Riv. Mat. Univ. Parma (2) **11** (1970), 251-270

V. P. MADAN (*)

Dynamical Analysis of Viscoelastic Beams. (**)

Introduction.

Free and forced vibrations of elastic beams have been investigated extensively [1], [2], [3]. Several papers have also been published on the vibration of viscoelastic beams [4], [5], [6]. However the present work is confined to the study of viscoelastic beams subjected to impulsive excitations. In contrast with the preceding works, this investigation has been done using a correspondence principle [8]. It is known that in the transform plane, the equations for the elastic and viscoelastic dynamical systems are same except that the elastic constants in one system are replaced by the corresponding viscoelastic transform moduli in the other system. The solution for a viscoelastic beam may thus be obtained from that of the elastic beam by taking the LAPLACE transform of the elastic solution, replacing elastic constants by corresponding viscoelastic moduli in transform parameter and finally by inverting the transform. It appears at the outset that one can obtain solutions to many viscoelastic problems by merely applying this principle. This, however, is not true; because the solutions of not many dynamical problems in elasticity are known up-to-date and secondly the inversion of resulting transforms also becomes extremely difficult for even the simple type of viscoelastic materials. The present investigation has offered us an exact solution to an identical problem with the elastic BERNOULLI-EULER beam, but the inversion of the transformed solution for the general viscoelastic beam is however not simple, to yield an exact solution for the viscoelastic problem. On the other hand solutions to the viscoelastic beams offered by FAVRE [4], FLAHERTY [5], PAN [6] and others are of different nature. Their solutions are in the series form and do not enable one to perform quantitative analysis with

(*) Indirizzo: Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada.

(**) Ricevuto: 13-V-1969.

much ease. The present investigation has advantage over the others in so much as this work gives the asymptotic solutions to the otherwise unmanageable problem. Furthermore, the solutions have been presented here for unit impulse so that the response of beams can be determined for an arbitrary excitation by using DUHAMEL's integral.

We first consider the transverse motion of a viscoelastic beam of the BERNOULLI-EULER type: the motion is due to an impulse at $x = 0$, $t = 0$. As first step, we write the integral solution by applying the correspondence principle to the solution of identical elastic problem. Approximations to this solution are then obtained for small and large values of time using asymptotic methods. These solutions determine the instantaneous and the long term response of beam due to an impulse loading and are useful for the stability purposes. The effects of rotatory inertia and shear are studied by considering motions according to the RAYLEIGH and TIMOSHENKO beams. The longitudinal and torsional motions have also been considered. These motions are governed by the one-dimensional wave equation.

1. - Constitutive relationships and equations of motion.

For a linear isotropic viscoelastic material, the stress-strain relationship can be represented as follows:

$$\left\{\begin{aligned} P_s(D)s_{ij} &= Q_s(D)e_{ij} \\ P_v(D)\sigma_{ii} &= Q_v(D)\varepsilon_{ii}\,, \end{aligned}\right. \tag{1.1}$$

in which σ_{ii} is three times the average hydrostatic tension and ε_{ii} is the dilatation, s_{ij} and e_{ij} are the stress and strain deviators respectively. They are related to the stress and strain tensors in the following manner:

$$\left\{\begin{aligned} s_{ij} &= \sigma_{ij} - \tfrac{1}{3}\,\sigma_{kk}\,\delta_{ij} \\ e_{ij} &= \varepsilon_{ij} - \tfrac{1}{3}\,\varepsilon_{kk}\,\delta_{ij}\,. \end{aligned}\right. \tag{1.2}$$

The P_s, Q_s and P_v, Q_v are differential operators of the form $\sum_{n=0}^{m} a_n D^n$, where D is the time derivative $\partial/\partial t$. The coefficients a_n and the number m are in general different for each operator.

Comparing the expressions in equation (1.1) with the corresponding results for an elastic material, it may be concluded that the viscoelastic operators $\frac{Q_s(D)}{P_s(D)}$ and $\frac{Q_v(D)}{P_v(D)}$ correspond to 2μ and $3k$, where μ and k are the shear and bulk moduli respectively. Since most experimental data on actual viscoelastic materials provide information on the behaviour in shear or simple tension and as the dilatation test is difficult to perform; several assumptions are usually made regarding the dilatational behaviour of these materials, each one in certain circumstances proves to be a good approximation to the acutal behaviour. We assume incompressibility as a condition for dilatation, so that

$$\frac{Q_v(D)}{P_v(D)} = \infty .$$

We may now establish the following correspondence between the elastic constants and the viscoelastic operators:

Modulus of elasticity	$E \longleftrightarrow \frac{3}{2}\,\frac{Q_s(D)}{P_s(D)}$,
Shear modulus	$\mu \longleftrightarrow \frac{1}{2}\,\frac{Q_s(D)}{P_s(D)}$,
Bulk modulus	$k = \infty$,
POISSON's ratio	$\nu = \frac{1}{2}$.

The approximate equations for the dynamic flexure of elastic beams are [10]:

The BERNOULLI-EULER equation

$$y_{tt} + c_0^2\, k^2\, y_{xxxx} = F(x,\ t)\ , \tag{1.3}$$

where $c_0^2 = E/\varrho$, $k^2 = I/A$.

RAYLEIGH's equation

$$y_{tt} + c_0^2\, k^2\, y_{xxxx} - k^2\, y_{xxtt} = F(x,\ t) \tag{1.4}$$

and the TIMOSHENKO beam equation

$$y_{tt} + c_0^2\, k^2\, y_{xxxx} - k^2\,(1 + \varepsilon')\, y_{xxtt} + \frac{\varepsilon'\, k^2}{c_0^2}\, y_{tttt} = F(x,\ t)\ , \tag{1.5}$$

ε' is a non-dimensional constant equal to $2(1+\varepsilon)/R'$, R' being a constant depending upon the shape of the cross-section of the bar. The differential equations of motion for longitudinal vibrations of rods and for torsional vibrations of cylindrical bars are identical viz

$$\varphi_{tt} = c^2 \varphi_{xx} \tag{1.6}$$

for the case of longitudinal vibration of a bar φ represents the longitudinal displacement and $c = (E/\varrho)^{1/2}$, for torsional vibration of a cylindrical bar φ represents the angular rotation of a cross-section and $c = (\mu/\varrho)^{1/2}$. In both these cases c has the dimensions of a linear velocity (LT^{-1}).

2. - Solution for Bernoulli-Euler elastic beam.

The BERNOULLI-EULER equation is an approximation to flexural vibrations in beams. It is assumed that the length of the beam is large compared with the cross-sectional dimensions and that the vibration occurs in the principle plane of bending. The differential equation governing the transverse displacement $y(x, t)$ in this case is

$$\left(\frac{\partial^2}{\partial t^2} + \gamma^2 \frac{\partial^4}{\partial x^4}\right) y(x, t) = \partial(x)\, \delta(t)\,, \tag{2.1}$$

where $\gamma^2 = c_0^2 k^2$.

Denote the LAPLACE transform of $y(x, t)$ with respect to t by $\bar{y}(x, p)$:

$$\bar{y}(x, p) = \int_0^\infty y(x, t)\, e^{-pt}\, \mathrm{d}t\,.$$

Apply a LAPLACE transform to equation (2.1) to get

$$(p^2 + \gamma^2\, \partial^4/\partial x^4)\, \bar{y}(x, p) = \delta(x)\,. \tag{2.2}$$

Consider the homogeneous equation

$$(p^2 + \gamma^2\, \partial^4/\partial x^4)\, \bar{y}(x, p) = 0\,. \tag{2.3}$$

The solution of equation (2.3) is

$$(2.4)\quad \bar{y}(x,\ p) \equiv \begin{cases} A(x) = \alpha_1\, e^{\sqrt{p\gamma^{-1}}\cdot(1+i)x/\sqrt{2}} + \alpha_2\, e^{-\sqrt{p\gamma^{-1}}\cdot(1+i)x/\sqrt{2}} + \\ \quad + \alpha_3\, e^{\sqrt{p\gamma^{-1}}\cdot(-1+i)x/\sqrt{2}} + \alpha_4\, e^{-\sqrt{p\gamma^{-1}}\cdot(-1+i)x/\sqrt{2}} & \text{for } x > 0 \\ B(x) = \beta_1\, e^{\sqrt{p\gamma^{-1}}\cdot(1+i)x/\sqrt{2}} + \beta_2\, e^{-\sqrt{p\gamma^{-1}}\cdot(1+i)x/\sqrt{2}} + \\ \quad + \beta_3\, e^{\sqrt{p\gamma^{-1}}\cdot(-1+i)x/\sqrt{2}} + \beta_4\, e^{-\sqrt{p\gamma^{-1}}\cdot(-1+i)x/\sqrt{2}} & \text{for } x < 0\ , \end{cases}$$

where α_i, β_i, $(i = 1, 2, 3, 4)$ are determined so that (2.4) represents the solution of (2.2). It is sufficient to require that at $x = 0$ (cf. [11])

$$(2.5)\quad A(0) = B(0),\quad A'(0) = B'(0),\quad A''(0) = B''(0);\quad \gamma^2\,[A'''(0) - B'''(0)] = 1\ .$$

Thus, for $x > 0$,

$$\bar{y}(x,\ p) = \frac{1+i}{4\sqrt{2\gamma}}\,\frac{1}{p^{3/2}}\, e^{-\sqrt{p\gamma^{-1}}\cdot(1+i)x/\sqrt{2}} + \frac{1-i}{4\sqrt{2\gamma}}\,\frac{1}{p^{3/2}}\, e^{\sqrt{p\gamma^{-1}}\cdot(-1+i)x/\sqrt{2}},$$

and, for $x < 0$,

$$(2.6)\qquad \bar{y}(x,\ p) = \frac{1+i}{4\sqrt{2\gamma}}\,\frac{1}{p^{3/2}}\, e^{\sqrt{p\gamma^{-1}}\cdot(1+i)x/\sqrt{2}} + \frac{1-i}{4\sqrt{2\gamma}}\,\frac{1}{p^{3/2}}\, e^{-\sqrt{p\gamma^{-1}}\cdot(-1+i)x/\sqrt{2}}\ .$$

The solution for $x < 0$ can be obtained from the case $x > 0$ by reversing the sign of x, we may therefore study the solution for $x > 0$:

$$(2.7)\quad \begin{cases} y(x,\ t) = \dfrac{1+i}{4\sqrt{2\gamma}}\,\dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} e^{pt}\, p^{-3/2}\, e^{-\sqrt{p\gamma^{-1}}\cdot(1+i)x/\sqrt{2}}\, dp + \\ \qquad + \dfrac{1-i}{4\sqrt{2\gamma}}\,\dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} e^{pt}\, p^{-3/2}\, e^{\sqrt{p\gamma^{-1}}\cdot(-1+i)x/\sqrt{2}}\, dp, \qquad x > 0\ , \end{cases}$$

which by ≠ 7, p. 246 [13] becomes

$$(2.8)\qquad y(x,\ t) = \frac{x}{2\gamma}\left[S\left(\frac{x}{\sqrt{2\pi\gamma t}}\right) - C\left(\frac{x}{\sqrt{2\pi\gamma t}}\right)\right] + \left(\frac{t}{\pi\gamma}\right)^{1/2}\sin\left(\frac{x^2}{4\,\gamma\,t} + \frac{\pi}{4}\right)$$

where S, C are FRESNEL integrals: $S(z) = \int\limits_0^z \sin\left(\frac{\pi}{2}\,t^2\right)\,dt$, $C(z) = \int\limits_0^z \cos\left(\frac{\pi}{2}\,t^2\right)dt$.

3.1. - Solution for a viscoelastic beam.

The solution of the BERNOULLI-EULER elastic beam in (2.8) does not give one easily the solution for a viscoelastic beam by correspondence principle. Accordingly, the solution for viscoelastic beam is obtained by applying the correspondence principle to the transform of the elastic solution in equation (2.6). However this solution assumes a simple form for $x = 0$, which can be treated with the correspondence principle to yield the corresponding viscoelastic solution.

From (2.8)

$$(3.1)\qquad y(0,\ t) = \left(\frac{t}{2\pi\gamma}\right)^{1/2} = \left(\frac{\varrho A}{4\pi^2\,I}\right)^{1/4}\frac{t^{1/2}}{E^{1/4}}\,,$$

$$(3.2)\qquad \bar{y}(0,\ p) = \left(\frac{\varrho A}{4\pi^2\,I}\right)^{1/4}\frac{1}{E^{1/4}}\,\frac{\Gamma(3/2)}{p^{3/2}}\,.$$

The viscoelastic solution $\bar{y}(0,\ p)$ is now obtained by writing the p-operator for E in equation (3.2). For an incompressible viscoelastic material $E \longleftrightarrow \frac{3}{2}\,\frac{Q_s(D)}{P_s(D)}$. Hence

$$(3.3)\qquad y(0,\ t) = \Gamma\left(\frac{3}{2}\right)\left(\frac{\varrho A}{4\pi^2\,I}\right)^{1/4}\frac{1}{2\,\pi\,i}\int\limits_{r-i\infty}^{r+i\infty}\frac{e^{pt}}{p^{3/2}}\left\{\frac{2}{3}\,\frac{P_s(p)}{Q_s(p)}\right\}^{1/4}dp\,.$$

We may examine the solution for small values of t by expanding the integrand for large p (cf. [14]). For general linear viscoelastic material exhibiting instantaneous elasticity

$$(3.4)\qquad \frac{1}{E(p)} = J(p) = \frac{2}{3}\,\frac{P_s(p)}{Q_s(p)} = \frac{p_n\,p^n + p_{n-1}\,p^{n-1} + \ldots + p_0}{q_n\,p^n + q_{n-1}\,p^{n-1} + \ldots + q_0}\,,$$

$$J(p) = \frac{p_n}{q_n}\left[1 + \frac{1}{p}\left(\frac{p_{n-1}}{p_n} - \frac{q_{n-1}}{q_n}\right) + \right. \tag{3.5}$$
$$\left. + \frac{1}{p^2}\left(\frac{p_{n-2}}{p_n} - \frac{q_{n-2}}{q_n} - \frac{q_{n-1}\,p_{n-1}}{p_n\,q_n} + \frac{q_{n-1}^2}{q_n^2}\right) + o\left(\frac{1}{p^3}\right)\right], \qquad p \to \infty .$$

Hence

$$y(0,\ t) = \Gamma(3/2)\left(\frac{\varrho A}{4\pi^2 I}\,\frac{p_n}{q_n}\right)^{1/4}\left[\frac{t^{1/2}}{\Gamma(3/2)} + \frac{A\,t^{3/2}}{\Gamma(5/2)} + \ldots\right], \qquad t \sim 0 , \tag{3.6}$$

where $A = \frac{1}{4}\left(\frac{p_{n-1}}{p_n} - \frac{q_{n-1}}{q_n}\right)$. Since $\lim\limits_{p\to\infty} J(p) = \frac{p_n}{q_n} = J_1$, the first term of above solution (3.6) approaches $\left(\frac{\varrho A}{4\pi^2 IE}\right)^{1/4} t^{1/2}$ which is the solution for an elastic beam with YOUNG's modulus E. For a material not exhibiting instantaneous elasticity

$$J(p) = \frac{p_n\,p^n + p_{n-1}\,p^{n-1} + \ldots + p_0}{q_{n+1}\,p^{n+1} + q_n\,p^n + \ldots + q_0},$$

$$y(0,\ t) = \left(\frac{\varrho A}{4\pi^2 I}\,\frac{p_n}{q_{n+1}}\right)^{1/4}\Gamma(3/2)\left[\frac{t^{3/4}}{\Gamma(7/4)} + \frac{A\,t^{7/4}}{\Gamma(11/4)} + \ldots\right], \qquad t \sim 0 , \tag{3.7}$$

where now $A = \frac{1}{4}\left(\frac{p_{n-1}}{p_n} - \frac{q_n}{q_{n+1}}\right)$. Since $\frac{p_n}{q_{n+1}} = \lim\limits_{p\to\infty}\{p\ J(p)\}$, the first term in (3.7) is the deflection in a viscous material, ($p\ J(p) = 1/\eta$). Hence for small values of time, the deflection (at $x = 0$) in a beam of viscoelastic material not exhibiting instantaneous elasticity in tension is the same as that for a material which is purely viscous in tension.

The correspondence principle will now be used to examine the solution further. The solution for a viscoelastic beam by the correspondence principle is

$$\left\{\begin{aligned} y(x,\ t) &= \frac{k(1+i)}{4\sqrt{2}}\,\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}\,\frac{J^{1/4}(p)}{p^{3/2}}\,e^{-\sqrt{p}\,k\{(1+i)/\sqrt{2}\}\,J^{1/4}(p)x}\,\mathrm{d}p + \\ &+ \frac{k(1-i)}{4\sqrt{2}}\,\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}\,\frac{J^{1/4}(p)}{p^{3/2}}\,e^{\sqrt{p}\,k\{(-1+i)/\sqrt{2}\}\,J^{1/4}(p)x}\,\mathrm{d}p. \end{aligned}\right. \tag{3.8}$$

We examine first the solution for small values of t. We theorefore consider the expansions of the integrands for large values of p. For general viscoelastic material exhibiting instantaneous elasticity

$$J^{1/4}(p) = \alpha_0 + \frac{\beta_0}{p} + o\left(\frac{1}{p^2}\right), \text{ where } \alpha_0 = \left(\frac{p_n}{q_n}\right)^{1/4} \text{ and } \beta_0 = \frac{1}{4}\left(\frac{p_n}{q_n}\right)^{1/4}\left(\frac{p_{n-1}}{p_n} - \frac{q_{n-1}}{q_n}\right).$$

Thus

$$(3.9)\quad \begin{cases} y(x,t) = \dfrac{k(1+i)}{4\sqrt{2}}\,\dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} e^{pt}\left\{\frac{\alpha_0}{p^{3/2}} - \frac{k(1+i)}{\sqrt{2}}\,\frac{x\alpha_0\beta_0}{p^2} + \frac{\beta_0}{p^{5/2}} + o\left(\frac{1}{p^3}\right)\right\}\cdot \\ \cdot\, e^{-k\{(1+i)/\sqrt{2}\}x\alpha_0 p^{1/2}}\,\mathrm{d}p + \dfrac{k(1-i)}{4\sqrt{2}}\,\dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} e^{pt}\left\{\frac{\alpha_0}{p^{3/2}} + \frac{k(-1+i)}{\sqrt{2}}\,\frac{x\alpha_0\beta_0}{p^2} + \right. \\ \left. + \dfrac{\beta_0}{p^{5/2}} + o\left(\dfrac{1}{p^3}\right)\right\}\cdot e^{k\{(-1+i)/\sqrt{2}\}x\alpha_0 p^{1/2}}\,\mathrm{d}p\,. \end{cases}$$

Hence for large values of $x/t^{1/2}$, we find

$$(3.10)\quad \begin{cases} y(x,t) \approx \left(\dfrac{t}{\pi}\right)^{1/2} k\,\alpha_0 \sin\left(\dfrac{x^2k^2\alpha_0^2}{4t} + \dfrac{\pi}{4}\right) + \dfrac{x\,k^2\,\alpha_0^2}{2}\left[S\left(\dfrac{x\,k\,\alpha_0}{\sqrt{2\pi t}}\right) - C\left(\dfrac{x\,k\,\alpha_0}{\sqrt{2\pi t}}\right)\right] + \\ + x\,\alpha_0\,\beta_0\dfrac{k^2}{2}\left[\dfrac{x\alpha_0\beta_0}{\sqrt{\pi}}\,t^{1/2}\sin\left(\dfrac{x^2\alpha_0^2k^2}{4t} - \dfrac{\pi}{4}\right) + \left(\dfrac{x^2\alpha_0^2k^2}{2} + t\right)C\left(\dfrac{x\,\alpha_0\,k}{\sqrt{2\pi t}}\right) + \right. \\ \left. + \left(\dfrac{x^2\alpha_0^2k^2}{2} - t\right)S\left(\dfrac{x\,\alpha_0\,k}{\sqrt{2\pi t}}\right)\right] + \dfrac{8\beta_0\,t^{7/2}}{\pi^{1/2}K^3\alpha_0^4x^4}\sin\left(\dfrac{k^2\alpha_0^2x^2}{4t} + \dfrac{\pi}{4}\right) + o\left(\dfrac{t^{9/2}}{x^4}\right). \end{cases}$$

For a perfectly elastic material $p_n = 1$, $q_n = E$ and p_{n-1}, q_{n-2}, q_{n-1}, etc. are zero,

$$\alpha_0 = (1/E)^{1/4}, \qquad \beta_0 = 0\,.$$

Hence the solution for an elastic beam may be recovered exactly from (3.10).

3.2. - Solution for large values of t.

The solution for large t is now examined by considering the expansions for small values of p (cf. [15])

$$J^{1/4}(p) = \alpha_1 + \beta_1 p + o(p^2)\,,$$

where

$$\alpha_1 = \left(\frac{p_0}{q_0}\right)^{1/4} \quad \text{and} \quad \beta_1 = \frac{1}{4}\left(\frac{p_0}{q_0}\right)^{1/4}\left(\frac{p_1}{p_0} - \frac{q_1}{q_0}\right), \tag{3.11}$$

$$\left\{\begin{aligned} y(x,\,t) &= \frac{k\,(1+i)}{4\sqrt{2}}\,\frac{1}{2\pi i}\int\limits_{r-i\infty}^{r+i\infty} e^{pt}\left[\frac{\alpha_1}{p^{3/2}} + \frac{\beta_1}{p^{1/2}} - kx\alpha_1\,\beta_1\,\frac{(1+i)}{\sqrt{2}} - kxp\,\beta_1^2\,\frac{(1+i)}{\sqrt{2}} + \right.\\ &\left. + \ldots\right] e^{-kx\alpha_1 p^{1/2}\,(1+i)/\sqrt{2}}\,\mathrm{d}p + \frac{k\,(1-i)}{4\sqrt{2}}\,\frac{1}{2\pi i}\int\limits_{r-i\infty}^{r+i\infty} e^{pt}\left[\frac{\alpha_1}{p^{3/2}} + \frac{\beta_1}{p^{1/2}} + \right.\\ &\left. + kx\alpha_1\,\beta_1\,\frac{-1+i}{\sqrt{2}} + kxp\,\beta_1^2\,\frac{-1+i}{\sqrt{2}} + \ldots\right] e^{kx\alpha_1 p^{1/2}\,(-1+i)/\sqrt{2}}\,\mathrm{d}p\,, \end{aligned}\right. \tag{3.12}$$

which gives

$$\left\{\begin{aligned} y(x,\,t) &= k\alpha_1\left(\frac{t}{\pi}\right)^{1/2}\sin\!\left(\frac{x^2\,k^2\,\alpha_1^2}{4t} + \frac{\pi}{4}\right) + \frac{x\alpha_1^2\,k^2}{2}\left[S\!\left(\frac{x\alpha_1\,k}{\sqrt{2\pi t}}\right) - C\!\left(\frac{x\alpha_1\,k}{\sqrt{2\pi t}}\right)\right]\\ &+ \frac{k\beta_1}{2\pi^{1/2}}\cdot\frac{1}{t^{1/2}}\,\sin\!\left(\frac{x^2\,k^2\,\alpha_1^2}{4t} + \frac{\pi}{4}\right) + \\ &+ \frac{x^2\,\alpha_1^2\,\beta_1\,k^3}{4\pi^{1/2}}\,\frac{1}{t^{3/2}}\,\cos\!\left(\frac{x^2\,\alpha_1^2\,k^2}{4t} + \frac{\pi}{4}\right) + o\!\left(\frac{x^2}{t^{5/2}}\right). \end{aligned}\right. \tag{3.13}$$

The corresponding result for a VOIGT-KELVIN beam may now be deduced from (3.13). For a VOIGT-KELVIN material the stress-strain relationship is

$$\sigma = E\varepsilon + \eta\,D\,\varepsilon\,,$$

$$J(p) = \frac{1}{E+\eta p}\,, \qquad \text{so that } \alpha_1 = \left(\frac{1}{E}\right)^{1/4} \text{ and } \beta_1 = -\frac{\eta}{4E^{5/4}}\,.$$

The result for an elastic beam follows, when $\eta \to 0$. The solution valid for large x and large t ($x > t$) can be obtained by the method of steepest descent (cfr. [12]).

From (3.8) we have

$$(3.14)\quad \begin{cases} y(x,\ t) = \dfrac{k\,(1+i)}{4\sqrt{2}}\,\dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty}\dfrac{J^{1/4}(p)}{p^{3/2}}\exp\left\{p-\dfrac{k\,(1+i)}{\sqrt{2}}\,p^{1/2}\,J^{1/4}(p)\,\alpha\right\}t\,\mathrm{d}p\,+ \\[2ex] \quad +\dfrac{k\,(1-i)}{4\sqrt{2}}\,\dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty}\dfrac{J^{1/4}(p)}{p^{3/2}}\exp\left\{p+\dfrac{k\,(-1+i)}{\sqrt{2}}\,p^{1/2}\,J^{1/4}(p)\,\alpha\right\}t\,\mathrm{d}p\,, \end{cases}$$

where $\alpha = x/t$.

Consider a general viscoelastic material, so that

$$J(p) = \frac{p_n\,p^n + p_{n-1}\,p^{n-1} + \ldots + p_0}{q_{n+1}\,p^{n+1} + q_n\,p^n + \ldots + q_0},$$

then $J(p) = (a/p)\{1 + b/p + o(1/p^2)\}$ where $a = p_n/q_{n+1}$ and $b = (p_{n-1}/p_n + - q_n/q_{n+1})$, $p \to \infty$.

Now

$$p - \frac{k\,(1+i)}{\sqrt{2}}\,p^{1/2}\,J^{1/4}(p)\,\alpha = p - \frac{k\,(1+i)}{\sqrt{2}}\,\alpha\,a^{1/4}\,p^{1/4} + o\left(\frac{1}{p^{3/4}}\right),$$

$$p + \frac{k\,(-1+i)}{\sqrt{2}}\,p^{1/2}\,J^{1/4}(p)\,\alpha = p + \frac{k\,(-1+i)}{\sqrt{2}}\,\alpha\,a^{1/4}\,p^{1/4} + o\left(\frac{1}{p^{3/4}}\right)$$

and

$$\frac{J^{1/4}(p)}{p^{3/2}} = \frac{a^{1/4}}{p^{7/4}}\left\{1 + \frac{b}{4p} + o\left(\frac{1}{p^2}\right)\right\}.$$

We thus consider

$$I(x,\ t) = \int_{r-i\infty}^{r+i\infty} f_1(p)\,e^{t\,g_1(p)}\,\mathrm{d}p + \int_{r-i\infty}^{r+i\infty} f_2(p)\,e^{t\,g_2(p)}\,\mathrm{d}p,$$

where

$$f_1(p) = \frac{1}{2\pi i}\,\frac{k\,(1+i)}{4\sqrt{2}}\,\frac{a^{1/4}}{p^{7/4}},\qquad g_1(p) = p - \frac{k\,(1+i)}{\sqrt{2}}\,\alpha\,a^{1/4}\,p^{1/4}$$

and

$$f_2(p) = \frac{1}{2\pi i}\,\frac{k\,(1-i)}{4\sqrt{2}}\,\frac{a^{1/4}}{p^{7/4}}\,, \qquad g_2(p) = p + \frac{k\,(-1+i)}{\sqrt{2}}\,\alpha\, a^{1/4}\, p^{1/4}\,.$$

Now $g_1'(p) = 0 \Longrightarrow p = p_0 = \left(\frac{k}{4}\right)^{4/3} a^{1/3}\, e^{i(\pi/3)}\, \alpha^{4/3}$, hence the large p calculation is consistent with large α. The direction of steepest descent φ_1 is such that $g_1''(p_0)\; r^2\, e^{2i\varphi_1}$ is real and negative, thus $\varphi_1 = -\pi/3$. Similarly $g_2'(p) = 0 \Longrightarrow p = \bar{p}_0 = \left(\frac{k}{4}\right)^{4/3} a^{1/3}\, e^{-i\pi/3}\;\alpha^{4/3}$ and $\varphi_2 = \frac{\pi}{3}$. The integrands have a branch point at $p = 0$. We take the branch cut along the imaginary axis and consider a two sheeted RIEMANN surface. Deform the first integral into the path of steepest descent through the saddle point p_0 and the second integral into a steepest path through the saddle point $\bar{p}_0$. The integral (3.14) is now estimated by the contributions from these saddle points. Applying LAPLACE's formula (cf. [12]), we have

$$(3.15)\quad \begin{cases} y(x,\,t) = \sqrt{\dfrac{2}{3\pi}}\left(\dfrac{4}{k}\right)^{2/3} \dfrac{e^{-3\,(4/k)^{4/3}\,a^{1/3}\,\alpha^{4/3}\,t}}{\alpha^{5/3}\,t^{1/2}\,a^{1/6}} \left\{\sqrt{3}\cos\left(\dfrac{3\sqrt{3}}{2}\left(\dfrac{4}{k}\right)^{4/3} a^{1/3}\,\alpha^{4/3}\,t\right)\right. \\ \qquad \left. -\sin\left(\dfrac{3\sqrt{3}}{2}\left(\dfrac{4}{k}\right)^{4/3} a^{1/3}\,\alpha^{4/3}\,t\right)\right\} \qquad (\alpha > 1,\; t \to \infty)\,. \end{cases}$$

4. - Dynamical response of a viscoelastic beam including the effects of rotatory inertia and shear.

When the correction for rotatory inertia is applied, the impulsive motion is governed by

$$(4.1)\qquad y_{tt} - k^2\, y_{xxtt} + c_0^2\, k^2\, y_{xxxx} = \delta(x)\,\delta(t)\,.$$

The solution for viscoelastic beams can be obtained by the correspondence principle as before

$$(4.2)\quad \begin{cases} y(x,\,t) = \dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} \dfrac{e^{pt}\, e^{-[(\varrho/2)\,Jp^2 - (p/2k)\,(\varrho J)^{1/2}\sqrt{\varrho Jk^2 p^2 - 4}\,]^{1/2}\,x}}{\sqrt{2}\;p\sqrt{\varrho Jk^2\,p^2 - 4}\,[p^2\,k^2 - (kp/\sqrt{\varrho J})\sqrt{\varrho Jk^2\,p^2 - 4}]^{1/2}}\,\mathrm{d}p \\ \qquad - \dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} \dfrac{e^{pt}\, e^{-[(\varrho/2)\,Jp^2 + (p/2k)\,(\varrho J)^{1/2}\sqrt{\varrho Jk^2 p^2 - 4}\,]^{1/2}\,x}}{\sqrt{2}\;p\sqrt{\varrho Jk^2\,p^2 - 4}\,[p^2\,k^2 + (pk/\sqrt{\varrho J})\sqrt{\varrho Jk^2\,p^2 - 4}]^{1/2}}\,\mathrm{d}p\,, \end{cases}$$

where $J = J(p)$.

We examine the solution at first for small values of t. We thus consider the expansions of the integrands for large values of p. For the general viscoelastic materials exhibiting instantaneous elasticity, the asymptotic expansion of $J(p)$, $p \to \infty$, is

$$J(p) = a\left(1 + \frac{b}{p} + \frac{c}{p^2} + \ldots\right), \quad \text{where} \quad a = \frac{p_n}{q_n},$$

$$b = \frac{p_{n-1}}{p_n} - \frac{q_{n-1}}{q_n}, \qquad c = \frac{p_{n-2}}{p_n} - \frac{q_{n-2}}{q_n} - \frac{q_{n-1}\,p_{n-1}}{p_n\,q_n} + \frac{q_{n-1}^2}{q_n^2}, \ \text{etc.},$$

we thus find

$$(4.3) \quad \begin{cases} y(x,\ t) = \dfrac{e^{-x/k}}{2k} \cdot t - \dfrac{e^{-(b/2)(\ \varrho)^{1/2} x}}{2k^2(a\varrho)^{1/2}}\, \dfrac{(t-\sqrt{a\varrho}\,x)^2}{2\,!}\, H(t-\sqrt{a\varrho}\,x) + \\ \qquad + o(t^2) + o(t-\sqrt{\alpha\varrho}\,x)^3\, H(t-\sqrt{a\varrho}\,x)\,, \qquad t \sim 0\,, \end{cases}$$

where H is the unit step function defined by

$$H(t) = \begin{cases} 0\,, & t < 0 \\ 1/2\,, & t = 0\,. \\ 1\,, & t > 0 \end{cases}$$

The solution for large t is examined by expanding the integrands for small values of p. The asymptotic expansion of $J(p)$ for small p is

$$J(p) = \alpha_1\,[1 + \beta_1\,p + \gamma_1\,p^2 + \ldots]\,, \qquad p \to 0\,,$$

where

$$\alpha_1 = \frac{p_0}{q_0}\,, \qquad \beta_1 = \frac{p_1}{p_0} - \frac{q_1}{q_0}\,, \qquad \gamma_1 = \frac{p_2}{p_0} - \frac{q_2}{q_0} - \frac{p_1\,q_1}{p_0\,q_0} + \frac{q_1^2}{q_0^2}\,, \quad \text{etc.}\,.$$

We may now show that

$$(4.4) \quad \begin{cases} y(x,\ t) \approx \dfrac{(\varrho\alpha_1)^{1/4}}{\sqrt{k\pi}}\, t^{1/2} \sin\left(\dfrac{\sqrt{\varrho\alpha_1}\,x^2}{4t} + \dfrac{\pi}{4}\right) + \dfrac{(\varrho\alpha_1)^{1/2}}{2k}\, x \left[S\left(\dfrac{x(\varrho\alpha_1)^{1/4}}{\sqrt{2\pi k t}}\right) - \right. \\ \left. \qquad - C\left(\dfrac{x(\varrho\alpha_1)^{1/4}}{\sqrt{2\pi k t}}\right)\right] + \dfrac{(\varrho\alpha_1)^{1/4}}{2\sqrt{\pi k}}\, \dfrac{1}{t^{1/2}} \left[\dfrac{\beta_1}{4} \sin\left(\dfrac{\varrho\alpha_1\,x^2}{4kt} + \dfrac{\pi}{4}\right) + \right. \\ \left. \qquad + \dfrac{k}{4}\sqrt{\varrho\alpha_1}\ \sin\left(\dfrac{\varrho\alpha_1\,x^2}{4\,k\,t} - \dfrac{\pi}{4}\right)\right] + O\left(\dfrac{x^2}{t^{3/2}}\right) \qquad (x \ll t^{3/4}\,, \quad t \gg 1)\,. \end{cases}$$

The asymptotic estimate of the solution for large values of x can be obtained by the method of steepest descent. Let

$$u = \varrho^{1/2}\, k\, p\,, \quad x' = \frac{x}{\sqrt{2}\, k} \quad \text{and} \quad t' = \frac{t}{\sqrt{2}\, k},$$

using the above relations and rewriting the results in the original variables p, x and t, the integrals in (4.2) assume the form

$$\pm \frac{\varrho^{1/2}}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{J^{1/2}\, e^{x[\beta p - f(p)]}}{\sqrt{2}\, p \sqrt{p^2 J - 4 f(p)}}\, dp\,,$$

where

$$f(p) = [p^2 J \pm p\, J^{1/2} \sqrt{p^2 J - 4}]^{1/2}, \qquad \beta = \frac{t}{x}\,.$$

We can now find the asymptotic estimate for large x by the method of steepest descent. The saddle points are at $f'(p) = \beta$. We consider at first the case of β small, to do this we find approximations to $f(p)$ for large p.

For a KELVIN material $J(p) = 1/(a + bp)$ where $a = E$, $b = \eta/\sqrt{\varrho}\, k$,

$$y(x,\ t) \approx \frac{2\varrho\, b^{5/2}}{\sqrt{\pi}}\, \beta^3\, \frac{t^{3/2}}{x^2}\, e^{-(1/2b)(x^2/t)} \qquad (\beta \sim 0, \quad x \to \infty)\,. \tag{4.5}$$

Also the small p calculuation is consistent with large β. In this case, we may show that

$$y(x,\ t) \approx \frac{\varrho^{1/2}\, a^{1/4}}{\sqrt{\pi}}\, \frac{t^{5/4}}{x^{3/2}} \cos\left(\frac{x}{2\sqrt{a}} + \frac{5\pi}{4}\right) \qquad (\beta > 1, \quad x \gg 1)\,. \tag{4.6}$$

5. – If the correction due to shear is also taken into account the motion is then governed by the TIMOSHENKO theory. For impulsive motion, equation (1.5) becomes

$$y_{tt} - k^2 (1 + \varepsilon')\, y_{xxtt} + c_0^2\, k^2\, y_{xxxx} + \frac{\varepsilon'\, k^2}{c_0^2}\, y_{tttt} = \delta(x)\, \delta(t)\,. \tag{5.1}$$

Although the analysis is very tedious this time, the solution for viscoelastic beam utilizing equation (5.1) can be obtained in the same way as for the BERNOULLI-EULER and RAYLEIGH beams treated earlier. In this case, the corresponding results are (cf. [7])

$$(5.2)\quad \begin{cases} y(x,\ t) \approx \dfrac{e^{-(b/2)(a\varrho)^{1/2}x}}{2(\varepsilon' a\varrho)^{1/2}(1-\varepsilon')\,k^2}\,\dfrac{(t-\sqrt{a\varrho\varepsilon'}\,x)^2}{2!}\,H(t-\sqrt{a\varrho\varepsilon'}\,x)\,+ \\ \quad -\dfrac{e^{-(b/2)(a\varrho)^{1/2}x}}{2(a\varrho)^{1/2}(1-\varepsilon')\,k^2}\,\dfrac{(t-\sqrt{a\varrho}\,x)^2}{2!}\,H(t-\sqrt{a\varrho}\,x)\,+ \\ \quad +O(t-\sqrt{a\varrho\varepsilon'}\,x)^3\,H(t-\sqrt{a\varrho\varepsilon'}\,x)+O(t-\sqrt{\varrho a}\,x)^3\,H(t-\sqrt{a\varrho}\,x) \end{cases}$$

and

$$(5.3)\quad \begin{cases} y(x,\ t) = \dfrac{(\varrho\alpha_1)^{1/2}x}{2k}\left[S\left(\dfrac{x\,(\varrho\alpha_1)^{1/4}}{\sqrt{2\pi kt}}\right)-C\left(\dfrac{x\,(\varrho\alpha_1)^{1/4}}{\sqrt{2\pi kt}}\right)\right]+\dfrac{(\varrho\alpha_1)^{1/4}}{\sqrt{\pi k}}\,t^{1/2}\,\cdot \\ \quad\cdot\,\sin\left(\dfrac{\sqrt{\varrho\alpha_1}\,x^2}{4kt}+\dfrac{\pi}{4}\right)+\dfrac{(\varrho\alpha_1)^{1/4}}{2\sqrt{\pi k}}\,\dfrac{1}{t^{1/2}}\left[\left(\dfrac{\beta_1}{4}-\dfrac{k\sqrt{\varrho\alpha_1}}{4}(1+\varepsilon')\right)\cdot\right. \\ \quad\left.\cdot\,\cos\left(\dfrac{\sqrt{\varrho\alpha_1}\,x^2}{4kt}\right)+\left(\dfrac{\beta_1}{4}+\dfrac{k}{4}\sqrt{\varrho\alpha_1}\,(1+\varepsilon')\right)\sin\left(\dfrac{\sqrt{\varrho\alpha_1}\,x^2}{4kt}\right)\right]+O\left(\dfrac{x^2}{t^{3/2}}\right) \\ \qquad\qquad (x \ll t^{3/4}, \qquad t \gg 1)\,. \end{cases}$$

The method of steepest descent may be used to find the solution for large values of x as for the RAYLEIGH's theory in (4). In all these cases if the impulse is applied at an arbitrary point x_1 then the results can be obtained by writing $x - x_1$ for x in the corresponding solutions for the impulse at $x = 0$.

6. - The propagation of longitudinal and torsional waves in a semi-infinite bar.

The propagation of longitudinal waves in a semi-infinite viscoelastic bar has been treated in [8]. The present investigation includes the study of longitudinal displacement and also obtains the final shape of the rod. The solution for the torsional motion being of identical nature can be examined similarly.

The governing equations for a viscoelastic bar undergoing longitudinal vibrations are

$$(6.1)\qquad \frac{\partial\sigma}{\partial x} = \varrho\,\frac{\partial^2 u}{\partial t^2}\,, \qquad \varepsilon = \frac{\partial u}{\partial x} \qquad \text{and} \qquad P(D) = Q(D)\varepsilon\,.$$

Eliminating σ and ε we have

$$\varrho\, D^2\, P(D)\, u = Q(D)\, \frac{\partial^2 u}{\partial x^2}\,. \tag{6.2}$$

The corresponding equation for the elastic rod is

$$D^2\, u = c_0^2\, \frac{\partial^2 u}{\partial x^2}\,, \qquad \text{where} \qquad c_0^2 = \frac{E}{\varrho}\,. \tag{6.3}$$

Thus the solution for viscoelastic problem can be obtained by applying correspondence principle. If an impulse of unit magnitude is applied at $x = 0$ and $t = 0$, then the solution of viscoelastic bar is given by

$$u(x,\ t) = \frac{1}{2\pi i}\, \frac{1}{\varrho} \int_{r-i\infty}^{r+i\infty} e^{pt}\, \frac{(\varrho J)^{1/2}}{p}\ \exp\{-\, xp(\varrho J)^{1/2}\}\, \mathrm{d}p$$

and

$$\sigma(x,\ t) = -\frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{pt} \exp\{-\, xp(\varrho J)^{1/2}\}\, \mathrm{d}p\,.$$

Now for large p,

$$[\varrho\, J(p)]^{1/2} = \sum_{n=0}^{\infty} \frac{a_n}{p^n}\,, \qquad \text{where} \qquad a_0 = \left[\varrho\, \frac{p_n}{q_n}\right]^{1/2},$$

$$a_1 = \frac{1}{2}\left(\varrho\, \frac{p_n}{q_n}\right)^{1/2} \left\{\frac{p_{n-1}}{p_n} - \frac{q_{n-1}}{q_n}\right\}, \quad \text{etc.}\ ;$$

$$\sigma(x,\ t) = -\frac{e^{-a_1 x}}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{p\,(t-a_0 x)} \left\{1 - \frac{x a_2}{p} + \frac{1}{p^2}\left(\frac{a_2^2\, x^2}{2} - a_3\, x\right) + o\left(\frac{1}{p^3}\right)\right\} \mathrm{d}p\,.$$

When the exponential factor in the integrand is $\exp(pt)$, expanding the integrand for large values of p, gives the solution for small values of t. In this case the factor is $\exp(t - a_0\, x)\, p$; the expansion for large p therefore gives the solution for small values of $t - a_0\, x$, i.e. it gives the solution near the wave front

$$(6.4)\quad \begin{cases} \sigma(x,\ t) = -e^{-a_1 x}\,\delta(t - a_0\, x) - e^{-a_1 x}\cdot \\ \qquad \cdot\left\{-a_2\, x + \left(\dfrac{a_2^2\, x^2}{2} - a_3\, x\right)(t - a_0\, x) + O(t - a_0\, x)^2\right\} \end{cases}$$

near the wave front. Thus the impulse applied to the end of a semi-infinite viscoelastic rod exhibiting instantaneous elasticity in tension, is transmitted down the rod with velocity $1/a_0$ and attenuation a_1. The attenuated impulse is followed immediately by a wave of finite amplitude whose magnitude at its head is $a_2\, x \exp(-a_1\, x)$. Also

$$u(x,\ t) = \frac{e^{-a_1 x}}{\varrho}\,\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \exp(t - a_0\, x)\cdot \cdot\left\{\frac{a_0}{p} + \frac{1}{p^2}(a_1 - a_0\, a_2\, x) + \frac{1}{p^3}(a_2 - a_1\, a_2\, x) + \ldots\right\} dp\,,$$

or

$$(6.5)\quad \begin{cases} u(x,\ t) = \dfrac{1}{\varrho} H(t - a_0\, x)\, e^{-a_1 x}\left\{a_0 + (a_1 - a_0\, a_2\, x)(t - a_0\, x) + \right. \\ \qquad \left. + (a_2 - a_1\, a_2\, x)\dfrac{(t - a_0\, x)^2}{2\,!} + O(t - a_0\, x)^3\right\}, \qquad t - a_0\, x \sim 0\,. \end{cases}$$

For a material not exhibiting instantaneous elasticity

$$\{\varrho\, J(p)\}^{1/2} = \frac{A}{p^{1/2}}\left\{1 + \frac{B}{p} + O\left(\frac{1}{p^2}\right)\right\},$$

where $A = \left(\dfrac{\varrho\, p_n}{q_{n+1}}\right)^{1/2}$ and $B = \dfrac{1}{2}\left(\dfrac{p_{n-1}}{p_n} - \dfrac{q_n}{q_{n+1}}\right)$,

$$(6.6)\quad \begin{cases} \sigma(x,\ t) = -\dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} e^{pt}\exp(-Axp^{1/2})\cdot\left\{1 - \dfrac{ABx}{p^{1/2}} + O\left(\dfrac{1}{p}\right)\right\} dp \\ \quad = \dfrac{-Ax}{2\pi^{1/2}}\exp\left(\dfrac{-A^2\, x^2}{4t}\right)\cdot[t^{-3/2} - 2B\, t^{-1/2} + O(t^{1/2})]\,, \quad t \text{ small.} \end{cases}$$

The first term is the stress distribution in a viscous material ($B = 0$). Hence for small values of the time, the stress distribution in a material not exhibiting instantaneous elasticity in tension, is the same as that for a material which is purely viscous in tension. Also, for small t,

$$(6.7)\quad \begin{cases} u(x,\ t) = \dfrac{A}{\varrho}\left[2\pi^{-1/2}\, t^{1/2}\, e^{-A^2 x^2/(4t)} - A\, x\ \operatorname{Erfc}\dfrac{Ax}{2t^{1/2}}\right] \\ \quad - \dfrac{A^2 B x}{\varrho}\left[\left(t + \dfrac{A^2 x^2}{2}\right)\operatorname{Erfc}\dfrac{Ax}{2t^{1/2}} - \dfrac{Ax}{\pi^{1/2}}\, t^{1/2}\, e^{-A^2 x^2/(4t)}\right] + \dots, \qquad t \sim 0\,. \end{cases}$$

We now examine the solution for large values of time. For the general viscoelastic material, the asymptotic expansion of $J(p)$ for small p is

$$J(p) = \alpha_1\,(1 + \beta_1\, p + \gamma_1\, p^2 + \dots)\,, \qquad p \sim 0\,,$$

where

$$\alpha_1 = \frac{p_0}{q_0}\,, \qquad \beta_1 = \frac{p_1}{p_0} - \frac{q_1}{q_0}\,, \qquad \text{etc.}\,.$$

Hence

$$\bar{\sigma}(x,\ p) = -1 + (\varrho\alpha_1)^{1/2}\, xp - \frac{p^2}{2}\,(x^2\, \varrho\alpha_1 - x\beta_1\sqrt{\varrho\alpha_1}) + O(p^3)\,.$$

And

$$(6.8)\qquad \sigma(x,\ t) \to 0 \quad \text{as} \quad t \to \infty\,.$$

Also since

$$\bar{u}(x,\ p) = \frac{(\varrho\alpha_1)^{1/2}}{\varrho}\left[\frac{1}{p} + \left(\frac{\beta_1}{2} - x\sqrt{\varrho\alpha_1}\right) + p\left(\frac{x^2}{2}\,\varrho\alpha_1 - x\beta_1\sqrt{\varrho\alpha_1}\right) + \dots\right],$$

$$(6.9)\qquad u(x,\ t) \to \left(\frac{\alpha_1}{\varrho}\right)^{1/2} \qquad \text{as} \quad t \to \infty\,, \quad (q_0 \neq 0)\,.$$

For large values of t the stress tends to zero while the displacement approaches a finite limit $(\alpha_1/\varrho)^{1/2}$ which vanishes if $p_0 = 0$.

We now consider the effect of torque M applied at the end $x = 0$ and time $t = 0$. The solution of the torsional motion may be obtained by applying correspondence principle.

Angular twist:

$$\theta = \frac{1}{2\pi i}\,\frac{M}{\varrho\, I_1}\int_{r-i\infty}^{r+i\infty}\frac{e^{pt}}{p}\,\{3\varrho\, J(p)\}^{1/2}\exp[-px\,(3\varrho\, J(p))^{1/2}]\,dp\,. \tag{6.10}$$

Shearing stress:

$$s = -\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{pt}\exp[-px\,(3\varrho\, J(p))^{1/2}]\,dp\,. \tag{6.11}$$

The integrals in (6.10), (6.11) are of the same form as those obtained for $u(x, t)$ and $\sigma(x, t)$ for the longitudinal vibrations of viscoelastic bars examined earlier. Hence the solutions for $\theta(x, t)$ and $s(x, t)$ can be written down by comparison.

Conclusions.

Investigation of the asymptotic solutions for the lateral motion reveals that the solution of a viscoelastic beam, exhibiting instantaneous elastic response consists of two parts; the first part is a contribution mainly due to the elastic nature of the beam, while the second part is a characteristic of the viscoelastic material, from which the beam is made. The maximum deflection of beam is observed at the position where the impulse is applied; and the deflection goes off to zero away from this point. For small values of time, the viscosity effects are appreciable while for large times ($t \to \infty$), the viscous part of the solution goes to zero as $O(1/t^{1/2})$. The effect of rotatory inertia and shear deformation on the beam deflection is considerable for small t but negligible at large t. Impulse applied to the end of a viscoelastic bar which exhibits instantaneous elasticity in tension is transmitted with velocity $1/a_0$ and attenuation a_1, (6.4). The attenuated impulse is followed immediately by a wave of finite amplitude whose magnitude at its head is $a_1 x \exp(-a_1 x)$. The displacement u vanishes exponentially with x. For small t, the stress distribution in a material not exhibiting instantaneous elasticity in tension is the same as that for a material which is purely viscous in tension. For large t ($t \to \infty$), the stress tends to zero while the displacement approaches a finite limit $[p_0/q_0\varrho]^{1/2}$. For torsional motion, however, the solution is mathematically identical with that for the longitudinal motion.

Acknowledgements. I am grateful to Professor R. A. Ross for his constant help and encouragement during the supervision of this work which forms part of the Ph. D. thesis (cf. [7]).

References.

[1] R. A. ANDERSON, *Flexural vibrations in uniform beams according to the Timoshenko theory*, J. Appl. Mech. **20** (1953), 504-510.

[2] C. L. DOLPH, *On tho Timoshenko theory of transverse beam vibrations*, Quart. Appl. Math. **12** (1954), 175-187.

[3] T. C. HUANG, *Effect of rotatory inertia and shear on the vibrations of beams treated by the approximate method of Ritz and Galerkin*, Proceedings, 3rd U. S. Natl. Congress of Applied Mechanics (ASME), pp. 189-194, New York 1958.

[4] H. FAVRE, *Sur la propagation des vibrations transversales le long d'une barre prismatique viscoélastique à comportement linéaire*, J. Mécanique **3** (1964), 251-276.

[5] F. T. FLAHERTY, *Viscoelastic Timoshenko Beam Vibration*, Thesis presented to New York University, New York 1963.

[6] H. H. PAN, *Vibration of a viscoelastic Timoshenko beam*, J. Eng. Mech. Division (ASCE) **92** (1966), 213-234.

[7] V. P. MADAN, *Impulsive Excitation of Viscoelastic Beams*, Ph. D. Thesis presented to Univ. of Toronto, Tcronto 1968.

[8] D. R. BLAND, *The Theory of Linear Viscoelasticity*, Pergamon Press, New York 1960 (cf. p. 96).

[9] H. KOLSKY, *The propagation of stress pulses in viscoelastic solids*, Phil. Mag. HI (1956), p. 693.

[10] H. KOLSKY, *Stress Waves in Solids*, Claredon Press, Oxford 1953.

[11] I. M. GEL'FAND and G. E. SHILOV, *Generalized Functions*, Vol. 1, Academic Press, New York 1964.

[12] A. ERDÉLYI, *Asymptotic Expansions*, Dover Publications, New York 1956.

[13] A. ERDÉLYI, *Tables of Integral Transforms*, Vol. 1, McGraw-Hill, New York 1954.

[14] H. S. CARSLAW and J. C. JAEGER, *Operational Methods in Applied Mathematics*, Oxford University Press, Oxford 1949.

[15] N. W. McLACHLAN, *Complex Variable Theory and Transform Calculus*, Cambridge University Press, Cambridge 1963 (cf. pp. 119-120).

Abstract.

The problem considered is that of an infinitely long viscoelastic beam subjected to an impulsive excitation. The dynamical motion so set up is analysed according to the models of Bernoulli-Euler, Rayleigh and the Timoshenko beams. The small and large time solutions are obtained in all these cases using asymptotic methods. The process of expanding integrands and the method steepest descents are found suitable for carrying out asymptotics. The effects of the rotatory inertia and shear are studied. The effect of impulsive pressure applied to the end of a semi-infinite beam is also studied by considering the longitudinal motion. The results for torsional motion can be obtained by comparison.

* * *