Riv. Mat. Univ. Parma (2) **10** (1969), 157-163

A. H. SIDDIQI (*)

On the Walsh-Fourier Coefficients of Certain Classes of Functions. (**)

1. – Let the RADEMACHER function be defined by

$$\varphi_0(x) = \begin{cases} 1 & \text{for } 0 \leqslant x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \leqslant x < 1 \end{cases}, \qquad \varphi_0(x+1) = \varphi_0(x),$$

$$\varphi_n(x) = \varphi_0(2^n x) \qquad (n = 1,\ 2,\ \ldots).$$

The WALSH functions $\psi_n(x)$ is defined as follows:

(a) $\psi_0(x) = 1$.

(b) If n has the unique dyadic expansion $\sum_{i=0}^{\infty} 2^i x_i$, where $x_i = \{ {0 \atop 1}$ and $x_i = 0$ for $i > m_r$, then

$$\psi_n(x) = \varphi_{m_1}(x)\, \varphi_{m_2}(x) \ldots \varphi_{m_r}(x)\,,$$

where $m_1, m_2, \ldots, m_r$ correspond to the coefficients $x_{m_i} = 1$. Every function $f(x)$ which is of period 1 and LEBESGUE integrable on $[0, 1]$ may be expanded in a WALSH-FOURIER series: $f(x) \sim \sum_{k=0}^{\infty} c_k\, \psi_k(x)$, where $c_k = \int_0^1 f(x)\ \psi_k(x)\ dx$ $(k = 0,\ 1,\ 2,\ \ldots)$.

A function $f(x)$ is said to belong to the class B of essentially bounded functions if $|f(x)| \leqslant M$ almost everywhere.

Let S′ (see [5]) denote the class of series $\sum_{k=0}^{\infty} c_k\, \psi_k(x)$ with coefficients $c_k = \int_0^1 \psi_k(t)\ dF(t)$, where $F(t)$ is continuous ando f bounded variation. We denote by α_k the k-th (C, 1) mean of the sequence of WALSH-FOURIER coefficients c_k.

(*) Indirizzo: Department of Mathematics and Statistics, Aligarh Muslim University, Aligarh (U P), India.

(**) Ricevuto: 5-IV-1968.

2. - The proving theorems.

HARDY [4] proved the following result concerning the FOURIER coefficients of a function of L^p class, $p \geqslant 1$.

Theorem A. If $a_1, a_2, \dots, a_n$ be the FOURIER coefficients of a function $f(x) \in L^p$, $p \geqslant 1$, then $A_1, A_2, \dots, A_n$ are also FOURIER coefficients of a function of L^p class, where

$$A_n = (1/n) \sum_{k=0}^{n-1} a_k .$$

In the present paper we shall examine how for the above result of HARDY remains the true for WALSH-FOURIER series of functions of L^p, B and S′ classes. The first three theorems are connected with WALSH-FOURIER coefficients while the last theorem is based on FOURIER series with respect to orthonormal system of functions. In what follows we prove the following theorems:

Theorem 1. *If* $\sum_0^\infty c_k \psi_k(x)$ *is the Walsh-Fourier series of a function* $f(x) \in L^p(0, 1)$, $1 < p < 2$, *then* $\sum_{k=0}^\infty \alpha_{2^k} \psi_k(x)$ *is the Walsh-Fourier series of a function* $F(x) \in L^q$, $(1/p) + (1/q) = 1$.

Theorem 2. *If* $\sum_0^\infty c_k \psi_k(x)$ *is the Walsh-Fourier series of a function* $f(x) \in B$, *then* $\sum_{k=0}^\infty \alpha_{2^k} \psi_k(x)$ *is the Walsh-Fourier series of a function* $F(x) \in B$.

Theorem 3. *If* $\sum_{k=0}^\infty c_k \psi_k(x)$ *is the series of class* S′ *with function* $F(x)$ *satisfying the condition*

$$\int_0^t | \mathrm{d}F(x) | = O(t) , \qquad t \to 0 ,$$

then

$$\sum_{k=0}^\infty \alpha_{2^k} \psi_k(x)$$

belongs to S′.

Theorem 4. *Suppose* $f(x) \in L^2$ *and* $f(x) \sim \sum_{1}^{\infty} c_k^* \; \varphi_k(x)$, $\varphi_k(x)$ *is an orthonormal system, then* $\alpha_n^* = (1/n) \sum_{0}^{n-1} c_k^*$ *is also Fourier coefficient of a function* $F(x) \in \mathrm{L}^2$ *with respect to this orthonormal system.*

3. - Lemmas for the proof of theorems.

We require the following lemmas for the proof of these theorem.

Lemma 1 (see [3]).

$$\sum_{k=0}^{2^n-1} \psi_k(x) = \begin{cases} 2^n & \text{in } (0,\ 2^{-(n+1)}) \\ 2^n & \text{in } (2^{-(n+1)},\ 2^{-n}) \\ 0 & \text{in } (2^{-n},\ 1)\,. \end{cases}$$

Lemma 2 (see [5]). A necessary and sufficient condition that a Walsh series should belong to the class B of essentially bounded periodic functions on $[0, 1]$, is the existence of a constant M such that the (C, 1) mean σ_k of the series satisfies $|\sigma_k(x)| \leqslant M$ for all k and all x.

Lemma 3 (see [5]). $\sum_{k=0}^{\infty} a_k \; \psi_k(x) \in \mathrm{S}'$ if and only if

$$\int_0^1 |\sigma_n(x)| \, \mathrm{d}x = O(1)$$

and

$$(1/n) \sum_{k=0}^{n} a_k \; \psi_k(x) \to 0,$$

uniformly in $[0, 1]$, σ_k is the k-th (C, 1) mean of the Walsh series.

Lemma 4 (see [2]). If $\sum_{1}^{\infty} b_k^2 < \infty$, then $\sum_{1}^{\infty} (B_k)^2 < \infty$, where

$$B_k = (1/k) \sum_{v=0}^{k-1} b_v\,.$$

4. - Proof of Theorem 1.

We have

$$c_k = \int_0^1 f(x)\, \psi_k(x)\, \mathrm{d}x$$

and therefore, by virtue of Lemma 1,

$$\begin{aligned}
\alpha_{2^n} &= 2^{-n} \sum_{k=0}^{2^n-1} c_k \\
&= 2^{-n} \sum_{k=0}^{2^n-1} \int_0^1 f(x)\, \psi_k(x)\, \mathrm{d}x \\
&= 2^{-n} \int_0^1 f(x) \sum_{k=0}^{2^n-1} \psi_k(x)\, \mathrm{d}x \\
&= 2^{-n} \left[\int_0^{2^{-(n+1)}} + \int_{2^{-(n+1)}}^{2^{-n}} + \int_{2^{-n}}^1 \right] f(x) \sum_{k=0}^{2^n-1} \psi_k(x)\, \mathrm{d} \\
&= 2^{-n}\, 2^n \int_0^{2^{-(n+1)}} f(x)\, \mathrm{d}x + 2^{-n}\, 2^n \int_{2^{-(n+1)}}^{2^{-n}} f(x)\, \mathrm{d}x \\
&= \int_0^{2^{-n}} f(x)\, \mathrm{d}x\,.
\end{aligned}$$

Applying HOLDER's inequality, we have

$$\begin{aligned}
|\, \alpha_{2^n} | &\leqslant \left[\int_0^{2^{-n}} |\, f(x)\, |^p\, \mathrm{d}x \right]^{1/p} \left[\int_0^{2^{-n}} 1\, \mathrm{d}x \right]^{1/p}, \qquad (1/p) + (1/q) = 1, \\
&= O(2^{-(n/q)})\,,
\end{aligned}$$

since $f(x) \in L^p(0,\ 1)$, $1 < p < 2$, so that

$$\left[\sum_{k=1}^{\infty} |\, \alpha_{2^n} |^p \right]^{1/p} = \left[\sum_{n=1}^{\infty} \{ O(2^{-(n/q)}) \}^p \right]^{1/p} < \infty\,.$$

By virtue of REISZ's theorem (see [1]) we conclude that α_{2^n} is the WALSH-FOURIER coefficient of a function belonging to class L^q.

This completes the proof of Theorem 1.

5. - Proof of Theorem 2.

Consider the series $\sum_0^\infty \alpha_{2^k}\, \psi_k(x)$ with

$$\alpha_{2^n} = 2^{-n} \sum_0^{2^n-1} c_k .$$

Denoting the m-th (C, 1) mean and m-th partial sum of this series by $\sigma_m(x, f)$ and $S_m(x, f)$ respectively, we have

$$| \sigma_m(x, f) | = (1/m) \left| \left[\sum_{v=0}^{m-1} S_v(x, f)\right] \right|$$

$$= (1/m) \left[\sum_{v=0}^{m-1} \sum_{k=0}^{v-1} \alpha_{2^k}\, \psi_k(x) \right|$$

$$\leqslant (1/m) \left[\sum_{v=0}^{m-1} \sum_{k=0}^{v-1} 2^{-k} \left| \sum_{r=0}^{2^k-1} c_r \right| | \psi_k(x) | \right]$$

$$\leqslant (1/m) \left[\sum_{v=0}^{m-1} \sum_{k=0}^{v-1} 2^{-k} \left| \sum_{r=0}^{2^k-1} \int_0^1 f(t)\, \psi_r(t)\, dt \right| \right]$$

$$\leqslant (1/m) \left[\sum_{v=0}^{m-1} \sum_{k=0}^{v-1} 2^{-k} \int_0^1 | f(t) | \left| \sum_{r=0}^{2^k-1} \psi_r(t) \right| dt \right]$$

$$\leqslant (M/m) \left[\sum_{v=0}^{m-1} \sum_{k=0}^{v-1} 2^{-k} \int_0^1 \left| \sum_{r=0}^{2^k-1} \psi_r(t) \right| dt \right]$$

$$= (M/m) \left| \sum_{v=0}^{m-1} \sum_{k=0}^{v-1} 2^{-k} \left\{ \int_0^{2^{-(k+1)}} + \int_{2^{-(k+1)}}^{2^{-k}} + \int_{2^{-k}}^1 \right\} \right| \sum_{r=0}^{2^k-1} \psi_r(t)\, dt,$$

since $f(x) \in B$.

By virtue of Lemma 1 we conclude that

$$| \sigma_m(x, f) | \leqslant (M/m) \left[\sum_{v=0}^{m-1} \sum_{k=0}^{v-1} \int_0^{2^{-k}} dt \right] =$$

$$= (M/m) \left[\sum_{v=0}^{m-1} \sum_{k=0}^{v-1} 2^{-k} \right] = (M/m) \left[\sum_{v=0}^{m-1} O(1) \right] = (M/m) [O(m)] = O(1) ,$$

for all m and x.

Appling Lemma 2, we get the required result.

6. - Proof of Theorem 3.

Denoting by $\sigma_m(x)$ the (C, 1) mean of the series $\sum_0^\infty \alpha_{2^k}\ \psi_k(x)$, we have

$$|\sigma_m(x)| = (1/m)\left|\sum_{v=0}^{m-1}\sum_{k=0}^{v-1}\left\{2^{-k}\sum_{r=0}^{2^k-1}\int_0^1 \psi_r(t)\ \mathrm{d}F(t)\right\}\psi_k(x)\right|$$

so that

$$\int_0^1 |\sigma_m(x)|\ \mathrm{d}x \leqslant (1/m)\int_0^1 \sum_{v=0}^{m-1}\sum_{k=0}^{v-1}\left\{\left|2^{-k}\sum_{r=0}^{2^k-1}\int_0^1 \psi_r(t)\ \mathrm{d}F(t)\right|\mathrm{d}x\right\}$$

$$= (1/m)\sum_{v=0}^{m-1}\sum_{k=0}^{v-1}\left\{2^{-k}\left|\int_0^{2^{-(k+1)}}\sum_{r=0}^{2^k-1}\psi_r(t)\ \mathrm{d}F(t) + 2^{-k}\int_{2^{-(k+1)}}^{2^{-k}}\sum_{r=0}^{2^k-1}\psi_r(t)\ \mathrm{d}F(t) + \right.\right.$$

$$\left.\left. + 2^{-k}\int_{2^{-k}}^1\sum_{r=0}^{2^k-1}\psi_r(t)\ \mathrm{d}F(t)\right|\right\}.$$

By virtue of Lemma 1 and the hypothesis we have

$$\int_0^1 |\sigma_m(x)|\ \mathrm{d}x \leqslant (1/m)\left\{\sum_{v=0}^{m-1}\sum_{k=0}^{v-1}\int_0^{2^{-k}}|\ \mathrm{d}F(t)\ |\right\} =$$

$$= (1/m)\sum_{v=0}^{m-1}\sum_{k=0}^{v-1} O(2^{-k}) = (1/m)\sum_{v=0}^{m-1} O(1) = O(1).$$

Also

$$\left|(1/n)\sum_{k=0}^{n-1}\alpha_{2^k}\ \psi_k(x)\right| = (1/n)\left|\sum_{k=0}^{n-1} 2^{-k}\ \left(\sum_{r=0}^{2^k-1} c_r\right)\ \psi_k(x)\right|$$

$$\leqslant (1/n)\sum_{k=0}^{n-1} 2^{-k}\left|\sum_{r=0}^{2^k-1}\left(\int_0^1 \psi_r(t)\ \mathrm{d}F(t)\right|\ |\ \psi_k(x)\ | \leqslant (1/n)\sum_{k=0}^{n-1} 2^{-k}\left(\int_0^1\left|\sum_{r=0}^{2^k-1}\psi_r(t)\right|\ |\ \mathrm{d}F(t)\ |\right)$$

$$= (1/n)\sum_{k=0}^{n-1} O(2^{-k}) = o(1)\ , \qquad n\to\infty,$$

uniformly in x.

Hence, by virtue of Lemma 3, $\sum_0^\infty \alpha_{2^k}\ \psi_k(x)$ belongs to S′.

This completes the proof of Theorem 3.

7. - Proof of Theorem 4.

Since $f(x) \in L^2$, by BESSEL's inequality the series

$$\sum_{k=1}^{\infty} c_{k^2}^{*} < \infty .$$

Applying Lemma 2 we obtain that

$$\sum_{k=1}^{\infty} \alpha_k^{*2} < \infty .$$

Now applying RIESZ-FISCHER theorem we conclude that α_n^* is FOURIER coefficient with respect to orthonormal system $\varphi_k(x)$ of a function $F(x) \in L^2$.

References.

[1] N. K. BARY, *A Treatise on Trigonometric Series*, Vol. 1, Pergamon Press, Oxford 1964 (see pp. 218-219).

[2] E. B. ELLIOTT, *A simple exposition of some recently proved facts as to convergency*, J. London Math. Soc. **1** (1926), 93-96.

[3] N. J. FINE, *On the Walsh-functions*, Trans. Amer. Math. Soc. **65** (1949), 372-414.

[4] G. H. HARDY, *Note on some points in integral calculus*, Messenger **58** (1929), 50-52.

[5] G. W. MORGENTHALER, *On Walsh-Fourier series*, Trans. Amer. Math. Soc. **84** (1957), 472-507.

* * *