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On the Order and Type of Entire Functions of Several Complex Variables. (**)

1. - Introduction.

In this paper we shall obtain the relations among entire functions of finite non-zero orders and types and study the relations among the coefficients in the TAYLOR expansion of entire functions and their orders and types. For simplicity we confine our selves to the case of two complex variables. The case of an arbitrary finite number of variables is examined in the same way.

Let

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n \tag{1.1}$$

be a function of two complex variables z_1 and z_2, where the coefficients $a_{m,n}$ are complex numbers. The series (1.1) represents an entire function of two complex variables z_1, z_2 if it converges absolutely for all values of $|z_1| < \infty$ and $|z_2| < \infty$.

M. M. DŽRBAŠYAN ([1], p. 1) has shown that the necessary and sufficient condition for the series (1.1) to represent an entire function of variables z_1 and z_2 is

$$\limsup_{m+n\to\infty} |a_{m,n}|^{1/(m+n)} = 0. \tag{1.2}$$

Let $\overline{G}_r$ be a family of closed polycircular domains in space (z_1, z_2) depen-

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dent on parameter $r > 0$ and possess the property that $(z_1, z_2) \in \overline{G}_r$ if and only if $(z_1/r, z_2/r) \in \overline{G}_r$. The maximum modulus of the entire function $f(z_1, z_2)$ is denoted by

$$M_G(r, f) = \max_{(z_1, z_2) \in \overline{G}_r} | f(z_1, z_2) |,$$

and the function will be called G-order and G-type respectively, if

$$\varrho_G = \limsup_{r \to \infty} \frac{\log \log M_G(r, f)}{\log r}, \tag{1.3}$$

$$T_G = \limsup_{r \to \infty} \frac{\log M_G(r, f)}{r^{\varrho_G}}. \tag{1.4}$$

Set

$$\Phi = \Phi_G(m, n) = \max_{(z_1, z_2) \in \overline{G}_r} (| z_1 |^m | z_2 |^n), \tag{1.5}$$

A. A. GOL'DBERG ([2], p. 146) has proved the following theorems:

Theorem A. All orders ϱ_G be equal and

$$\varrho = \varrho_G = \limsup_{m+n \to \infty} \frac{(m+n) \log (m+n)}{\log (1/| a_{m,n} |)}. \tag{1.6}$$

Theorem B. G-type T_G satisfies the correlation

$$(e \varrho T_G)^{1/\varrho} = \limsup_{m+n \to \infty} [(m+n)^{1/\varrho} \{\Phi_G(m, n) | a_{m,n} |\}^{1/(m+n)}], \tag{1.7}$$

or, by (1.5),

$$(e \varrho T)^{1/\varrho} = \limsup_{m+n \to \infty} [(m+n)^{1/\varrho} \{\Phi \ | a_{m,n} |\}^{1/(m+n)}]. \tag{1.8}$$

2. – Theorem 1. *Let*

$$f_k(z_1, z_2) = \sum_{m,n=0}^{\infty} (a_{m,n})_k z_1^m z_2^n \qquad (k = 1, 2, \ldots, p)$$

be p entire functions of finite non-zero orders $\varrho_1, \varrho_2, \ldots, \varrho_p$ respectively. Then *the function*

$$f(z_1\,,\ z_2) = \sum_{m,\,n=0}^{\infty} a_{m,n}\, z_1^m\, z_2^n\,,$$

where

$$\{\log(1/|\,a_{m,n}\,|)\}^{-1} \sim \sum_{k=1}^{p} \alpha_k \{\log(1/|\,(a_{m,n})_k\,|)\}^{-1}$$

$$(0 < \alpha_k < 1, \qquad \alpha_1 + \alpha_2 + \ldots + \alpha_p = 1)\,,$$

is an entire function such that

$$\varrho \leqslant \sum_{k=1}^{p} \alpha_k\, \varrho_k \qquad [\varrho \text{ is the order of } f(z_1, z_2)].$$

Proof. Since $f_k(z_1\,,\ z_2)$ is an entire function. Therefore, using (1.2), we have, for an arbitrary ε and enough large R,

$$1/|\,(a_{m,n})_k\,| > (R-\varepsilon)^{m+n} \qquad \text{for } m+n > k_k\,,$$

or

$$\log\{1/|\,(a_{m,n})_k\,|\} > (m+n)\log(R-\varepsilon) \qquad \text{for } m+n > k_k\,,$$

or

$$\alpha_k\{\log(1/|\,(a_{m,n})_k\,|)\}^{-1} < \alpha_k/\{(m+n)\log(R-\varepsilon)\} \qquad \text{for } m+n > k_k\,.$$

Putting $k = 1, 2, \ldots, p$ and adding the p inequalities thus obtained, we get, for large $m+n$,

$$\sum_{k=1}^{p} \alpha_k\{\log(1/|\,(a_{m,n})_k\,|)\}^{-1} < (\alpha_1+\alpha_2+\ldots+\alpha_p)/\{(m+n)\log(R-\varepsilon)\}$$

$$[m+n > k = \max(k_1\,,\ k_2\,,\ \ldots,\ k_p)]\,.$$

Thus, if

$$\{\log(1/|\,a_{m,n}\,|)\}^{-1} \sim \sum_{k=1}^{p} \alpha_k\{\log(1/|\,(a_{m,n})_k\,|)\}^{-1}\,, \qquad \sum_{k=1}^{p} \alpha_k = 1\,.$$

Then, for large $m+n$,

$$\{\log(1/|a_{m,n}|)\}^{-1} < 1/\{(m+n)\log(R-\varepsilon)\} \qquad \text{for } m+n > k\,,$$

or

$$\log(1/|a_{m,n}|) > (m+n)\log(R-\varepsilon) \qquad \text{for } m+n > k\,,$$

or

$$\limsup_{m+n\to\infty} |a_{m,n}|^{1/(m+n)} = 0\,.$$

Hence $f(z_1, z_2)$ is an entire function.

Now using (1.6) for the function $f_k(z_1, z_2)$, we have

$$\limsup_{m+n\to\infty} \frac{(m+n)\log(m+n)}{\log\{1/|(a_{m,n})_k|\}} = \varrho_k\,.$$

Therefore, for an arbitrary ε we get

$$\alpha_k/\{\log(1/|(a_{m,n})_k|)\} < \alpha_k(\varrho_k+\varepsilon)\{(m+n)\log(m+n)\}^{-1} \qquad \text{for } m+n > k_k\,,$$

or

$$\alpha_k\{\log(1/|(a_{m,n})_k|)\}^{-1} < \alpha_k(\varrho_k+\varepsilon)\{(m+n)\log(m+n)\}^{-1} \qquad \text{for } m+n > k_k\,.$$

Putting $k=1, 2, \ldots, p$ and adding the p inequalities thus obtained, we get

$$\sum_{k=1}^{\infty}\alpha_k\{\log(1/|(a_{m,n})_k|)\}^{-1} < \sum_{k=1}^{p}\alpha_k(\varrho_k+\varepsilon)\{(m+n)\log(m+n)\}^{-1}$$

$$[m+n > k = \max(k_1, k_2, \ldots, k_p)]\,.$$

Since

$$\{\log(1/|a_{m,n}|)\}^{-1} \sim \sum_{k=1}^{p}\alpha_k\{\log(1/|(a_{m,n})_k|)\}^{-1}\,,$$

and we have

$$\limsup_{m+n\to\infty} \frac{(m+n)\log(m+n)}{\log(1/|a_{m,n}|)} \leqslant \sum_{k=1}^{p}\alpha_k\varrho_k\,.$$

Hence we get

$$\varrho \leqslant \sum_{k=1}^{p} \alpha_k \varrho_k ,$$

where ϱ is the order of $f(z_1, z_2)$.

3. – Theorem 2. *Let*

$$f_k(z_1, z_2) = \sum_{m,n=0}^{\infty} (a_{m,n})_k z_1^m z_2^n \qquad (k = 1, 2, ..., p)$$

be p entire functions of finite non-zero orders $\varrho_1, \varrho_2, ..., \varrho_p$ respectively. Then the function

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n ,$$

where

$$\log (1/|a_{m,n}|) \sim \prod_{k=1}^{p} \{\log (1/|(a_{m,n})_k|)\}^{\alpha_k} \qquad (0 < \alpha_k < 1, \quad \sum_{k=1}^{p} \alpha_k = 1) ,$$

is an entire function such that

$$\varrho \leqslant \prod_{k=1}^{p} \varrho_k^{\alpha_k} ,$$

where ϱ is the order of $f(z_1, z_2)$.

Proof. Since $f_k(z_1, z_2)$ is an entire function, therefore, using (1.2), we have for arbitrary ε and enough large R

$$1/|(a_{m,n})_k| > (R - \varepsilon)^{m+n} \qquad \text{for } m + n > k_k ,$$

or

$$\log \{1/|(a_{m,n})_k|\} > (m + n) \log(R - \varepsilon) \qquad \text{for } m + n > k_k ,$$

or

$$\{\log (1/|(a_{m,n})_k|)\}^{\alpha_k} > (m + n)^{\alpha_k} [\log(R - \varepsilon)]^{\alpha_k} \qquad \text{for } m + n > k_k .$$

Putting $k = 1, 2, \ldots, p$ and multiplying the p inequalities thus obtained, we get, for large $m+n$,

$$\prod_{k=1}^{p}\{\log(1/|(a_{m,n})_k|)\}^{\alpha_k} > (m+n)\log(R-\varepsilon)\,.$$

Therefore, if

$$\log(1/|a_{m,n}|) \sim \prod_{k=1}^{p}\{\log(1/|(a_{m,n})_k|)\}^{\alpha_k}\,,$$

then, for large $m+n$,

$$\log(1/|a_{m,n}|) > (m+n)\log(R-\varepsilon)\,.$$

Hence

$$\limsup_{m+n\to\infty}|a_{m,n}|^{1/(m+n)} = 0\,,$$

and $f(z_1, z_2)$ is an entire function.

Now using (1.6) for the function $f_k(z_1, z_2)$, we have

$$\limsup_{m+n\to\infty}\frac{(m+n)\log(m+n)}{\log\{1/|(a_{m,n})_k|\}} = \varrho_k\,.$$

Therefore, for an arbitrary ε we get

$$1/\{\log(1/|(a_{m,n})_k|)\} < (\varrho_k+\varepsilon)\{(m+n)\log(m+n)\}^{-1} \qquad \text{for } m+n > k_k\,,$$

or

$$\{\log(1/|(a_{m,n})_k|)\}^{-\alpha_k} < (\varrho+\varepsilon)^{\alpha_k}\{(m+n)\log(m+n)\}^{-\alpha_k} \qquad \text{for } m+n > k_k\,.$$

Putting $k = 1, 2, \ldots, p$ and multiplying the p inequalities thus obtained, we get, for large $m+n$,

$$\prod_{k=1}^{p}\{\log(1/|(a_{m,n})_k|)\}^{-\alpha_k} < \prod_{k=1}^{p}(\varrho_k+\varepsilon)^{\alpha_k}\{(m+n)\log(m+n)\}^{-1}\,,$$

or

$$\limsup_{m+n\to\infty}\frac{(m+n)\log(m+n)}{\prod_{k=1}^{p}[\log\{1/|(a_{m,n})_k|\}]^{\alpha_k}} \leqslant \prod_{k=1}^{p}\varrho_k^{\alpha_k}.$$

Thus, if

$$\log(1/|a_{m,n}|) \sim \prod_{k=1}^{p}\{\log(1/|(a_{m,n})_k|)\}^{\alpha_k},$$

then

$$\limsup_{m+n\to\infty}\frac{(m+n)\log(m+n)}{-\log|a_{m,n}|}\leqslant\prod_{k=1}^{p}\varrho_k^{\alpha_k}.$$

Hence

$$\varrho\leqslant\prod_{k=1}^{p}\varrho_k^{\alpha_k},$$

where ϱ is the order of $f(z_1, z_2)$.

4. – Theorem 3. *Let*

$$f_k(z_1, z_2)=\sum_{m,n=0}^{\infty}(a_{m,n})_k z_1^m z_2^n \qquad (k=1, 2, ..., p)$$

be p *entire functions of finite non-zero orders* $\varrho_1, \varrho_2, ..., \varrho_p$ *and finite non-zero types* (1) $T_1, T_2, ..., T_p$ *respectively. Then the function*

$$f(z_1, z_2)=\sum_{m,n=0}^{\infty}a_{m,n} z_1^m z_2^n,$$

where

$$\log(1/|\Phi\, a_{m,n}|)\sim\prod_{k=1}^{p}\{\log(1/|\Phi\cdot(a_{m,n})_k|)\}^{\alpha_k} \qquad \left(0<\alpha_k<1, \quad \sum_{k=1}^{p}\alpha_k=1\right),$$

is an entire function such that

$$T\leqslant\prod_{k=1}^{p}(T_k)^{\alpha_k},$$

where ϱ *and* T *are the order and type of* $f(z_1, z_2)$ *respectively provided*

$$\varrho=\prod_{k=1}^{p}\varrho_k^{\alpha_k}.$$

(1) The types $T_1, T_2, ..., T_p$ correspond the same family of polycircular domains $\overline{G}_r$.

Proof. It is easy to prove that $f(z_1, z_2)$ is an entire function. Further, using (1.8), for the function $f_k(z_1, z_2)$ we have

$$\limsup_{m+n\to\infty} \left[(m+n)^{1/\varrho_k}\{\Phi \mid (a_{m,n})_k \mid\}^{1/(m+n)}\right] = (e\,\varrho_k\,T_k)^{1/\varrho_k}.$$

For arbitrary $\varepsilon_k > 0$ and sufficiently large $m + n$, we have

$$(m+n)^{1/\varrho_k}\{\Phi \mid (a_{m,n})_k \mid\}^{1/(m+n)} < \{e\,\varrho_k\,(T_k + \varepsilon_k)\}^{1/\varrho_k} \qquad \text{for } m+n > k_k.$$

Put $e\,\varrho_k\,(T_k + \varepsilon_k) = A_k$,

$$(m+n)^{1/\varrho_k}\{\Phi \mid (a_{m,n})_k \mid\}^{1/(m+n)} < (A_k)^{1/\varrho_k} \qquad \text{for } m+n > k_k,$$

or

$$\log \frac{1}{\Phi \mid (a_{m,n})_k \mid} > \frac{m+n}{\varrho_k} \log \frac{m+n}{A_k} \qquad \text{for } m+n > k_k,$$

or

$$\left\{\log \frac{1}{\Phi \mid (a_{m,n})_k \mid}\right\}^{\alpha_k} > \frac{(m+n)^{\alpha_k}}{\varrho_k^{\alpha_k}} \left\{\log \frac{m+n}{A_k}\right\}^{\alpha_k}$$

for sufficiently large $m + n$.

Putting $k = 1, 2, \ldots, p$ and multiplying p inequalities thus obtained, we have

$$\prod_{k=1}^{p} \left\{\log \frac{1}{\Phi \mid (a_{m,n})_k \mid}\right\}^{\alpha_k} > \prod_{k=1}^{p} \frac{m+n}{\varrho_k^{\alpha_k}} \left\{\log \frac{m+n}{A_k}\right\}^{\alpha_k}$$

for sufficiently large $m + n > k$, or

$$\prod_{k=1}^{p} \left\{\log \frac{1}{\Phi \mid (a_{m,n})_k \mid}\right\}^{\alpha_k} > \frac{m+n}{\varrho} \prod_{k=1}^{p} \left\{\log \frac{m+n}{A_k}\right\}^{\alpha_k} \qquad \text{for } m+n > k$$

$$> \frac{m+n}{\varrho} \prod_{k=1}^{p} \left\{1 - \frac{\log A_k}{\log (m+n)}\right\}^{\alpha_k} \log(m+n)$$

$$> \frac{m+n}{\varrho} \left[1 - \sum_{k=1}^{p} \frac{\log A_k^{\alpha_k}}{\log (m+n)} + O(\log (m+n))^{-2}\right] \log(m+n) \qquad \text{for } m+n > k.$$

Thus, if

$$\log \frac{1}{\Phi \mid a_{m,n} \mid} \sim \prod_{k=1}^{p} \left\{ \log \frac{1}{\Phi \left| (a_{m,n})_k \right|} \right\}^{\alpha_k} \qquad (0 < \alpha_k < 1, \quad \sum_{k=1}^{p} \alpha_k = 1),$$

we obtain, for sufficiently large $m + n$,

$$\log \frac{1}{\Phi \mid a_{m,n} \mid} > \frac{m+n}{\varrho} \left[1 - \sum_{k=1}^{p} \frac{\log A_k}{\log (m+n)} + O\big(\log(m+n)\big)^{-2} \right] \log(m+n),$$

or

$$(m+n) \left\{ \Phi \mid a_{m,n} \mid \right\}^{\varrho/(m+n)} < (m+n)^{\beta}$$

with

$$\beta = \sum_{k=1}^{p} \frac{\log A_k^{\alpha_k}}{\log (m+n)} + O\big(\log(m+n)\big)^{-2}.$$

Since

$$\lim_{m+n \to \infty} (m+n)^{\beta} = \prod_{k=1}^{p} A_k'^{\alpha_k}, \qquad \text{where} \qquad A_k' = e\, \varrho_k\, T_k\,,$$

we obtain

$$\limsup_{m+n \to \infty} \left\{ (m+n)\, (\Phi \mid a_{m,n} \mid)^{\varrho/(m+n)} \right\} \leqslant \prod_{k=1}^{p} (e\, \varrho_k\, T_k)^{\alpha_k}.$$

Hence

$$e\, \varrho\, T \leqslant \prod_{k=1}^{p} (e\, \varrho_k\, T_k)^{\alpha_k},$$

or

$$T \leqslant \prod_{k=1}^{p} (T_k)^{\alpha_k}, \qquad \text{where } T \text{ is type of } f(z_1, z_2).$$

5. – Theorem 4. *Let*

$$f_k(z_1, z_2) = \sum_{m,n=0}^{p} (a_{m,n})_k\, z_1^m\, z_n^z \qquad (k = 1, 2, \ldots, p)$$

be p entire functions of finite non-zero orders $\varrho_1, \varrho_2, \ldots, \varrho_k$ and finite non-zero types $T_1, T_2 \ldots, T_p$ respectively. Then the function

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} (a_{m,n})_1^{\alpha_1} (a_{m,n})_2^{\alpha_2} \ldots (a_{m,n})_p^{\alpha_p} z_1^m z_2^n$$

$$(0 < \alpha_k < 1, \quad \alpha_1 + \alpha_2 + \ldots + \alpha_p = 1)$$

is an entire function such that $(\varrho T)^{1/\varrho} \leqslant \prod_{k=1}^{p} (\varrho_k T_k)^{\alpha_k/\varrho_k}$, where ϱ and T are the order and type of $f(z_1, z_2)$ respectively provided $1/\varrho = \sum_{k=1}^{p} \alpha_k/\varrho_k$.

Proof. It is easy to prove that $f(z_1, z_2)$ is an entire function. Using (1.8) for the function $f_k(z_1, z_2)$, we have

$$\limsup_{m+n\to\infty} [(m+n)^{1/\varrho_k} \{\Phi \mid (a_{m,n})_k \mid\}^{1/(m+n)}] = (e \varrho_k T_k)^{1/\varrho_k},$$

or

$$\limsup_{m+n\to\infty} [(m+n)^{1/\varrho_k} \{\Phi \mid (a_{m,n})_k \mid\}^{1/(m+n)}]^{\alpha_k} = (e \varrho_k T_k)^{\alpha_k/\varrho_k}.$$

Putting $k = 1, 2, \ldots, p$ and multiplying p inequalities thus obtained, we get

$$\prod_{k=1}^{p} \limsup_{m+n\to\infty} [(m+n)^{1/\varrho_k} \{\Phi \mid (a_{m,n})_k \mid\}^{1/(m+n)}]^{\alpha_k} = \prod_{k=1}^{p} (e \varrho_k T_k)^{\alpha_k/\varrho_k},$$

or

$$\limsup_{m+n\to\infty} \prod_{k=1}^{p} [(m+n)^{\alpha_k/\varrho_k} \{\Phi \mid (a_{m,n})_k \mid\}^{\alpha_k/(m+n)}] \leqslant \prod_{k=1}^{p} (e \varrho_k T_k)^{\alpha_k/\varrho_k},$$

or

$$\limsup_{m+n\to\infty} [(m+n)^{1/\varrho_k} \{\Phi \prod_{k=1}^{p} |(a_{m,n})_k^{\alpha_k} |\}^{1/(m+n)}] \leqslant \prod_{k=1}^{p} (e \varrho_k T_k)^{\alpha_k/\varrho_k}.$$

Hence again using (1.8) for $f(z_1, z_2)$, we obtain

$$(e \varrho T)^{1/\varrho} \leqslant \prod_{k=1}^{p} (e \varrho_k T_k)^{\alpha_k/\varrho_k},$$

or

$$(\varrho\, T)^{1/\varrho} \leqslant \prod_{k=1}^{p} (\varrho_k\, T_k)^{\alpha_k/\varrho_k} ,$$

where ϱ and T are order and type of $f(z_1, z_2)$.

Corollary. *If all $f_k(z_1, z_2)$ are of same finite non-zero order, then*

$$T \leqslant \prod_{k=1}^{p} (T_k)^{\alpha_k} .$$

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References.

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Summary.

In this paper we obtain certain relationships among entire functions of finite non-zero orders and types. Further, we study the relations among the coefficients in the Taylor expansion of entire functions and their orders and types.

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