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Squeeze Properties for Graphs. (**)

1. - Introduction.

Let P be any property that a graph may possess. A graph G is said to be *critical with respect to property* P if G has property P but the graph $G-v$ does not have property P, for each point v of G. Similarly, a graph G is said to be *minimal with respect to property* P if G has property P but the graph $G-e$ does not, for each line e of G. DUKE [7] and YOUNGS [12] have studied genus-minimal graphs; CHARTRAND, KAUGARS and LICK ([3], [8]) have considered critically n-connected graphs; and LICK ([8], [9]) has examined minimally n-line-connected graphs. The renowned Four Color Problem has stimulated a great deal of research into the chromatic number for graphs. For example, DIRAC (see [4], [5], [6]) has studied graphs which are critical with respect to chromatic number n.

In contrast to the situations described above, it may be that the removal of an arbitrary point (or line) yields a new graph which still has property P. A graph G is said to be *durable with respect to property* P if G has property P, and so does the graph $G-v$, for each point v of G. Similarly, G is said to be *permanent with respect to property* P if the graphs G and $G-e$ both have property P, for each line e of G. Graphs which are durable (or permanent) with respect to having chromatic number n have been studied in [10].

The purpose of the paper is to extend the investigation of [10] to a large class of properties of graphs, including the chromatic number, for which some general results may be obtained. To this end we introduce the notion of a squeeze property.

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2. - Squeeze properties for graphs.

The notation $H \leqslant G$ indicates that H is a subgraph (not necessarily induced) of the graph G. A property P is said to be a *squeeze property* if, whenever H and G have property P and $H \leqslant H' \leqslant G$, then H' also has property P. Parameters which give rise to squeeze properties include the chromatic number (the property P would be: $\chi(G) = n$), genus, BETTI number, clique number, point-arboricity (see [2]), and point-partition number (see [10]). Connectivity and line-connectivity are parameters which do not give rise to squeeze properties. The following theorem relates minimality to criticality, and durability to permanence, for the class of properties we are considering.

Theorem 1. *With respect to any squeeze property P:*

(i) *If G has no isolated points and is minimal, then G is critical.*

(ii) *If G is durable, then G is permanent.*

Proof. (i) Let v be any point of G. Since G has no isolated points, there is a line e incident with v. Since G is minimal with respect to property P, the graph $G - e$ does not have property P. But $G - v \leqslant G - e \leqslant G$, and since P is a squeeze property, $G - v$ does not have property P; i.e., G is critical with respect to property P.

(ii) Let e be any line of G, and let v be a point incident with e. Since G is durable, $G - v$ has property P. But again $G - v \leqslant G - e \leqslant G$, and since P is a squeeze property, $G - e$ has property P. Hence G is permanent with respect to property P.

Further examples of squeeze properties abound, as the next theorem and the comments following it indicate.

Theorem 2. *The property of «being durable with respect to the squeeze property P» is itself a squeeze property.*

Proof. Let H and G both be durable with respect to the squeeze property P, with $H \leqslant H' \leqslant G$. Since H and G both have property P, it follows that H' also has property P. It remains to show that H' is durable in this respect. Let v be any point of H'. Then $H - v \leqslant H' - v \leqslant G - v$. But H and G are both durable, so that $H - v$ and $G - v$ still have property P; hence $H' - v$ has property P. This completes the proof.

The above theorem remains valid if the word «durable» is replaced with the word «permanent». Furthermore, if the original squeeze property P is defined in terms of a parameter taking on integral values, the theorem is also true for «durable» replaced by «critical» or «minimal». The proofs, being routine, are omitted.

3. - Critically durable and critically permanent graphs.

A graph G is said to be *critically durable with respect to property* P if G is durable with respect to property P and critical with respect to the property of being durable. That is: (i) G has property P; (ii) $G-v$ has property P for each point v of G; and (iii) for each point v of G, there is a point u of $G-v$ such that the graph $G-v-u$ does not have property P. Correspondingly, a graph G is said to be *critically permanent with respect to property* P if G is permanent with respect to property P and critical with respect to the property of being permanent; specifically: (i) G has property P; (ii) $G-e$ has property P for each line e of G; and (iii) for each point v of G, there is a line e of $G-v$ such that the graph $G-v-e$ does not have property P.

The following theorem will be helpful in providing examples of critically durable graphs.

Theorem 3. *The property « being critically durable with respect to the squeeze property P » is itself a squeeze property.*

Proof. Let H and G be critically durable with respect to the squeeze property P, with $H \leqslant H' \leqslant G$. Then H and G are both durable with respect to P, and it follows from Theorem 2 that H' is also. Now let v be a point of H', and hence also of G. Then there exists a point u of $G-v$ such that $G-v-u$ does not have property P. Also, u must be a point of $H'-v$, for otherwise $H'-v \leqslant G-v-u \leqslant G$, which would imply that $G-v-u$ has property P. The relationship $H'-v-u \leqslant G-v-u \leqslant G$ now shows that $H'-v-u$ does not have property P. Hence H' is critically durable with respect to P, thus completing the proof.

Similarly, one may show that « being critically permanent with respect to a squeeze property » is also a squeeze property. In order to give examples of critically durable and critically permanent graphs, we define the following graphs. The *complete n-partite graph* $K(p_1, \dots, p_n)$ has its point set V partitioned into n disjoint nonempty subsets V_i, where $|V_i| = p_i$ $(i = 1, \dots, n)$, such that a line joins two points u and v of V if and only if $u \in V_j$ and $v \in V_k$, where $j \neq k$. If $p_i = 1$ $(i = 1, \dots, n)$, then the resulting graph is the *complete graph* K_n with n points. If $p_i = 2$ $(i = 1, \dots, n)$, we will designate the resulting graph by $K_{n(2)}$. We use the symbol $2K_n$ to denote the disconnected graph having two components, each isomorphic to K_n. The graph $2K_n + e$ is $2K_n$ with one line added; note that this gives a connected graph. See Fig. 1 for the graph $2K_4 + e$. The *one point union* of two complete graphs K_n, denoted by $K_n \cdot K_n$, is two copies of K_n with two points identified, one from each copy. See Fig. 2 for $K_4 \cdot K_4$.

For each positive integer $n \geqslant 3$, we define the graph D_n as follows: let $H_1 = H_2 = K_{n-1}$ and $H_3 = H_4 = \{v\}$. To the disjoint union

$$\bigcup_{i=1}^{4} H_i ,$$

add $n-1$ lines joining H_1 to H_i $(i = 3, 4)$; add $n-2$ lines joining H_2 to H_i

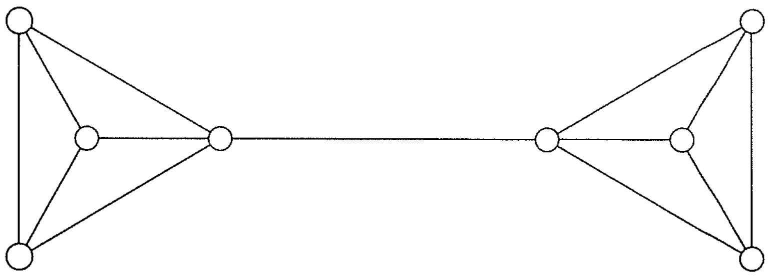

Fig. 1.

$(i = 3, 4)$, but in such a manner that in the graph D_n two points of H_2 have degree $n-1$. See Fig. 3 for the graph D_4.

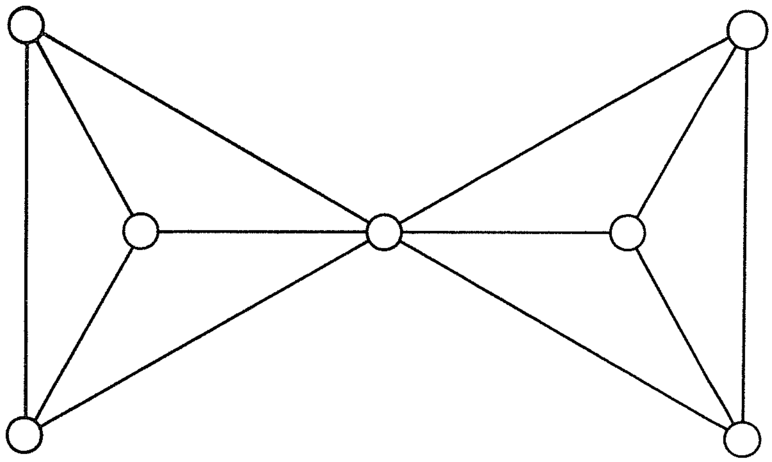

Fig. 2.

The graph $2K_n$ is critically durable and critically permanent with respect to having chromatic number n. However, this graph is not connected. The following examples will be connected graphs. The graph $K_{n(2)}$ is critically durable, but not critically permanent with respect to chromatic number n. The graph $2K_n + e$ is both critically durable and critically permanent with respect to chromatic number n. With respect to the number of points and lines, $2K_n + e$ is the unique smallest connected critically durable n-chromatic graph (see [**10**]). It is easy to observe that any critically durable n-chromatic graph is a subgraph of $K_{n(2)}$. Theorem 3 shows that, if $2K_n + e \leqslant G \leqslant K_{n(2)}$,

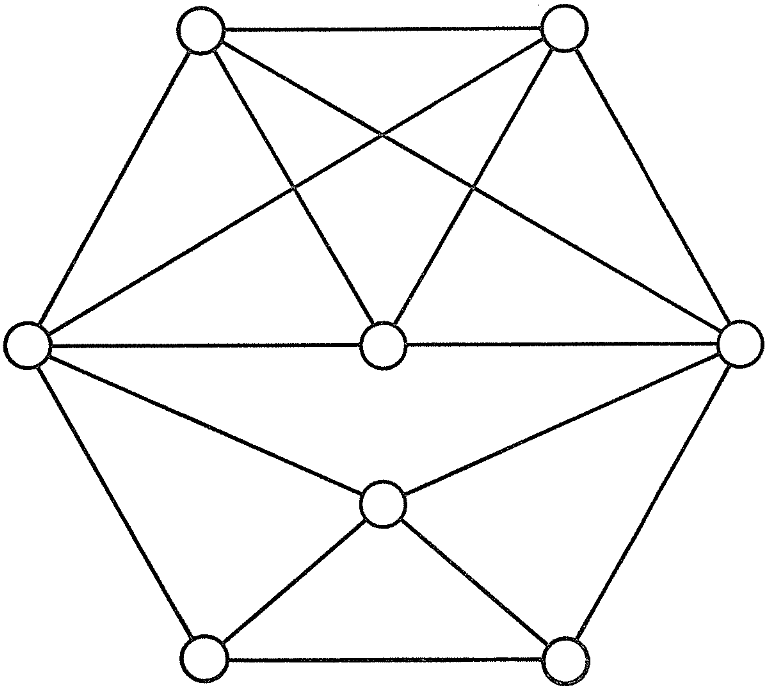

Fig. 3.

then G is critically durable with respect to having chromatic number n. One might suspect that any critically durable n-chromatic graph is a supergraph of $2K_n + e$, but this is not the case. The graph D_n is critically durable with respect to chromatic number n, but is not a supergraph of $2K_n + e$. The graph D_n is also critically permanent with respect to chromatic number n. The graph $K_n \cdot K_n$ is critically permanent with respect to chromatic number n. With respect to the number of points and lines, this is the unique smallest such graph (see [**10**]).

4. - Minimally durable and minimally permanent graphs.

A graph G is said to be *minimally durable with respect to property* P if G is durable with respect to property P and minimal with respect to the property of being durable. Similarly, G is said to be *minimally permanent with respect* *to property* P if G is permanent with respect to property P and minimal with respect to the property of being permanent. For example, $2K_n$ is both minimally durable and minimally permanent with respect to having chromatic number n, as is the graph D_n defined in Section 3. The graph $K_n \cdot K_n$ is minimally permanent, but not minimally durable, with respect to chromatic number n. The graphs $K_{n(2)}$ and $2K_n + e$ are neither minimally durable nor minimally permanent with respect to chromatic number n.

The method of proof employed in Theorem 3 now shows that the property « being minimally durable (or minimally permanent) with respect to the squeeze property P » is itself a squeeze property.

5. - Durably critical and permanently minimal graphs.

Let P_n be a sequence of distinct properties indexed by the nonnegative integers. A graph G is said to be *durably critical with respect to* P_n if: (i) G has property P_n; (ii) $G - v$ has property $P_{m(v)}$, where $m(v) \neq n$, for each point v of G; and (iii) for each two distinct points u and v of G, $G - v - u$ has property $P_{m(u,v)}$, where $m(u, v) \neq m(v)$. Similarly, G is said to be *permanently minimal with respect to* P_n if: (i) G has property P_n; (ii) $G - e$ has property $P_{m(e)}$, where $m(e) \neq n$, for each line e of G; and (iii) for each two distinct lines e and f of G, $G - e - f$ has property $P_{m(e,f)}$, where $m(e, f) \neq m(e)$.

The complete graph K_n, $n \geqslant 2$, is durably critical with respect to chromatic number n, but not permanently minimal. The *n-star* $K(1, n)$ is permanently minimal with respect to line-chromatic number n.

Theorem 4. *The graph G is durably critical with respect to chromatic number n, $n \geqslant 2$, if and only if $G = K_n$.*

Proof. Clearly, K_n is durably critical with respect to chromatic number n. Conversely, let G be a durably critical graph with respect to chromatic number n, and let u and v be any two points of G. Then $\chi(G - v - u) = n - 2$. Color $G - v - u$ with a proper $n - 2$ coloring. Now reintroduce the point u together with the lines incident with u in $G - v$. Since $\chi(G - v) = n - 1$, a new color is required for u. Then no other point in $G - v$ has the same color as u. Next reintroduce the point v together with the lines incident with v in G. Since $\chi(G) = n$, v is adjacent to at least one point of each color in $G - v$. Hence u and v are adjacent in G, and G is a complete graph. Since $\chi(G) = n$, $G = K_n$.

To correspond with the above theorem, we have the following: the graph G is permanently minimal with respect to line-chromatic number n, $n \geqslant 2$, $n \neq 3$, if and only if $G = K(1, n)$. This will follow directly from the next theorem, noting that $K(1, n)$ is the unique graph with the line graph K_n, for $n \neq 3$. (Both K_3 and $K_{1,3}$ have line graph K_3). The line-chromatic number of the graph G is denoted by $\chi_1(G)$, and the line graph of G is denoted by $L(G)$.

Theorem 5. *The graph G is permanently minimal with respect to $\chi_1(G)=n$ if and only if $L(G)$ is durably critical with respect to $\chi(L(G))=n$, $n \geqslant 2$.*

Proof. (i) Let G be permanently minimal with respect to $\chi_1(G)=n$; then $\chi_1(G)=\chi(L(G))=n$. Let v be a point of $L(G)$, with corresponding edge e in G; then $\chi(L(G-e))=\chi_1(G-e)=n-1$, so that $L(G)$ is critical with respect to $\chi(L(G))=n$. Now let u be a point of $L(G)-v$, with corresponding edge f in $G-e$. We see that $\chi(L(G)-v-u)=\chi(L(G-e-f))=\chi_1(G-e-f)=n-2$, so that $L(G)$ is durably critical with respect to $\chi(L(G))=n$.

(ii) The converse is established similarly.

We observe that the properties: « being durably critical with respect to the squeeze property P_n » and « being permanently minimal with respect to the squeeze property P_n » are themselves squeeze properties. The proofs are routine and are omitted.

6. - Durable and critical graphs with respect to the Betti number and genus.

The *Betti number* of a graph G is given by the equation

$$\beta(G)=E-V+n,$$

where G has n components, E lines, and V points. The observations of the following theorem are immediate upon noting that the BETTI number counts the number of independent cycles in G. This theorem typifies the contrast between the concepts of durability and criticality, and between permanence and minimality.

Theorem 6. *With respect to the property « having Betti number n »:*

(i) *G is durable if and only if no point of G is in a cycle.*

(ii) *G is permanent if and only if no line of G is in a cycle.*

(iii) *G is critical if and only if every point of G is in a cycle.*

(iv) *G is minimal if and only if every line of G is in a cycle.*

It is evident that the conditions of statements (i) and (ii) above are both equivalent to the condition that G be a forest. If G has no isolated points, then the condition of statement (iv) implies that of statement (iii). (See Theorem 1.) The graph $2K_n + e$, $n \geq 3$, illustrates that condition (iii) does not imply the condition of (iv).

BATTLE, HARARY, KODAMA, and YOUNGS [1] proved that if G has n blocks $B_1, \ldots, B_n$, then the genus, $\gamma(G)$, of G is:

$$\gamma(G) = \sum_{i=1}^{n} \gamma(B_i) .$$

Let G_n^* be a connected graph with n blocks, each isomorphic to G. Then:

(i) $(K_5)_n^*$ is both critical and minimal with respect to having genus n.

(ii) $(K_6)_n^*$ is permanent (but not minimally permanent) and critically durable with respect to having genus n.

It follows from the equation

$$\gamma\big(K(p, q, r)\big) = \left\{\frac{(p-2)(q+r-2)}{4}\right\},$$

for $p \geq q \geq r$ and $q + r \leq 6$ (see [**11**]), that $K(n+2, 3, 3)$ is durably critical with respect to having genus n, for $n \geq 5$, $n \not\equiv 0 \pmod 4$. It is evident that examples may be given of graphs which are minimally permanent with respect to having genus n. We close with the following question: do there exist permanently minimal graphs with respect to having genus n? (It is easy to show that any such graph must contain no triangles.)

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Summary.

A property P that a graph may have is said to be a squeeze property if, whenever two graphs H and G have property P and H' is a subgraph of G and a supergraph of H, then H' must have property P also. Several results are established for squeeze properties in general, and some specialized results are given for particular squeeze properties, such as the chromatic number.

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