Riv. Mat. Univ. Parma (2) 9 (1968), 155-172

RONALD B. GUENTHER (*)

Representation, Uniqueness and Isolated Singularity Theorems for Solutions to Linear, Parabolic Equations of Higher Order. (**)

In this paper we first derive representation and uniqueness theorems for solutions to linear, parabolic equations of higher order with variable coefficients. These theorems are proved by making use of some rather precise estimates on the fundamental solution. Using these representation and uniqueness theorems, we are then able to give theorems concerning the behavior of solutions to parabolic equations in the neighborhood of an isolated singular point.

1. - Notation and Assumptions.

Let R_n denote n-dimensional Euclidean space and let x, ξ, etc. be elements of R_n with the respective coordinates $(x_1, \ldots, x_n)$, $(\xi_1, \ldots, \xi_n)$, etc.. We let $x \cdot \xi = \sum_{i=1}^{n} x_i \xi_i$ and $|x| = (x \cdot x)^{1/2}$. For all $\xi \in R_n$, we set $d\xi = d\xi_1 \ldots d\xi_n$.

Let t, τ, etc. be elements of R_1. Open intervals in R_1, e. g., $\{t_0 < t < t_1\}$ will be denoted by (t_0, t_1), closed intervals, e.g ., $\{t_0 \leqslant t \leqslant t_1\}$ by $[t_0, t_1]$, etc.. Let I be an interval in R_1. Denote the topological product of R_n with I by $R_n \times I$. We shall often speak of a strip, e. g., the strip $t_0 < t < t_1$, and mean $R_n \times (t_0, t_1)$.

(*) Indirizzo: Department of Mathematics, Oregon State University, Corvallis, Oregon 97331, U.S.A. .

(**) Ricevuto: 16-IV-1968.

We shall always use α and β to denote multi-indices, i. e., n-tuples of non-negative integers. For multi-indices, we define $|\alpha| \equiv \sum_{i=1}^{n} \alpha_i$,

$$D^\alpha \equiv \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n} \qquad \text{and, for all } \xi \in R_n, \quad \xi^\alpha \equiv \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}.$$

Let us now define the differential operator, L, by

$$\text{(1)} \qquad \mathrm{L}u \equiv \sum_{|\alpha| \leq 2p} a_\alpha(x, t)\, D^\alpha u - \partial u/\partial t,$$

where $p \geqslant 1$ is an integer.

In order to formulate the assumptions on L, it is convenient to introduce the following class of functions.

Definition. Let $I = [t_0, T]$, where $0 \leqslant t_0 < T$, and let $R = R_n \times I$. A function, $f(x, t)$, is said to belong to the class $H(x; \nu; R)$, if it is defined, continuous, and uniformly bounded on R, and if there exist positive constants H, ν, $0 < \nu \leqslant 1$, such that

$$|f(x', t) - f(x'', t)| \leqslant H\, |x' - x''|^\nu,$$

for all (x', t), $(x'', t) \in R$.

We now make the following assumptions on the operator L defined by (1).

(A 1) The functions $D^\beta a_\alpha(x, t)$, $0 \leqslant |\beta| \leqslant |\alpha|$, $|\alpha| = 0, 1, \dots, 2p-1$, and $0 \leqslant |\beta| \leqslant 2p - 1$ for $|\alpha| = 2p$, belong to the class $H(x; \nu; R)$. The functions $D^\beta a_\alpha(x, t)$, $|\alpha| = |\beta| = 2p$, are continuous and uniformly bounded.

(A 2) L is uniformly parabolic, i. e., there exists a constant $\gamma > 0$ such that

$$\gamma\, |\xi|^{2p} \leqslant (-1)^{p+1} \sum_{|\alpha| = 2p} a_\alpha(x, t)\, \xi^\alpha \leqslant (1/\gamma)\, |\xi|^{2p}$$

for all $(x, t) \in R$, $\xi \in R_n$.

We shall always assume that L satisfies (A 1) and (A 2) without explicitly stating this each time.

2. - General Comments.

In this article we shall treat the CAUCHY problem for the equation $Lu = 0$, i. e., we shall study the problem

$$(2) \qquad Lu = 0, \qquad x \in R_n, \qquad t_0 < t \leqslant T, \qquad u(x, t_0) \text{ known.}$$

After deriving some estimates on the fundamental solution (defined precisely below), we obtain a representation theorem for solutions to the problem (2). With the help of the representation theorem, we obtain conditions under which it is possible to represent solutions to $Lu = 0$ in a strip $t_0 < t < T$ in the form

$$(3) \qquad u(x, t) = \int_{R_n} \Gamma(x, t; \xi, t_0)\, \varrho(d\xi),$$

where Γ is the fundamental solution of (1) and ϱ is a signed measure: From the representation (3), we derive a uniqueness theorem. These results extend the earlier work of TYCHONOFF [25], KRZYŻAŃSKI [15], [16], ROSENBLOOM [23] and others (see in this respect also [24]), as well as the more recent work of FRIEDMAN [11], ARONSON [4] (and also [1] when specialized to a single equation), and GUENTHER [13]. In a sense these results are also extensions to higher order equations of similar results obtained for positive solutions to parabolic equations of second order. See in this respect the work of ARONSON [3], [4], FRIEDMAN [11], KRZYŻAŃSKI [15], [16], WIDDER [27], and others. These authors have shown that the assumpion of positivity implies an integral growth condition of the type stated in Theorem 2, below. In our case, such a growth condition must be postulated, but then we are able to obtain results similar to those for positive solutions. Finally, by a modification of the proof of the uniqueness theorem, we obtain theorems on isolated singularities for solutions to $Lu=0$ in a strip $t_0 < t < T$. These results extend to classical solutions to $Lu = 0$, the work of ARONSON [2], ĚIDEL'MAN [8] (and when specialized to a single equation [9]), and PINI [22]. Again, these results are similar to those obtained for positive solutions of parabolic equations of the second order by ARONSON [3] and KRZYŻAŃSKI [17], [18].

3. - Preliminary Resultus.

We begin with the following

Definition. A function, $\Gamma(x, t; \xi, \tau)$, is said to be a *fundamental solution* for $Lu = 0$ if:

i) Γ is defined and continuous for all $(x, t), (\xi, \tau) \in R$ except for $(x, t) = (\xi, \tau)$,

ii) Γ is $2p$ times continuously differentiable with respect to the x variables and once with respect to the t variable except at $(x, t) = (\xi, \tau)$ and satisfies $\mathrm{L}\Gamma = 0$ except at $(x, t) = (\xi, \tau)$.

iii) For all bounded and continuous functions φ defined on R_n,

$$\lim_{(x,t)\to(x^0,\tau+)} \int_{R_n} \Gamma(x, t; \xi, \tau)\, \varphi(\xi)\, \mathrm{d}\xi = \varphi(x^0),$$

for all $x^0 \in R_n$.

Theorem 1. *There exists a unique, fundamental solution, $\Gamma(x, t; \xi, \tau)$, to $\mathrm{L}u = 0$. Further, there exist positive constants K, λ depending only on the constants of the assumptions* (A 1) *and* (A 2), *such that*

$$|\, \mathrm{D}^\alpha \Gamma(x, t; \xi, \tau)\,| \leqslant K\,(t-\tau)^{-(n+|\alpha|)/(2p)} \exp\left\{-\left(\frac{|x-\xi|^{2p}}{\lambda(t-\tau)}\right)^{1/(2p-1)}\right\},$$

for $|\alpha| = 0, 1, \ldots, 2p$, *and*

$$|\,(\partial/\partial t)\Gamma(x, t; \xi, \tau)\,| \leqslant K\,(t-\tau)^{-(n+2p)/(2p)} \exp\left\{-\left(\frac{|x-\xi|^{2p}}{\lambda(t-\tau)}\right)^{1/(2p-1)}\right\}.$$

In what is to follow, the positive number λ will always refer to the constant λ of Theorem 1.

The proof of Theorem 1 may be found, for example, in FRIEDMAN [11]. See in particular pages 251 and 255 there. The uniqueness part of this theorem follows easily from the definition of a fundamental solution and from the next theorem.

Theorem 2. *Let $u(x, t)$ be continuous in the strip $t_0 \leqslant t < T$ and satisfy $\mathrm{L}u = 0$ in the strip $t_0 < t < T$. Suppose $u(x, t_0) \equiv 0$ for all $x \in R_n$ and that there exists a positive constant μ such that*

$$\int_{t_0}^{T}\int_{R_n} \exp(-\mu\,|x|^q)\cdot|\,u(x, t)\,|\,\mathrm{d}x\,\mathrm{d}t < \infty, \qquad q = 2p/(2p-1).$$

Then $u(x, t) \equiv 0$ in the strip $t_0 \leqslant t < T$.

This theorem is a mild generalization due to ARONSON [1] of a theorem of FRIEDMAN [11] and SLOBODETSKI [25], although for parabolic equations of the form $\mathrm{L}u = 0$, where the coefficients depend only on t, it was first proven by LADYZHENSKAYA [21].

In the proofs of the theorems to follow, it will be apparent that we only make use of the fact that a fundamental solution exists and that a uniqueness theorem is available. Indeed, these requirements dictate the assumptions on the coefficients of L. As a consequence of this remark, the theory developed below may be extended in several ways. For example, we could make less stringent assumptions on the coefficients of L and more stringent assumptions on the solutions to $Lu = 0$ and make use of the uniqueness theory developed by ARONSON [1]. In the case where $p = 1$, we could make use of the uniqueness theory of ARONSON and BESALA [5] and the fundamental solution constructed either by KRZYŻAŃSKI and SZYBIAK [18], KRZYŻAŃSKI [17], or by ARONSON and BESALA [6], or we could make use of the uniqueness theory of ARONSON in [4] and further extend our results. For the sake of greater clarity and an effort to give a unified treatment of the problem considered here for all integral values of p making only minimal assumptions on the solutions u of $Lu = 0$, we shall only develop the theory using the assumptions (A 1) and (A 2) and the uniqueness Theorem 2.

4. - Fundamental Lemmas.

The purpose of this section is to derive some rather precise estimates on the fundamental solution for $Lu = 0$.

Lemma 1. *Let (x', t') be a fixed point in the strip $t_0 < t \leqslant T$ and let D be a compact subset of the strip $t_0 < t < t'$. Then for all $(x, t) \in D$,*

$$(4) \qquad (t - t_0)^{-(n+|\alpha|)/(2p)} \exp\left\{-\left(\frac{|x-\xi|^{2p}}{t-t_0}\right)^{1/(2p-1)}\right\} \leqslant$$

$$\leqslant N \exp\left\{-\left(\frac{|x'-\xi|^{2p}}{t'-t_0}\right)^{1/(2p-1)}\right\},$$

where N is a positive constant depending only on $|\alpha|$, D, n, p and (x', t').

Proof. Since D is a compact subset of the strip $t_0 < t < t'$, there exist numbers τ_1, τ_2, $t_0 < \tau_1 \leqslant \tau_2 < t'$, such that if $(x, t) \in D$, then $\tau_1 \leqslant t \leqslant \tau_2$. Let $\omega^{2p} = (t' - t_0)/(\tau_2 - t_0)$. Clearly $\omega^{2p} > 1$. Consider the set $E = \{\xi \mid \xi \in R_n, |x' - \xi| > \frac{\omega}{\omega - 1} \max_{(x,t)\in D} |x - x'|\}$. For $(x, t, \xi) \in D \times E$, we have $\omega|\xi - x| \geqslant \omega|\xi - x'| - \omega|x' - x| \geqslant |\xi - x'|$. Hence, $|\xi - x|/(t - t_0)^{1/(2p)} \geqslant \omega|\xi - x|/(t' - t_0)^{1/(2p)} \geqslant |\xi - x'|/(t' - t_0)^{1/(2p)}$. Thus, (4) holds for $(x, t, \xi) \in D \times E$ with $N = (\tau_1 - t_0)^{-(n+|\alpha|)/(2p)}$. Let C$E$ denote the complement of E. Set $m = \min_{\xi\in CE} \exp\left\{-\left(\frac{|x'-\xi|^{2p}}{t'-t_0}\right)^{1/(2p-1)}\right\}$. Since C$E$ is compact, $m > 0$ and it follows that (4) holds for $(x, t, \xi) \in D \times CE$ with $N = \frac{1}{m}(\tau_1 - t_0)^{-(n+|\alpha|)/(2p)}$.

Lemma 2. *For each $a>0$ and each compact subset D of the strip $t_0<t<$ $<\min\{T, t_0+1/(\lambda a^{2p-1})\}$, there exists a constant M depending on D, a, n and the constants of the assumptions* (A 1) *and* (A 2), *such that for $(x, t)\in D$,*

$$|\mathrm{D}^{\alpha}\Gamma(x, t;\ \xi, t_0)| \leqslant M \exp\{-a\,|\,\xi\,|^{2p/(2p-1)}\}, \qquad |\alpha| = 0, 1, \dots, 2p,$$
$$|(\partial/\partial t)\Gamma(x, t;\ \xi, t_0)| \leqslant M \exp\{-\alpha\,|\,\xi\,|^{2p/(2p-1)}\}.$$

Proof. First apply Theorem 1. Next, set $x'=0$ and $t'=\min\{T, t_0+$ $+1/(\lambda a^{2p-1})\}$ in Lemma 1 and use the properties of the exponential function.

In our investigation of isolated singularities for solutions to $Lu=0$ in a strip, we shall need a result stating that if $|x-\xi|\leqslant \varkappa\,(t-\tau)^{1/(2p)}$, where $\varkappa$ is a positive constant, then for $t-\tau$ sufficiently small, there exists a constant C such that $\Gamma(x, t;\ \xi, \tau)\geqslant C\,(t-\tau)^{-n/(2p)}$. This bound from below on the fundamental solution for $Lu=0$ generalizes a similar bound obtained by Il'in, Kalashnikov, and Oleinik ([**14**], page 82), for second order parabolic equations. To facilitate the discussion, we introduce the following notation.

Let $A_{\alpha}(\xi, t, \tau)=\int_{\tau}^{t} a_{\alpha}(\xi, \theta)\,\mathrm{d}\theta$, and let $G(x, t; \xi, \tau)=(2\pi)^{-n}\int_{R_n}\exp\{-i(x-\xi)\cdot$ $\cdot\eta+(-1)^p\sum_{|\alpha|=2p}A_{\alpha}(\xi, t, \tau)\,\eta^{\alpha}\}\,\mathrm{d}\eta$. Then the fundamental solution can be written in the form (see Friedman [**12**])

$$\Gamma(x, t;\ \xi, \tau) = G(x, t;\ \xi, \tau) + R(x, t;\ \xi, \tau), \tag{5}$$

where R satisfies the estimate

$$|R(x, t;\ \xi, \tau)| \leqslant B\,(t-\tau)^{-(n-\nu)/(2p)}. \tag{6}$$

Here B is a positive constant depending on the constants of the assumptions (A 1) and (A 2), and ν, $0<\nu\leqslant 1$, is the constant arising from assumption (A 1).

We now investigate $G(x, t;\ \xi, \tau)$. Making use of assumption (A 2) and integrating, we find that

$$G(\xi, t;\ \xi, \tau) = (2\pi)^{-n}\int_{R_n}\exp\{(-1)^p\sum_{|\alpha|=2p}A_{\alpha}(\xi, t, \tau)\eta^{\alpha}\}\,\mathrm{d}\eta \geqslant \tag{7}$$
$$\geqslant (2\pi)^{-n}\int_{R_n}\exp\{-(t-\tau)\gamma\,|\,\eta\,|^{2p}\}\,\mathrm{d}\eta = \frac{\omega_n\,\gamma^{-n/(2p)}}{2p\,(2\pi)^n}\,\Gamma\!\left(\frac{n}{2p}\right)(t-\tau)^{-n/2p},$$

where ω_n is the surface area of the unit sphere in n-dimensions, and $\Gamma(z)$ is the Eulerian gamma function.

Let us now write $G(x, t;\ \xi, \tau)=[G(x, t;\ \xi, \tau)-G(\xi, t;\ \xi, \tau)]+G(\xi, t;\ \xi, \tau)$

and estimate the difference in brackets.

$$(8)\qquad | G(x, t; \xi, \tau) - G(\xi, t; \xi, \tau) \leqslant$$

$$\leqslant (2\pi)^{-n} \int_{R_n} | \exp(-i(x-\xi)\cdot\eta) - 1 | \exp\{(-1)^p \sum_{|\alpha|=2p} A_\alpha(\xi, t, \tau)\, \eta^\alpha\}\, d\eta$$

$$\leqslant (2\pi)^{-n} \int_{R_n} | x-\xi | \, | \eta | \exp\{(-1)^p \sum_{|\alpha|=2p} A_\alpha(\xi, t, \tau)\, \eta^\alpha\}\, d\eta$$

$$\leqslant (2\pi)^{-n} | x-\xi | \int_{R_n} | \eta | \exp(-\gamma(t-\tau) | \eta |^{2p})\, d\eta$$

$$= \omega_n \Gamma((n+1)/2p)\, (2p)^{-1} \gamma^{-(n+1)/(2p)} | x-\xi | \, (t-\tau)^{-(n+1)/(2p)}.$$

Let us define the constants C' and C'' by

$$(9)\qquad \begin{cases} C' = \omega_n \gamma^{-n/(2p)} \Gamma(n/2p)\, (2p)^{-1} (2\pi)^{-n} \\ C'' = \omega_n \Gamma((n+1)/(2p))\, (2p)^{-1} (2\pi)^{-n} \gamma^{-(n+1)/(2p)}. \end{cases}$$

From (5) - (9) follows then

$$(10)\qquad \Gamma(x, t; \xi, \tau) \geqslant C'(t-\tau)^{-n/(2p)} - C'' | x-\xi | (t-\tau)^{-(n+1)/(2p)} - B\, (t-\tau)^{-(n-\nu)/(2p)}.$$

Using the constants defined in (6) and (9), we state the following lemma, the proof of which follows immediately from (10).

Lemma 3. *If* $| x-\xi | \leqslant \varkappa (t-\tau)^{1/(2p)}$, *where* $\varkappa$ *is any positive constant satisfying* $\varkappa \leqslant C'/(2C'')$ *and if* $(t-\tau) \leqslant (C'/(4B))^{2p/\nu}$, *then there exists a positive constant* C *depending only on* n *and the constants of the assumptions* (A 1) *and* (A 2) *such that*

$$\Gamma(x, t; \xi, \tau) \geqslant C(t-\tau)^{-n/(2p)}.$$

5. - Representation and Uniqueness Theorems.

From the properties of a fundamental solution and the estimates of Lemma 2, we obtain the following theorem.

Theorem 3. *Let* $\varphi(x)$ *be a continuous function defined on* R_n *and suppose for some constant* $\mu > 0$ *that*

$$\int_{R_n} \exp(-\mu | x |^q) | \varphi(x) | \, dx < \infty, \qquad \text{where } q = 2p/(2p-1).$$

Then the function

$$u(x,\ t) \equiv \int_{R_n} \Gamma(x,\ t;\ \xi,\ t_0)\, \varphi(\xi)\, d\xi$$

is continuous in the strip $t_0 \leqslant t < \min\{T,\ t_0 + 1/(\lambda\mu^{2p-1})\}$, *it satisfies* $Lu = 0$ *in the strip* $t_0 < t < \min\{t,\ t_0 + 1/(\lambda\mu^{2p-1})\}$, *and* $\lim\limits_{(x,t)\to(x^0,t_0+)} u(x,\ t) = \varphi(x^0)$, *for all* $x^0 \in R_n$.

Theorem 3 is known. See for example ÈIDEL'MAN [10]. However, the proof based on Lemma 2 is particularly simple.

Theorem 4. *Suppose* $u(x,\ t)$ *is continuous in the strip* $t_0 \leqslant t < T$, *satisfies* $Lu = 0$ *in the strip* $t_0 < t < T$, *and for some constant* $\mu > 0$ *satisfies the growth condition* $\int_{t_0}^{T}\int_{R_n} \exp(-\mu\,|\,x\,|^q)\,|\,u(x,\ t)\,|\, dx\, dt < \infty$, *where* $q = 2p/(2p-1)$. *Then in the substrip* $t_0 \leqslant t < \min\{T,\ t_0 + 1/(\lambda\mu^{2p-1})\}$, $u(x,\ t)$ *may be represented by*

$$u(x,\ t) = \int_{R_n} \Gamma(x,\ t;\ \xi,\ t_0)\, u(\xi,\ t_0)\, d\xi.$$

Proof. Let $T' = \min\{T,\ t_0 + 1/(\lambda\mu^{2p-1})\}$ and $v(x,\ t) \equiv \int_{R_n} \Gamma(x,\ t;\ \xi,\ t_0)$ $u(\xi,\ t_0)\, d\xi$. Form the function $w(x,\ t) \equiv u(x,\ t) - v(x,\ t)$. From the hypotheses of the theorem and Theorem 3, it follows that $w(x,\ t)$ is continuous in the strip $t_0 \leqslant t < T'$, $w(x,\ t_0) \equiv 0$ for all $x \in R_n$, and $Lw = 0$ in the strip $t_0 < t < T'$. We now show that in a certain substrip, $t_0 < t < h$, of the strip $t_0 < t < T$, the function $w(x,\ t)$ satisfies an integral growth condition of the type given in Theorem 2. We shall then be able to conclude by Theorem 2 that $w(x,\ t) \equiv 0$ in the substrip $t_0 \leqslant t < h$. By repeating the argument on parallel substrips in a standard way, one can show that $w(x,\ t) \equiv 0$ in the strip $t_0 \leqslant t < T'$, which will prove the theorem.

Let $q = 2p/(2p-1)$, and let $a = 2^q$. Integrate the inequality

$$\exp(-\mu a\,|\,\xi\,|^q)\cdot|\,w(\xi,\ \tau)\,| \leqslant \exp(-\mu a\,|\,\xi\,|^q)\cdot|\,u(\xi,\ \tau)\,| + \\ + \exp(-\mu a\,|\,\xi\,|^q)\cdot \int_{R_n} |\,\Gamma(\xi,\ \tau;\ s,\ t_0)\quad u(s,\ t_0)\,|\, ds$$

with respect to ξ over R_n. In the resulting inequality, interchange the order of integration in the second integral on the right hand side, observe that $a > 1$ implies $-\mu a < -\mu$, to obtain

$$\int_{R_n} \exp(-\mu a\,|\,\xi\,|^q)\cdot|\,w(\xi,\ \tau)\,|\, d\xi \leqslant \int_{R_n} \exp(-\mu\,|\,\xi\,|^q)\cdot|\,u(\xi,\ \tau)\,|\, d\xi + \\ + \int_{R_n} |\,u(s,\ t_0)\,|\, ds \int_{R_n} |\,\Gamma(\xi,\ \tau;\ s,\ t_0)\,|\, \exp(-\mu a\,|\,\xi\,|^q)\, d\xi\,. \tag{11}$$

We now estimate the integral

$$I \equiv \int_{R_n} | \Gamma(\xi, \tau; s, t_0) | \exp(-\mu a | \xi |^q) \, d\xi .$$

In the computations to follow, C will denote a positive constant which may change from formula to formula. We find

$$I \leqslant C (\tau - t_0)^{-n/(2p)} \int_{R_n} \exp\{- | \xi - s |^q (\lambda\tau - \lambda t_0)^{-1/(2p-1)} - \mu a | \xi |^q\} \, d\xi$$

$$= C (\tau - t_0)^{-n/(2p)} \left[\int_{|\xi| \leq \frac{1}{2}|s|} + \int_{|\xi| > \frac{1}{2}|s|} \right] \exp\{- | \xi - s |^q \cdot$$

$$\cdot (\lambda\tau - \lambda t_0)^{-1/(2p-1)} - \mu a | \xi |^q\} \, d\xi \equiv I_1 + I_2 ,$$

respectively. For I_2 we have

$$I_2 \leqslant C (\tau - t_0)^{-n/(2p)} \exp(-\mu | s |^q) \int_{R_n} \exp\{- | \xi - s |^q (\lambda\tau - \lambda t_0)^{-1/(2p-1)}\} \, d\xi$$

$$= C \cdot \exp(-\mu | s |^q) .$$

To estimate I_1, let us note that for $\tau - t_0$ sufficiently small, say $\tau - t_0 \leqslant h - t_0$, where h, $t_0 < h < t_0 + \frac{1}{\lambda} \left(\frac{1}{2^{q+1}\mu} \right)^{2p-1}$, is fixed, we have $\mu | s |^q \leqslant \{2^{-q}/2(\lambda\tau - \lambda t_0)\}^{1/(2p-1)} | s |^q$. Consequently, from the fact that $| \xi - s | \geqslant | s | - | \xi | \geqslant \frac{1}{2} | s |$ in the region under consideration, we find

$$- | \xi - s |^q / 2(\lambda\tau - \lambda t_0)^{1/(2p-1)} - \mu a | \xi |^q \leqslant - \mu | s |^q .$$

Consequently,

$$I_1 \leqslant C (\tau - t_0)^{-n/(2p)} \exp(-\mu | s |^q) \int_{|\xi| \leq \frac{1}{2}|s|} \exp\{- | \xi - s |^q / 2(\lambda\tau - \lambda t_0)^{1/(2p-1)}\} \, d\xi$$

$$\leqslant C (\tau - t_0)^{-n/(2p)} \exp(-\mu | s |^q) \int_{R_n} \exp\{- | \xi - s |^q / 2(\lambda\tau - \lambda t_0)^{1/(2p-1)}\} \, d\xi$$

$$= C \cdot \exp(-\mu | s |^q) .$$

From (11), we obtain

$$(12) \quad \left\{ \begin{aligned} \int_{R_n} \exp(-\mu a | \xi |^q) \cdot | w(\xi, \tau) | \, d\xi \leqslant & \int_{R_n} \exp(-\mu | \xi |^q) \cdot | u(\xi, \tau) | \, d\xi + \\ & + C \int_{R_n} \exp(-\mu | s |^q) \cdot | u(s, t_0) | \, ds . \end{aligned} \right.$$

Integrate (12) with respect to τ from t_0 to h. The right hand side of the resulting inequality is finite; hence, $w(x, t) \equiv 0$ in $t_0 \leqslant t < h$. From our remarks at the beginning or the proof, the theorem now follows.

The following theorem is an immediate consequence of Theorem 4.

Theorem 5. *Suppose $u(x, t)$ is continuous in the strip $t_0 \leqslant t < T$ and satisfies $Lu = 0$ in the strip $t_0 < t < T$. If there exist positive constants μ, m such that*

$$\int_{R_n} \exp(-\mu |x|^q) \cdot |u(x, t)| \, dx \leqslant m, \qquad q = 2p/(2p-1),$$

for all t in the interval $t_0 < t < T$, then in the strip $t_0 \leqslant t < \min\{T, t_0 + 1/(\lambda_l \mu^{2p-1})\}$, $u(x, t)$ may be represented by

$$u(x, t) = \int_{R_n} \Gamma(x, t; \xi, t_0) \, u(\xi, t_0) \, d\xi .$$

Theorem 6. *Suppose $u(x, t)$ is continuous in the strip $t_0 < t < T$ and satisfies $Lu = 0$ there. Suppose there exist positive constants μ, m such that*

$$\int_{R_n} \exp(-\mu |x|^q) \cdot |u(x, t)| \, dx \leqslant m, \qquad q = 2p/(2p-1), \tag{13}$$

for all t in the interval $t_0 < t < T$. Then in the subscript $t_0 < t < \min\{T, t + 1/(\lambda_l \mu^{2p-1})\}$, $u(x, t)$ may be represented in the form

$$u(x, t) = \int_{R_n} \Gamma(x, t; \xi, t_0) \, \varrho(dx), \tag{14}$$

where ϱ is a signed measure and the integral converges absolutely in this substrip.

Remark. This theorem is analogous to one proven by Krzyżański [18] for positive solutions to second order parabolic equations. Krzyżański showed that positive solutions to second order parabolic equations automatically satisfy a growth condition of the type (13). From this Krzyżański obtained a representation theorem. In our case, we postulate the growth condition (13) and then obtain a representation.

The proof of this theorem follows very closely the proof of Krzyżański's Theorem 4 (see [18]) and so will only be sketched here. Furthermore, we shall make use of the measure theoretic results contained in Krzyżański's paper.

Let $T' = \min\{T, t_0 + 1/(\lambda_l \mu^{2p-1})\}$, $q = 2p/(2p-1)$, and let δ, $0 < \delta < (T' - t_0)/4$ be arbitrary. Define for any Borel set $E \subset R_n$ the family of set functions $\{\nu_\delta\}$ by

$$\nu_\delta(E) \equiv \int_E \exp(-\mu\,|\,\xi\,|^q)\cdot u(\xi,\, t_0+\delta)\,\mathrm{d}\xi\,.$$

The $\nu_\delta(E)$ are uniformly bounded by m and we write

$$\nu_\delta(E) = \int_E \exp(-\mu\,|\,\xi\,|^q)\cdot|\,u(\xi,\, t_0+\delta)\,|\,\mathrm{d}\xi -$$

$$-\int_E \exp(-\mu\,|\,\xi\,|^q)\cdot\{\,|\,u(\xi,\, t_0+\delta)\,| - u(\xi,\, t_0+\delta)\}\,\mathrm{d}\xi \equiv$$

$$\equiv \zeta_\delta(E) - \eta_\delta(E)\,,\ \text{respectively.}$$

The non-negative measures $\{\zeta_\delta(E)\}$ and $\{\eta_\delta(E)\}$ are uniformly bounded with respect to δ. There exists, therefore, a sequence, call it $\{\delta_l\}$ with $\delta_l \to 0$, $l\to\infty$, such that each of the sequences $\{\zeta_{\delta_l}(E)\}$ and $\{\eta_{\delta_l}(E)\}$ converge to the non-negative measures $\zeta(E)$ and $\eta(E)$, respectively.

Now let $(x,\, t)$ be an arbitrary fixed point in the strip $t_0 < t < T'$. We may assume $\{\delta_l\}$ is such that $t_0+\delta_l < t$, $\quad l = 1,\, 2,\, \ldots$. Then the sequence $\{W_l(\xi)\}$ defined by

$$W_l(\xi) = \Gamma(x,\, t;\, \xi,\, t_0+\delta_l)\cdot\exp(\mu\,|\,\xi\,|^q)$$

is uniformly bounded by Lemma 2. Further, the functions $W_l(\xi)$ are continuous functions of ξ. By Theorem 5 and the definition of W_l and ν_{δ_l}, we have

$$\int_{R_n} W_l(\xi)\,\nu_{\delta_l}(\mathrm{d}\xi) = \int_{R_n} \Gamma(x,\, t;\, \xi,\, t_0+\delta_l)\, u(\xi,\, t_0+\delta_l)\,\mathrm{d}\xi = u(x,\, t)\,.$$

On the other hand

$$\lim_{l\to\infty}\int_{R_n} W_l(\xi)\,\nu_{\delta_l}(\mathrm{d}\xi) = \lim\int_{R_n} \Gamma(x,\, t;\, \xi,\, t_0+\delta_l)\ \exp(\mu\,|\,\xi\,|^q)\cdot\zeta_{\delta_l}(\mathrm{d}\xi)$$

$$-\lim\int_{R_n} \Gamma(x,\, t;\, \xi,\, t_0+\delta_l)\cdot\exp(\mu\,|\,\xi\,|^q)\,\eta_{\delta_l}(\mathrm{d}\xi) =$$

$$= \int_{R_n} \Gamma(x,\, t;\, \xi,\, t_0)\cdot\exp(\mu\,|\,\xi\,|^q)\cdot\{\zeta(\mathrm{d}\xi) - \eta(\mathrm{d}\xi)\}.$$

Setting $\exp(\mu\,|\,\xi\,|^q)\cdot\zeta - \exp(\mu\,|\,\xi\,|^q)\cdot\eta = \varrho$, we obtain (14). By the way in which ϱ was constructed, it is easily seen that (14) converges absolutely.

Theorem 6 enables us to extend the uniqueness theorem in the following manner.

Theorem 7. *Suppose $u(x,\, t)$ is continuous in the strip $t_0 < t < T$ and satisfies $Lu = 0$ there. Suppose there exist positive constants μ, m such that*

$$\int_{R_n} \exp(-\mu\,|\,x\,|^q)\cdot|\,u(x,\, t)\,|\,dx \leqslant m\,, \qquad q = 2p/(2p-1)\,,$$

for all t in the interval $t_0 < t < T$. If $\lim_{t\to t_0+} u(x,\, t) = 0$ for all $x \in R_n$, then $u(x,\, t) \equiv 0$ in the strip $t_0 < t < \min\{T,\; t_0 + 1/(\lambda\,\mu^{2p-1})\}$.

Remark. In the case of the heat equation, Theorem 7 is due to ROSENBLOOM [21].

Proof. By Theorem 6, $u(x,\, t)$ may be represented in the form

$$u(x,\, t) = \int_{R_n} \Gamma(x,\; t;\; \xi,\; t_0)\; \varrho(d\xi)\,,$$

where ϱ is a signed measure. Let $\varrho = \tau - \sigma$, where $\tau = \exp(\mu\,|\,\xi\,|^q)\cdot\zeta$ and $\sigma = \exp(\mu\,|\xi\,|^q)\cdot\eta$ and ζ and η are the measures constructed in the proof of Theorem 6. Let $D \subset R_n$ be an arbitrary, bounded domain. We show that $\varrho(D) = 0$. Let x be an arbitrary but fixed point in D. Let Q be the cube defined by the inequalities

$$|\,\xi_i - x_i\,| \leqslant \left[\frac{\varkappa}{n}\,(t - t_0)\right]^{1/(2p)},$$

where $\varkappa$ is the number from Lemma 3 and $t - t_0$ is chosen sufficiently small so that the conclusion of Lemma 3 holds, and so that $Q \subset D$. Then

$$u(x,\, t) = \int_Q \Gamma(x,\; t;\; \xi,\; t_0)\; \varrho(d\xi) + \int_{R_n - Q} \Gamma(x,\; t;\; \xi,\; t_0)\; \varrho(d\xi)\,.$$

Now $\int_{R_n - Q} \Gamma(x,\; t;\; \xi,\; t_0)\; \varrho\,(d\xi) = o(1)$ as $t \to t_0 +$. To see this, observe that from the proof of Theorem 6 and the definitions of τ and σ, we have

$$\int_{R_n} \exp(-\mu\,|\,\xi\,|^q)\cdot|\,\varrho(d\xi)\,| \leqslant \int_{R_n} \exp(-\mu\,|\,\xi\,|^q)\cdot\tau(d\xi) + \int_{R_n} \exp(-\mu\,|\,\xi\,|^q)\cdot\sigma(d\xi)$$
$$= \int_{R_n} \zeta(d\xi) + \int_{R_n} \eta(d\xi) \leqslant 3m\,.$$

Let d be the distance from x to $R_n - D$. Then using Theorem 1, we find

$$\left|\int_{R_n - Q} \Gamma(x,\; t;\; \xi,\; t_0)\; \varrho(d\xi)\right| \leqq K\,(t - t_0)^{-n/(2p)}\, \exp\left\{-\frac{1}{2}\left(\frac{d^{2p}}{\lambda t - \lambda t_0}\right)^{1/(2p-1)}\right\}\cdot$$
$$\cdot \int_{R_n - Q} \exp\left\{-\frac{1}{2}\left(\frac{|\,x - \xi\,|^{2p}}{\lambda t - \lambda t_0}\right)^{1/(2p-1)}\right\}\,|\,\varrho(d\xi)\,|\,.$$

Let $\tau' = t_0 + 1/\{\lambda\,(2\mu)^{2p-1}\}$ and take $t - t_0$ so small that $t - t_0 < \tau' - t_0$. Now apply Lemma 1 to the latter integral with $x' = 0$, $t' = \min(T, \tau')$, to obtain

$$\Big| \int_{R_n - Q} \Gamma(x,\ t;\ \xi,\ t_0)\, \varrho(\mathrm{d}\xi) \Big| \leqslant K\,(t - t_0)^{-n/(2p)} \exp\Big\{-\frac{1}{2}\Big(\frac{d^{2p}}{\lambda t - \lambda t_0}\Big)^{1/(2p-1)}\Big\} \cdot$$
$$\cdot \int_{R_n} \exp\{-\mu\,|\,\xi\,|^{2p/(2p-1)}\} \cdot |\,\varrho(\mathrm{d}\xi)\,|\,.$$

The integral on the right hand side is bounded. Consequently, as $t \to t_0 +$, the right hand side tends to zero, proving the assertion. Returning to the proof of the theorem, we observe that by Lemma 3, $\Gamma(s,\ t;\ \xi,\ t_0)$ is positive for $\xi \in Q$. Thus by the triangle inequality, we find

$$|\,u(x,\ t)\,| + o(1) \geqslant \Big|\int_Q \Gamma(x,\ t;\ \xi,\ t_0)\,\tau(\mathrm{d}\xi) - \int_Q \Gamma(x,\ t;\ \xi,\ t_0)\,\sigma(\mathrm{d}\xi)\Big|,$$

in other words

$$|\,u(x,\ t)\,| + o(1) \geqslant \int_Q \Gamma(x,\ t;\ \xi,\ t_0)\,\tau(\mathrm{d}\xi) - \int_Q \Gamma(x,\ t;\ \xi,\ t_0)\,\sigma(\mathrm{d}\xi), \tag{15}$$

$$|\,u(x,\ t)\,| + o(1) \geqslant \int_Q \Gamma(x,\ t;\ \xi,\ t_0)\,\sigma(\mathrm{d}\xi) - \int_Q \Gamma(x,\ t;\ \xi,\ t_0)\,\tau(\mathrm{d}\xi). \tag{16}$$

Applying Lemma 3 and Theorem 1 to (15) and (16) yields

$$|\,u(x,\ t)\,| + o(1) \geqslant C\,(t - t_0)^{-n/(2p)} \int_Q \tau(\mathrm{d}\xi) - K\,(t - t_0)^{-n/(2p)} \int_Q \sigma(\mathrm{d}\xi), \tag{17}$$

$$|\,u(x,\ t)\,| + o(1) \geqslant C\,(t - t_0)^{-n/(2p)} \int_Q \sigma(\mathrm{d}\xi) - K\,(t - t_0)^{-n/(2p)} \int_Q \tau(\mathrm{d}\xi), \tag{18}$$

where the positive constants C and K are the same in both (17) and (18). Let $m(Q)$ denote the LEBESGUE measure of the cube Q:

$$m(Q) = [\varkappa\, n^{-1}\,(t - t_0)]^{n/(2p)}\,.$$

Thus, there exist positive constants C_1 and C_2 so that (17) and (18) take the form

$$|\,u(x,\ t)\,| + o(1) \geqslant C_1\,\tau(Q)/m(Q) - C_2\,\sigma(Q)/m(Q), \tag{19}$$

$$|\,u(x,\ t)\,| + o(1) \geqslant C_1\,\sigma(Q)/m(Q) - C_2\,\tau(Q)/m(Q). \tag{20}$$

Without loss of generality, we may assume $C_2 > C_1$, since this may always be achieved by increasing the constant C_2 and the inequalities (19) and (20) still hold.

Define the upper and lower symmetric derivatives of a measure ζ by setting respectively

$$\mathrm{D}^*\zeta = \limsup_{m(E)\to 0} \frac{\zeta(E)}{m(E)}, \qquad \mathrm{D}_*\zeta = \liminf_{m(E)\to 0} \frac{\zeta(E)}{m(E)}.$$

In (19) and (20), let $t \to t_0 +$ to obtain the inequalities

$$0 \geqslant \delta\, \mathrm{D}^*\tau - \mathrm{D}^*\sigma, \qquad 0 \geqslant \delta\, \mathrm{D}^*\sigma - \mathrm{D}^*\tau, \tag{21}$$

$$0 \geqslant \delta\, \mathrm{D}_*\tau - \mathrm{D}_*\sigma, \qquad 0 \geqslant \delta\, \mathrm{D}_*\sigma - \mathrm{D}_*\tau, \tag{22}$$

where $\delta = C_1/C_2 < 1$. Note that by (21), if either $\mathrm{D}^*\tau$ or $\mathrm{D}^*\sigma$ is zero then both $\mathrm{D}^*\tau$ and $\mathrm{D}^*\sigma$ are zero. Similarly, (22) implies that if either $\mathrm{D}_*\tau$ or $\mathrm{D}_*\sigma$ is zero, then both are zero. Assume that neither $\mathrm{D}^*\tau = 0$ nor $\mathrm{D}^*\sigma = 0$. Then by (21) $\mathrm{D}^*\sigma \geqslant \delta\, \mathrm{D}^*\tau \geqslant \delta^2\, \mathrm{D}^*\sigma$, i. e., $\mathrm{D}_*\sigma > \mathrm{D}_*\sigma$, which is a contradiction. Hence, $\mathrm{D}^*\tau = \mathrm{D}^*\sigma = 0$. Similarly, using (22), one shows that $\mathrm{D}_*\sigma = \mathrm{D}_*\tau = 0$. Consequently, the symmetric derivatives of σ and τ are zero, which implies that the symmetric derivative of ϱ is zero. By a theorem of BESICOVITCH **[6]**, see also ROSENBLOOM **[21]**, we may conclude that $\varrho(D) = 0$. Since D was an arbitrary bounded domain in R_n, we conclude that $\varrho \equiv 0$, which proves the theorem.

6. - Theorems on Isolated Singularities of Solutions to $Lu = 0$.

We begin by proving the following simple lemma.

Lemma 4. *Let f be a continuous function defined on R_n and let ϱ be a signed measure. Suppose the integral $\int_{R_n} f(\xi)\,\varrho(d\xi)$ converges absolutely. If for every Borel set E, not containing the point $y \in R_n$, $\varrho(E) = 0$, then*

$$\int_{R_n} f(\xi)\,\varrho(d\xi) = f(y)\,\varrho(D),$$

where D is an arbitrary compact set containing y in its interior.

Proof. Let D be a compact set containing y in its interior. Let $\{D_l\}$ be a decreasing sequence of compact sets contained in D which converge to y and such that for each l, y is in the interior of D_l. Then note that by the hypotheses of the lemma, $\int_{R_n} f(\xi)\, \varrho(d\xi) = \int_{D_l} f(\xi)\, \varrho(d\xi)$. Since $D_l = D - (D - D_l)$, $\varrho(D_l) = \varrho(D)$. Hence, $\int_{D_l} f(\xi)\, \varrho(d\xi) = \int_{D_l} f(y)\, \varrho(d\xi) + \int_{D_l} [f(\xi) - f(y)]\, \varrho(d\xi) = f(y)\, \varrho(D) + \int_{D_l} [f(\xi) - f(y)]\, \varrho(d\xi)$. The term $\int_{D_l} [f(\xi) - f(y)]\, \varrho(d\xi) \to 0$, $l \to \infty$, since f is continuous and this proves the lemma.

A modification of the proof of Theorem 7 will enable us to prove the following theorem.

Theorem 8. *Suppose $u(x, t)$ is continuous in the strip $t_0 < t < T$ and satisfies $Lu = 0$ there. Suppose there exist positive constants m, μ such that*

$$\int_{R_n} \exp(-\mu\, |x|^q) \cdot |u(x, t)|\, dx \leqslant m\,, \qquad q = 2p/(2p-1)\,,$$

for all t in the interval $t_0 < t < T$. Finally, suppose $\lim_{t \to t_0+} u(x, t) = 0$ for all $x \in R_n$ except at $x = y$. Then there exists a constant A such that

$$u(x, t) = A\, \Gamma(x, t;\, y, t_0) \quad \textit{in the strip} \quad t_0 < t < \min\{T,\, t_0 + 1/(\lambda \mu^{2p-1})\}.$$

Proof. By Theorem 6, there exists a signed measure ϱ such that $u(x, t) = \int_{R_n} \Gamma(x, t;\, \xi, t_0)\, \varrho(d\xi)$. By the proof of Theorem 7, $\varrho(E) = 0$ for every bounded domain not containing the point y. By Lemma 4, we conclude that $u(x, t) = \varrho(D)\, \Gamma(x, t;\, y, t_0)$, where D is a compact set containing y in its interior. The assertion of the theorem now follows with $A = \varrho(D)$.

Theorem 9. *Suppose $u(x, t)$ is continuous in the strip $t_0 < t < T$ and satisfies $Lu = 0$ there. Suppose there exist constants m, μ such that*

$$\int_{R_n} \exp(-\mu\, |x|^q) \cdot |u\, x, t)|\, dx \leqslant m\,, \qquad q = 2p/(2p-1)\,,$$

for all t in the interval $t_0 < t < T$. Finally, suppose $\lim_{t \to t_0+} u(x, t) = 0$ for all $x \in R_n$ except possibly at $x = y$ and in a deleted neighborhood of y, $u(x, t)$ satisfies the estimate $|u(x, t)| \leqslant K\, (t - t_0)^{-(n-\eta)/(2p)}$ for $t > t_0$, where $K > 0$ and $\eta > 0$ are constants. Then $u(x, t) \equiv 0$ in the strip $t_0 < t < \min\{T,\, t_0 + 1/(\lambda \mu^{2p-1})\}$.

Proof. By Theorem 8, there exists a constant A such that $u(x, t) = A\, \Gamma(x, t;\, y, t_0)$. We show that $A = 0$. For suppose $A \neq 0$. Let $|x - y| \leqslant \varkappa\, (t - t_0)^{1/(2p)}$. Here $\varkappa$ is a positive constant and $\varkappa$ and $t - t_0$ are chosen so small that the conclusion of Lemma 3 holds. Then by Lemma 3, there exists a positive constant C such that $\Gamma(x, t; y, t_0) \geqslant C(t-t_0)^{-n/(2p)}$. Thus by Lemma 3 and the hypotheses of the theorem, we find $|A|\, C(t-t_0)^{-n/(2p)} \leqslant |A|\, K(t-t_0)^{-(n-\eta)/(2p)}$, or $(t-t_0)^{-\eta/(2p)} \leqslant K/C$, where K and C are some fixed constants. For $t-t_0$ sufficiently small, this inequality is impossible. The assumption that $A \neq 0$ leads to a contradiction and this proves the theorem.

Theorem 10. *Let $u^*(x, t)$ be continuous in the strip $t_0 \leqslant t < T$ and satisfy $\mathrm{L}u^* = 0$ in the strip $t_0 < t < T$. Let $u(x, t)$ be continuous in the strip $t_0 < t < T$ and satisfy $\mathrm{L}u = 0$ there. Suppose there exist positive constants m, μ such that*

$$\int_{R_n} \exp(-\mu\, |x|^q)\cdot |u(x, t)|\, dx \leqslant m\,, \qquad \int_{R_n} \exp(-\mu\, |x|^q)\cdot |u^*(x, t)|\, dx \leqslant m\,,$$

where $q = 2p/(2p-1)$, for all t in the interval $t_0 < t < T$. Finally, suppose that for every point $x \in R_n$ except at $x = y$, $\lim_{t\to t_0^+} u(x, t) = u^*(x, t_0)$. Then there exists constant A such that

$$u(x, t) = A\,\Gamma(x, t; y, t_0) + u^*(x, t) \quad \textit{in the strip } t_0 < t < \min\{T,\; t_0 + 1/(\lambda\mu^{2p-1})\}.$$

Proof. Let $w(x, t) \equiv u(x, t) - u^*(x, t)$ and apply Theorem 8 to $w(x, t)$. Similarly, from Theorem 9 follows

Theorem 11. *Let $u^*(x, t)$ be continuous in the strip $t_0 \leqslant t < T$ and satisfy $\mathrm{L}u^* = 0$ in the strip $t_0 < t < T$. Let $u(x, t)$ be continuous in the strip $t_0 < t < T$ and satisfy $\mathrm{L}u = 0$ there. Suppose there exist positive constants m, μ such that*

$$\int_{R_n} \exp(-\mu\, |x|^q)\cdot |u(x, t)|\, dx \leqslant m\,, \qquad \int_{R_n} \exp(-\mu\, |x|^q)\cdot |u^*(x, t)|\, dx \leqslant m,$$

where $q = 2p/(2p-1)$, for all t in the interval $t_0 < t < T$. Finally, suppose for every point $x \in R_n$, except possibly $x = y$, $\lim_{t\to t_0^+} u(x, t) = u^*(x, t_0)$, and in a deleted neigborhood of y, $u(x,t)$ satisfies the estimate $|u(x,t)| \leqslant K(t-t_0)^{-(n-\eta)/(2p)}$ for $t > t_0$, where $K > 0$, $\nu > 0$ are costants. Then $u(x, t) \equiv u^*(x, t)$ in the strip $t_0 < t < \min\{T,\; t_0 + 1/(\lambda\mu^{2p-1})\}$.

Remark. Theorem 8 can be extended as follows: Suppose u satisfies the hypotheses of Theorem 8 except that we suppose $\lim_{t\to t_0} u(x, t) = 0$ for all $x \in R_n$ except at a finite number of points, say $y_1, \ldots, y_k$. Then there exist constants $A_1, \ldots, A_k$ such that $u(x, t) = \sum_{i=1}^{k} A_i\, \Gamma(x, t; y_i, t_0)$.

The proof of this result is similar to the proof of Theorem 8, but it necessitates a slight extension of Lemma 4. In a similar way, Theorems 9-11 can also be extended.

Bibliography.

[1] D. G. ARONSON, *Uniqueness of solutions of the initial value problem for parabolic. systems of differential equations*, J. Math. Mech. **11** (1962), 403-420.

[2] D. G. ARONSON, *Removable singularities for linear parabolic equations*, Arch. Rat. Mech. Anal. **17** (1964), 79-84.

[3] D. G. ARONSON, *Isolated singularities of solutions of second order parabolic equations*, Arch. Rat. Mech. Anal. **19** (1965), 231-238.

[4] D. G. ARONSON, *Uniqueness of positive solutions of second order parabolic equations*, Ann. Polon. Math. **16** (1965), 285-303.

[5] D. G. ARONSON and P. BESALA, *Uniqueness of solutions of the Cauchy problem for parabolic equations*, J. Math. Anal. Appl. **13** (1966), 516-526.

[6] D. G. ARONSON and P. BESALA, *Parabolic equations with unbounded coefficients*, J. Diff. Equ. **3** (1967), 1-14.

[7] A. S. BESICOVITCH, *A general form of the covering principle and relative differentiation of additive functions*, Proc. Camb. Philos. Soc. **41** (1945), 103-110; **42** (1946), 1-10.

[8] S. D. ÈIDEL'MAN, *Behavior of solutions of the heat equation in the neighborhood of an isolated singular point*, Uspehi, Math. Nauk **11** (1956), 207-210.

[9] S. D. ÈIDEL'MAN, *The behavior of solutions of a parabolic system in the neighborhood of an isolated singular point*, Dokl. Akad. Nauk SSSR **125** (1959), 743-745.

[10] S. D. ÈIDEL'MAN, *Parabolic Systems*, Izdat. « Nauka », Moscow 1964.

[11] A. FRIEDMAN, *On the uniqueness of the Cauchy problem for parabolic equations*, Amer. J. Math. **81** (1959), 503-511.

[12] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, New York 1964.

[13] R. B. GUENTHER, *Representation theorems for linear, second-order, parabolic, partial differential equations*, J. Math. Anal. Appl. **17** (1967), 488-501.

[14] A. M. IL'IN, A. S. KALASHNIKOV and O. A. OLEINIK, *Linear second order parabolic equations*, Russian Math. Surveys **17** (1962), 1-143.

[15] M. KRZYŻAŃSKI, *Sur les solutions des équations du type parabolique determinées dans une région illimitée*, Bull. Amer. Math. Soc. **47** (1941), 911-915.

[16] M. KRZYŻAŃSKI, *Sur les solutions de l'équation linéaire du type parabolique determinées par les conditions initiales*, Ann. Soc. Polon. Math. **18** (1945), 145-156.

[17] M. KRZYŻAŃSKI, *Sur la solution élémentaire de l'équation de la chaleur*, Atti Accad. Naz. Lincei, Sci. Fis. Mat. Nat. (7) **8** (1950), 193-199; **13** (1952), 24-25.

[18] M. KRZYŻAŃSKI, *Sur les solutions non négatives de l'équation lineaire normale parabolique*, Revue Roumaine Math. Pures et Appl. **9** (1964), 393-408.

[19] M. KRZYŻAŃSKI, *Sur la solution fondamentale de l'équation linéaire normale du type parabolique dont le dernier coefficient est non borné*, Rend. Accad. Naz. Lincei, Sc. Fis. Mat. Nat. 32 (1962), 326-330.

[20] M. KRZYŻAŃSKI and A. SZYBIAK, *Construction et étude de la solution fondamentale de l'équation linéaire du type parabolique dont le dernier coefficient est non borné* (I and II), Rend. Accad. Naz. Lincei, Sc. Fis. Nat. Mat. **27** (1959), 26-30 and 113-117.

[21] O. A. LADYZHENSKAYA, *On the uniqueness of solutions of the Cauchy problem for linear parabolic equations*, Mat. Sbornik **27** (1950), 175-184.

[22] B. PINI, *Sui punti singolari delle soluzioni delle equazioni paraboliche lineari*, Ann. Univ. Ferrara (Sezione VII, Scienze Mat.) **2** (1953), 1-12.

[23] P. C. ROSENBLOOM, *Linear equations of parabolic type with constant coefficients* (Contributions to the theory of partial differential equations. Annals of Mathematics Studies, n. 33), Princeton 1954.

[24] P. C. ROSENBLOOM, *Linear partial differential equations*, Surveys Applied Math. (New York) **5** (1958), 42-204.

[25] L. E. SLOBODETSKI, *On the Cauchy problem for non-homogeneous parabolic equations* Dok. Akad. Nauk SSSR (N. S.) **101** (1955), 805-808.

[26] A. N. TYCHONOFF, *Theoremes d'unicité pour l'équation de la chaleur*, Mat. Sbornik (N. S.) **42** (1935), 199-216.

[27] D. V. WIDDER, *Positive temperatures on the infinite rod*, Trans. Amer. Math. Soc. **55** (1944), 85-95.

Riassunto.

Si ottengono delle condizioni sotto le quali le soluzioni $u(x, t)$ di equazioni paraboliche, lineari, di ordine superiore, e a coefficienti variabili, si possono rappresentare nella forma

$$u(x, t) = \int_{R_n} \Gamma(x, t; \xi, t_0)\, \alpha(d\xi) \qquad (x \in R_n, \quad t > t_0).$$

Qui $\Gamma(x, t; \xi, \tau)$ è la soluzione fondamentale dell'equazione parabolica considerata, e l'integrale è assolutamente convergente. Da questi teoremi di rappresentazione discendono teoremi di unicità e sulle singolarità isolate.

* * *