)\,S(2,\,nx+\beta,\,\beta)}{n} = \dfrac{x}{1-\beta} + \dfrac{\beta}{n(1-\beta)^2}, \end{cases}$$

which tends uniformly to x as $\beta = o(1)$.

Proceeding to the function $f(t) = t^2$ and applying (2.2), we have

$$(P_n t^2)(x) = (1-\beta)\sum_{k=0}^{\infty} \frac{(nx+\beta k)^k \exp(-(nx+\beta k))}{k!}\,\frac{k^2}{n^2} =$$

$$= (1-\beta)[S(3, nx+2\beta, \beta) + S(2, nx+\beta, \beta)]/n^2,$$

and a use of (2.6) and (2.7) yields

$$(2.10)\qquad (P_n t^2)(x) = \frac{x^2}{(1-\beta)^2} + \frac{x(1+2\beta)}{n(1-\beta)^3} + \frac{\beta(1+2\beta)}{n^2(1-\beta)^4}.$$

Thus if $\beta = o(1)$ the expression (2.10) tends uniformly to x^2.

Hence, by KOROVKIN's theorem, the proof of the Theorem (2.1) is complete.

3. - Order of approximation.

Theorem (3.1). *If $f \in C[0, \lambda]$ and $0 \leqslant \beta < 1$, then*

$$|f(x) - (P_n f)(x)| \leqslant \frac{3}{2}\, w(\sqrt{N_\beta}),$$

where

$$N_\beta = 4\left[\frac{\lambda\beta^2}{(1-\beta)^2} + \frac{1+2\beta^2}{n(1-\beta)^3} + \frac{\beta(1+2\beta)}{\lambda n^2(1-\beta)^4}\right]$$

and $w(\delta) = \sup |f(x'') - f(x')|$; $x', x'' \in [0, \lambda]$; being a +ve number such that $|x'' \cdot x| < \delta$.

Proof. By using the properties of modulus of continuity

$$(3.1)\qquad |f(x'') - f(x')| \leqslant w(|x'' - x'|);$$

$$(3.2)\qquad w(\gamma\delta) \leqslant (\gamma+1)\, w(\delta), \qquad \gamma > 0$$

and noting the fact that

$$\sum_{k=0}^{\infty} w_k(nx, \beta) = 1 \quad \text{and} \quad w_k(nx, \beta) \geq 0, \qquad \forall\, n, k$$

it can be seen, by the application of CAUCHY's inequality, that

$$(3.3)\quad \begin{cases} |f(x)-(P_n f)(x)| \leq \left\{1+\dfrac{1}{\delta}\sum\limits_{k=0}^{\infty}(1-\beta)\dfrac{(nx+\beta k)^k}{k!}\exp(-(nx+\beta k))\left|x-\dfrac{k}{n}\right|\right\} w(\delta) \\ \leq \left\{1+\dfrac{1}{\delta}\left[(1-\beta)\sum\limits_{k=0}^{\infty}\dfrac{(nx+\beta k)^k}{k!}\exp(-(nx+\beta k))\cdot\left(x-\dfrac{k}{n}\right)^2\right]^{1/2}\right\} w(\delta). \end{cases}$$

Now by linearity of the operator and by using (2.8), (2.9) and (2.10) we get

$$\sum_{k=0}^{\infty}(1-\beta)\frac{(nx+\beta k)^k}{k!}\exp(-(nx+\beta k))\cdot\left(x-\frac{k}{n}\right)^2 = x^2 P_n 1 - 2x(P_n t)(x) + (P_n t^2)(x)$$

$$= \frac{x^2\beta^2}{(1-\beta)^2}+\frac{x(1+2\beta^2)}{n(1-\beta)^3}+\frac{\beta(1+2\beta)}{n^2(1-\beta)^4} \leq \lambda\left[\frac{\lambda\beta^2}{(1-\beta)^2}+\frac{1+2\beta^2}{n(1-\beta)^3}+\frac{\beta}{\lambda n^2}\frac{\beta}{(1-\beta)^4}\right] = \frac{1}{4}N_\beta .$$

Hence (3.3) can be written as

$$(3.4)\qquad |f(x)-(P_n f)(x)| \leq \left[1+\frac{1}{2\delta}N_\beta^{1/2}\right] w(\delta) .$$

Choosing $\delta = N_\beta^{1/2}$, Theorem (3.1) is proved.

For $\beta = 0$, the expression (3.4) gets reduced to an inequality for SZASZ-MIRAKYAN operator obtained earlier by MÜLLER [7].

Theorem (3.2). *If $f \in C'[0, \lambda]$, then the following inequality holds:*

$$|f(x)-(P_n f)(x)| \leq \frac{3}{4}N_\beta^{1/2}\, w_1(N_\beta^{1/2}) ,$$

where $w_1(\delta)$ is the modulus of continuity of f'.

Proof. For definiteness we prove the theorem for $f'(x) \geq 0$ but it also applies to $f'(x) < 0$. By the mean value theorem of differential calculus, it is known that

$$f(x) - f(k/n) = \left(x - \frac{k}{n}\right) f'(\xi) ,$$

where $\xi = \xi_{n,k}(x)$ is an interior point of the interval determined by x and k/n.

Now

$$f(x)-f(k/n)\leqslant\left(\frac{x}{1-\beta}+\frac{\beta}{n(1-\beta)^2}-\frac{k}{n}\right)f'(x)+\left(x-\frac{k}{n}\right)[f'(\xi)-f'(x)]\,.$$

Multiplying both sides of the inequality by $w_k(nx,\beta)$, summing over k and using (2.9) we get

$$(3.6)\quad |f(x)-(P_nf)(x)|\leqslant\sum_{k=0}^{\infty}\left|x-\frac{k}{n}\right|\frac{(1-\beta)(nx+\beta k)^k}{k!}\exp\left(-(nx+\beta k)\right)\cdot|f'(\xi)-f(x)|\,.$$

Now by (3.1) and (3.2)

$$|f'(\xi)-f'(x)|\leqslant w_1(|\xi-x|)\leqslant\left(1+\frac{1}{\delta}|\xi-x|\right)w_1(\delta)\leqslant\left(1+\frac{1}{\delta}\left|\frac{k}{n}-x\right|\right)w_1(\delta)\,;$$

where δ is a +ve number not depending on k.

A use of this in (3.6) gives

$$|f(x)-(P_nf)(x)|\leqslant\left\{\sum_{k=0}^{\infty}\left|x-\frac{k}{n}\right|\frac{(1-\beta)(nx+\beta k)^k}{k!}\exp(-(nx+\beta k))+\right.$$
$$\left.+\frac{1}{\delta}\sum_{k=0}^{\infty}\left(x-\frac{k}{n}\right)^2\frac{(1-\beta)(nx+\beta k)^k}{k!}\exp(-(nx+\beta k))\right\}w_1(\delta)\,.$$

Hence by CAUCHY's inequality and by (3.4)

$$(3.7)\qquad |f(x)-(P_nf)(x)|\leqslant\tfrac{1}{2}N_\beta^{1/2}\left(1+\frac{1}{2\delta}N_\beta^{1/2}\right)w_1(\delta)\,.$$

Choosing $\delta=N_\beta^{1/2}$, Theorem (3.2) is proved.

We may put $\beta=0$, $\delta=1/\sqrt{n}$ in (3.7) to get the expression for SZASZ-MIRAKYAN operator.

Theorem (3.3). *If $f(x)$ is bounded and possesses a second derivative at a point x, and if $\beta n^2\to C$, then*

$$(3.8)\qquad n[(P_nf)(x)-f(x)]\to\tfrac{1}{2}f''(x)[x+x^2C]\,.$$

In order to prove (3.8) *we write*

$$(3.9)\qquad f\left(\frac{k}{n}\right)-f(x)=\left(\frac{k}{n}-x\right)f'(x)+\left(\frac{k}{n}-x\right)^2\left[\frac{1}{2}f''(x)+\theta\left(\frac{k}{n}-x\right)\right],$$

where $\theta(h)$ is bounded for all h and converges to zero with h.

Multiplying (3.9) by $w_k(nx, \beta)$ and summing, we get

$$n(P_n f)(x) - f(x)] =$$

$$= nf'(x)\,[(P_n t)(x) - x] + \frac{n}{2} f''(x)\,[(P_n t^2)(x) - 2x(P_n t)(x) + x^2] +$$

$$+ n(1-\beta) \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 \theta \left(\frac{k}{n} - x\right) \frac{(nx + \beta k)^k \exp(-(nx + \beta k))}{k!}.$$

From earlier results, we know that $n[(P_n t)(x) - x] \to 0$. If $\beta n^2 \to C$ we can show by using (2.9) and (2.10) that

$$n[(P_n t^2)(x) - 2x(P_n t)(x) + x^2] \to x^2 C + x.$$

Thus the Theorem is proved.

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