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On Generalized Stirling Numbers and Polynomials. (**)

1. - Introduction.

In a recent paper R. P. SINGH [5] has defined the generalised STIRLING numbers and polynomials as

$$S^{\alpha}(n,\ k,\ r) = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\alpha + rj)^n \tag{1.1}$$

and

$$T_n^{\alpha}(x,\ r,\ -p) = \sum_{k=0}^{n} S^{\alpha}(n,\ k,\ r)\, p^k\, x^{rk}\,, \tag{1.2}$$

which reduces to STIRLING numbers $S(n,\ k)$ when $\alpha = 0,\ r = 1$ and STIRLING polynomials $A_n(x)$ when $\alpha = 0,\ r = p = 1$ and generalised TRUESDEL polynomials on changing $-p$ to p.

Again numbers similar to the STIRLING numbers have been defined by NIELSEN [3] and CARLITZ [1] respectively as

$$C_t^k(\gamma) = \sum_{j=0}^{t} (-1)^j \binom{t}{j} (\gamma - j)^k \tag{1.3}$$

and

$$A_t^k(\gamma) = \binom{\gamma}{t} C_t^k(\gamma)\,. \tag{1.4}$$

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We immediately see that

$$C_t^k(\gamma) = (-1)^t \, t! \, S^\gamma(k, \; t, \; -1) \tag{1.5}$$

and

$$A_t^k(\gamma) = \binom{\gamma}{t} t! \, (-1)^t \, S^\gamma(k, \; t, \; -1) \, . \tag{1.6}$$

The object of this paper is to study some further properties of the generalised STIRLING numbers and polynomials.

2. – It is familiar that the formulas

$$g(n) = \sum_{d/n} f(d) \qquad (n = 1, \; 2, \; ...) \tag{2.1}$$

and

$$f(n) = \sum_{cd=n} \Gamma(c) \, g(d) \, , \tag{2.2}$$

where $\Gamma(c)$ is the MOBIOUS function are equivalent.

Now from (2.1) and (2.2), it can be easily verified that

$$g_r = \sum_{j=0}^{r} \binom{r}{j} f_j \qquad (r = 0, \; 1, \; 2, \; ...) \tag{2.3}$$

and

$$f_r = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} g_j \, , \tag{2.4}$$

which give us, with the help of (1.1),

$$(\alpha + r \, j)^n = \sum_{i=0}^{j} \binom{j}{i} i! \, S^\alpha(n, \; i, \; r) \tag{2.5}$$

and

$$(\alpha + rj)^n = \sum_{k=0}^{n} (-1)^k j^k \, S^\alpha(n, \; k, \; r) \, . \tag{2.6}$$

Now

$$
(2.7)\quad \begin{cases} S^{\alpha+1}(n,\ k,\ r) = \dfrac{(-1)^k}{k!}\displaystyle\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\alpha+1+r\,j)^n \\[2ex] \qquad = \dfrac{(-1)^k}{k!}\displaystyle\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{i=0}^{n}\binom{n}{i}(\alpha+r\,j)^i \\[2ex] \qquad = \displaystyle\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\alpha+rj)^i = \sum_{i=0}^{n}\binom{n}{i}S^{\alpha}(i,\ k,\ r)\,. \end{cases}
$$

Again

$$
\sum_{i=0}^{n} r^{n-i}\binom{n}{i}S^{\alpha}(i,\ k-1,\ r) =
$$

$$
= \sum_{i=0}^{n}\binom{n}{i}r^{n-i}\frac{(-1)^{k-1}}{(k-1)!}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(\alpha+r\,j)^i
$$

$$
= \frac{(-1)^{k-1}}{(k-1)!}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\sum_{i=0}^{n}\binom{n}{i}r^{n-i}(\alpha+r\,j)^i
$$

$$
= \frac{(-1)^{k-1}}{(k-1)!}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}[\alpha+r\,(j+1)]^n
$$

$$
= \frac{(-1)^{k-1}}{(k-1)!}\sum_{j=1}^{k}(-1)^{j-1}\frac{(k-1)!}{(k-j)!\,(j-1)!}(\alpha+r\,j)^n
$$

$$
= \frac{(-1)^{k-1}}{k!}\sum_{j=1}^{k}(-1)^{j-1}\frac{k!\,j}{(k-j)!\,j!}(\alpha+r\,j)^n
$$

$$
= \frac{(-1)^k}{k!\,r}\sum_{j=1}^{k}(-1)^j\binom{k}{j}(\alpha+rj-\alpha)(\alpha+r\,j)^n
$$

$$
= \frac{(-1)^k}{k!\,r}\sum_{j=1}^{k}(-1)^j\binom{k}{j}(\alpha+r\,j)^{n+1} - \frac{(-1)^k\,\alpha}{k!\,r}\sum_{j=1}^{k}(-1)^j\binom{k}{j}(\alpha+rj)^n
$$

$$
= \frac{(-1)^k}{k!\,r}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\alpha+rj)^{n+1} - \frac{(-1)^k}{k!}\frac{\alpha}{r}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(\alpha+rj)^n
$$

$$
= \frac{1}{r}S^{\alpha}(n+1,\ k,\ r) - \frac{\alpha}{r}S^{\alpha}(n,\ k,\ r)\,.
$$

Thus

$$r \sum_{i=0}^{n} \binom{n}{i} r^{n-i} S^{\alpha}(i,\ k-1,\ r) = S^{\alpha}(n+1,\ k,\ r) - \alpha\, S^{\alpha}(n,\ k,\ r)\,, \tag{2.8}$$

which is the generalization of the familiar result

$$S(n+1,\ k) = \sum_{i=0}^{n} \binom{n}{i} S(i,\ k-1)\,.$$

3. - Generating functions.

Consider

$$\sum_{n=0}^{\infty} S^{\alpha}(n,\ k,\ r) \frac{t^n}{n\,!} = \sum_{n=0}^{\infty} \frac{t^n}{n\,!} \frac{(-1)^k}{k\,!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (\alpha + r\,i)^n$$

$$= \frac{(-1)^k}{k\,!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} e^{(\alpha+ri)t} = \frac{(-1)^k}{k\,!} e^{\alpha t} (1 - e^{rt})^k\,.$$

Thus

$$\sum_{n=0}^{\infty} S^{\alpha}(n,\ k,\ r) \frac{t^n}{n\,!} = \frac{(-1)^k}{k\,!} e^{\alpha t} (1 - e^{rt})^k\,. \tag{3.1}$$

In a similar manner we can prove

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} S^{\alpha}(m+n,\ k,\ r) \frac{u^m v^n}{m\,!\, n\,!} = \frac{(-1)^k}{k\,!} e^{(u+v)\alpha} \left[1 - e^{r(u+v)}\right]^k. \tag{3.2}$$

4. - Some congruences.

With the help of (2.7) we get the following congruences:

$$S^{\alpha}(n+1,\ 2s,\ r) \equiv r\, S^{\alpha}(n,\ 2s-1,\ r) + \alpha\, S^{\alpha}(n,\ 2s,\ r)\,, \tag{4.1}$$

$$S^{\alpha}(n+1,\ 2s+1,\ r) \equiv r\, S^{\alpha}(n,\ 2s,\ r) + (\alpha + r)\, S^{\alpha}(n,\ 2s+1,\ r)\,, \tag{4.2}$$

and

$$S^\alpha(n+1,\ 2s+1,\ r) \equiv \tag{4.3}$$

$$\equiv r^2\, S^\alpha(n-1,\ 2s-1,\ r) + r\,\alpha\, S^\alpha(n-1,\ 2s,\ r) + (\alpha + r)\, S^\alpha(n,\ 2s+1,\ r)\,,$$

which are the generalizations of the congruences given by CARLITZ [2].

5. – In this section we prove

$$T_n^\alpha(n,\ r,\ -p) = e^{-px^r} \sum_{j=0}^{\infty} \frac{p^j\, x^{rj}}{j\,!}\,(\alpha + rj)^n\,. \tag{5.1}$$

To prove, we have

$$e^{-px^r} \sum_{j=0}^{\infty} \frac{p^j\, x^{rj}}{j\,!}\,(\alpha + rj)^n =$$

$$= \sum_{m=0}^{\infty} \frac{(-p)^m\, x^{rm}}{m\,!} \sum_{j=0}^{\infty} \frac{p^j\, x^{rj}}{j\,!}\,(\alpha + rj)^n = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} (-1)^m \frac{p^{m+j}}{m\,!\,j\,!}\, x^{r(m+j)}(\alpha + r\,j)^n$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{m} (-1)^{m-j} \frac{p^m\, x^{rm}}{j\,!\,(m-j)\,!}\,(\alpha + rj)^n = \sum_{m=0}^{\infty} \frac{p^m\, x^{rm}}{j\,!}\,\frac{(-1)^m}{m\,!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (\alpha + rj)^n$$

$$= \sum_{m=0}^{\infty} p^m\, x^{rm}\, S^\alpha(n,\ m,\ r) = T_n^\alpha(n,\ r,\ -p)\,.$$

Again R. P. SINGH [5] has proved

$$\sum_{n=0}^{\infty} T_n^\alpha(x,\ r,\ -p)\ \frac{t^n}{n\,!} = \exp\left[\alpha\, t - p\, x^r\,(1 - e^{rt})\right]\,. \tag{5.2}$$

Differentiating it with respect to x, we get

$$\mathrm{D} T_n^\alpha(x,\, r,\, -p) + r\, p\, x^{r-1}\, T_n^\alpha(x,\, r,\, -p) = r\, p\, x^{r-1} \sum_{m=0}^{n} \binom{n}{m} x\, r^{n-m}\, T_m^\alpha(x,\ r,\ -p)\,, \tag{5.3}$$

where $\mathrm{D} = \mathrm{d}/\mathrm{d}x$, and by iteration we can get a result for $\mathrm{D}^q\, T_n^\alpha(x,\ r,\ -p)$.

Also, if we differentiate both sides of (5.2) with respect to t, we have

$$(5.4) \qquad T_{n+1}^{\alpha}(x, r, -p) = \alpha\, T_n^{\alpha}(x, r, -p) + r\, p\, x^r \sum_{m=0}^{n} \binom{n}{m} r^{n-m}\, T_m^{\alpha}(n, r, -p),$$

which on combination with (5.3) gives

$$(5.5) \qquad T_{n+1}^{\alpha}(x, r, -p) = x\, \mathrm{D} T_n^{\alpha}(x, r, -p) + (\alpha + r\, p\, x^r)\, T_n^{\alpha}(x, r, -p).$$

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