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Summability in the Space $C(Y)$ and the Structure of Set-valued Additive Functions. (**)

1. - Introduction.

Let Y denote a reflexive BANACH space and let $C(Y)$ be the space of all non-empty, closed, convex, and bounded sets of the space Y. The space $C(Y)$ forms a semilinear space under the operation of algebraic addition of sets and multiplication of a set by a scalar. If V is a neighborhood of zero in a fixed locally convex topology of the space Y, then the family of sets

$$N(A) = \{B \in C(Y) : B \subset A + V \text{ and } A \subset B + V\}$$

constitutes a base of neighborhoods for the set A of $C(Y)$. This topology is said to be the weak (or strong) topology of the space $C(Y)$ if it is generated by the weak (or strong) topology of the space Y. Let T be an abstract set. Denote by D the directed set of all finite subsets J of the set T ordered by inclusion. A family of sets A_t $(t \in T)$ is said to be summable to an element $A = \sum_{t \in T} A_t$ if the net $S_J = \sum_{t \in J} A_t$ $(J \in D)$ is convergent to A. Define the norm of a set A by

$$\| A \| = \sup \{ \| x \| : x \in A \}.$$

2. - Summability in the space $C(Y)$.

In this paragraph we will give conditions for summability in either the strong or the weak topology of the space $C(Y)$. To this end the following theorem will

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be proven:

Theorem 1. *Let a sequence of sets $A_n \in C(Y)$ satisfy the condition that $\sum_{n=1}^{\infty} \| A_n \| < \infty$. Then there exists a set $A \in C(Y)$ such that the series $\sum_{n=1}^{\infty} A_n$ is convergent to A in the strong topology of the space $C(Y)$.*

Proof. Consider any series of the form $\sum_{i=1}^{\infty} y_i$, where $y_i \in A_i \subset Y$ for $i = 1, 2, \dots$. Since

$$\| y_i \| \leqslant \| A_i \| \qquad \text{and} \qquad \sum_{i=1}^{\infty} \| A_i \| < \infty ,$$

there exists an element $y \in Y$ such that $\sum_{i=1}^{n} y_i \to y$. This follows from the completeness of the Banach space Y. Consider now the set of all possible sums

$$B = \{ x = x_1 + x_2 + \dots ; \quad x_i \in A_i ; \quad i = 1, 2, \dots \} .$$

One may conclude from the estimation

$$\| x \| \leqslant \| x_1 \| + \| x_2 \| + \dots \leqslant \| A_1 \| + \| A_2 \| + \dots$$

that

$$\| B \| \leqslant \| A_1 \| + \| A_2 \| + \dots .$$

Therefore B is bounded. To show that the set B is also convex, consider any two elements $x^1, x^2 \in B$. Then:

$$x^1 = x_1^1 + x_2^1 + \dots , \qquad x_i^1 \in A_i ,$$

$$x^2 = x_1^2 + x_2^2 + \dots , \qquad x_i^2 \in A_i .$$

Consider any two numbers $t, s \geqslant 0$ such that $s + t = 1$. Then

$$x = s\, x^1 + t\, x^2 = x_1 + x_2 + x_3 + \dots ,$$

where

$$x_i = s\, x_i^1 + t\, x_i^2 \in A_i .$$

This means that the element x belongs to the set B. Let A be the closure of the set B. Then $A = \overline{B} \in C(Y)$, since B is non-empty, bounded and convex, then the set A is also non-empty, bounded and convex and therefore is an element of the space $C(Y)$. Denote by

$$S_n = \sum_{j=1}^{n} A_j$$

and consider any neighborhood of zero in the space Y, where

$$V = \{y \in Y: \ \|y\| < \varepsilon\}.$$

Let

$$U = \{y \in Y: \ \|y\| \leqslant \varepsilon/2\}.$$

One can choose a k in such a way that

$$\sum_{j=k}^{\infty} \|A_j\| \leqslant \varepsilon/2 .$$

One shows now that

$$A \subset S_n + V \qquad \text{for} \qquad n \geqslant k .$$

Consider an arbitrary element x in B. Then

$$x = s_n + r_n, \qquad s_n = x_1 + \ldots + x_n, \qquad r_n = x_{n+1} + x_{n+2} + \ldots$$

and

$$\|r_n\| \leqslant \|x_{n+1}\| + \|x_{n+2}\| + \ldots \leqslant \sum_{j=k}^{\infty} \|A_j\| \leqslant \varepsilon/2 ,$$

provided that n is greater or equal to k. This means that the element x belongs to the set S_n, and

$$x = s_n + r_n \in S_n + U .$$

Since S_n and U are elements of the semilinear space $C(Y)$, then the element $S_n + U$ also is an element of $C(Y)$ and therefore is a closed set. Since it contains B, it must also contain its closure, that is the set A.

In the same manner one obtains also the inequality

$$\text{(a)} \qquad A \subset S_n + U \subset S_n + V \qquad \text{for} \qquad n \geqslant k .$$

To show finally that $S_n \subset A + V$ for $n \geqslant k$ one considers the elements

$s_n \in S_n$ and $r_n = x_{n+1} + x_{n+2} + \dots, \quad x_i \in A_i$. Then

$$x = s_n + r_n \in B .$$

Therefore

$$s_n = x - r_n \in B + U \subset B + V \qquad \text{for} \qquad n \geqslant k ,$$

and therefore:

$$\text{(b)} \qquad S_n \subset B + V \subset A + V \qquad \text{for} \qquad n \geqslant k.$$

Conditions (a) and (b) together are equivalent to

$$S_n = \sum_{j=1}^{n} A_j \in N_V(A) \qquad \text{if} \qquad n \geqslant k ,$$

and this by definition of the strong topology in $C(Y)$ means the series is convergent to an element A in that topology, that is $\sum_{j=1}^{\infty} A_j = A$, which concludes the proof of the theorem.

A necessary and sufficient condition for a family of sets to be summable in the weak topology of the space $C(Y)$ will now be given.

Theorem 2. *A family of sets A_α $(\alpha \in T)$ is summable in the weak topology of $C(Y)$ to an element of that space if and only if there exists a number $M > 0$ such that $\| \sum_{\alpha \in J} A_\alpha \| \leqslant M$ for all finite $J \subset T$.*

Proof of Sufficient Condition. One uses the following

Lemma 1. *Let A_α $(\alpha \in T)$ be as in Theorem 2. Then any family $x_\alpha \in A_\alpha$ $(\alpha \in T)$ is weakly summable in Y.*

To prove the Lemma 1, let $x_\alpha \in A_\alpha$ $(\alpha \in T)$. To show that $\sum_{\alpha \in J} | y'(x) | \leqslant 2M \| y' \|$ for any finite subset J of T and for any linear continuous functional y' on the space Y. Indeed, consider

$$J^+ = \{\alpha \in J : \quad y'(x_\alpha) > 0\}, \qquad J^- = J - J^+ .$$

Then

$$| \sum_{J^+} y'(x_\alpha) | = | y'(\sum_{J^+} x_\alpha) | \leqslant \| y' \| \, \| \sum_{J^+} x_\alpha \| \leqslant M \| y' \| ,$$

and similarly

$$\left| \sum_{J^-} y'(x_\alpha) \right| \leqslant M \| y' \| .$$

Therefore

$$\sum_J | y'(x_\alpha) | = \sum_{J^+} | y'(x_\alpha) | + \sum_{J^-} | y'(x_\alpha) | =$$

$$= \sum_{J^+} y'(x_\alpha) - \sum_{J^-} y'(x_\alpha) \leqslant \left| \sum_{J^+} y'(x_\alpha) \right| + \left| \sum_{J^-} y'(x_\alpha) \right| \leqslant 2M \| y' \| .$$

This means that the family of numbers for any linear continuous functional y', namely $y'(x_\alpha)$ $(\alpha \in T)$ is summable. Let us denote its sum by

$$F(y') = \sum_T y'(x_\lambda) .$$

It follows from the estimation that

$$| F(y') | \leqslant 2M \| y' \| ,$$

for any y' in the conjugate space Y' of the space Y. Since the functional $F(y')$, as a functional of y', is linear on the space Y' and it is also bounded, it must be continuous. Since the space Y' is reflexive, we conclude that it is of the from $F(y') = y'(x)$. This means that

$$y'(x) = \sum_T y'(x_\lambda)$$

and hence

$$\left| y'(x) - \sum_J y'(x_\alpha) \right| < \varepsilon \qquad \text{if} \qquad J \supset J_0$$

for any finite subset J of the set T and for some finite subset J_0 of the set T. But this is equivalent to the following condition:

$$\left| y'\left(x - \sum_J x_\alpha\right) \right| < \varepsilon \qquad \text{if} \qquad J \supset J_0 .$$

Now consider a weak neighborhood of zero, in the space Y,

$$V = \{ y \in Y, \quad | y_i'(y) | < \varepsilon \quad (i = 1, 2, \ldots, k) \}.$$

Then by similar means as used previously we obtain the following:

$$\left| y_i'\left(x - \sum_J x_\alpha\right) \right| < \varepsilon \qquad \text{for} \qquad i = 1, 2, \ldots, k; \; J \supset J_0 .$$

Hence the following condition is satisfied

$$x - \sum_J x_\alpha \in V \qquad \text{if} \qquad J \supset J_0 ,$$

and therefore we have

$$\sum_J x_\alpha \in x + V \qquad \text{if} \qquad J \supset J_0 .$$

This by the definition of summation means that the elements x_α are summable on the set T in the weak topology of the space Y to the element x, i. e.

$$\sum_T x_\alpha = x .$$

Now let us consider the set of all sums of this form, that is

$$B = \{x : \ x \in \sum_T x_\alpha , \ x_\alpha \in A_\alpha\} .$$

The set B is bounded. To prove it let us take any element x from the set, then we have for it the following estimation

$$|\, y'(x) \,| \leqslant 2M \| \, y' \| \qquad \text{for} \qquad x \in B, \ \ y' \in Y' .$$

Therefore according to the BANACH and STEINHAUS theorem we have:

$$\| \, x \| \leqslant 2M \qquad \text{for any} \qquad x \in B .$$

But this means that

$$\| \, B \, \| \leqslant 2M .$$

The set B is also convex, for let x^1 and x^2 be two points of B, then

$$x^1 = \sum_T x^1_\alpha , \qquad x_2 = \sum_T x^2_\alpha , \qquad x^i_\alpha \in A_\alpha .$$

Let $t_1 , \ t_2 \geqslant 0$ be any two real numbers such that $t_1 + t_2 = 1$. Then

$$x = t_1 \, x^1 + t_2 \, x^2 = \sum_T x_\alpha ,$$

where

$$x_\alpha = t_1 \, x^1 + t_2 \, x^2 \in A_\alpha \qquad (\alpha \in T) .$$

Hence we see that $x \in B$. Now let us define the set A to be the closure of the set B. Since the set B is convex, the closure in the weak topology coincides with the closure in the strong topology. Since the set B is non-empty, convex and bounded then the set A is non-empty, convex, bounded and closed and is therefore an element of the space $C(Y)$, i. e.

$$A = \overline{B} \in C(Y).$$

Lemma 2. *Let A be a set compact in the weak topology of the space Y and let V be a closed set in the weak topology of Y. Then the set $A+V$ is closed in the weak topology of Y.*

Proof. To show that the set is closed we will show that it contains the limit of any convergent net which has values in this set.

Let us consider a net

$$z_\beta \in A + V, \qquad \lim z_\beta = z.$$

To show now that $z \in A+V$. The net is of the form

$$z_\beta = x_\beta + y_\beta, \qquad x_\beta \in A, \qquad y_\beta \in V.$$

Since the set A is compact, we may conclude that there exists a subset x_γ of the net x_β which converges to an element of the set A. For this subset we have

$$z_\gamma = x_\gamma + y_\gamma$$

and

$$\lim_\gamma x_\gamma = x \in A.$$

Since a subset of a net which converges to a point c also converges to the point c, we conclude that $\lim_\gamma z_\gamma = z$. Now we see that the net $y_\gamma = z_\gamma - x_\gamma$ must be convergent in the space Y and since the set V is closed it must converge to a point of V, i. e.

$$\lim_\gamma y_\gamma = z - x = y \in V.$$

Now we have

$$\lim_\gamma z_\gamma = \lim_\gamma x_\gamma + \lim_\gamma y_\gamma = x + y.$$

This is true since the operation of addition is continuous in the weak topology of the space $C(Y)$. We see that $x \in A$, $y \in V$ and therefore

$$z = x + y \in A + V$$

and this means the set $A + V$ is closed in the weak topology of Y.

Lemma 3. *For any weak neighborhood V of zero in the weak topology of the space Y there exists a finite set $J_0 \subset T$ such that*

$$\sum_{T/J} x_\alpha \subset V \qquad \text{if} \qquad J \supset J_0, \quad x_\alpha \in A_\alpha \ (\alpha \in T).$$

Proof. Let us consider any neighborhood of zero in the weak topology of the space Y, that is

$$V = \{y \in Y: \quad |y_i'(y)| < \varepsilon \qquad (i = 1, 2, \ldots, k)\}$$

and let us denote

$$\sup \{ |y_i'(x)|: \quad x \in A_\alpha\} = a_{i\alpha} \qquad (i = 1, 2, \ldots, k).$$

Since the set A is bounded then the numbers $a_{i\alpha}$ are finite. Since the functional $|y_i'(x)|$ for $x \in Y$, considered as a function of x, is continuous in the weak topology of the space Y and the sets A_α are compact in the weak topology of the space Y, we conclude that the supremum of the functional must be taken on at a point of the set A. Let us denote this point x_α^i. Then

$$|y_i'(x_\alpha^i)| = a_{i\alpha} \qquad (\alpha \in T; \quad i = 1, 2, \ldots, k).$$

It follows from the proof of Lemma 1, that for any finite subset J of the set T we have the following estimation

$$\sum_J |y_i'(x_\alpha^i)| \leqslant 2M \| y_i' \|$$

and therefore

$$\sum_{\alpha \in J} a_{i\alpha} \leqslant 2M \| y_i' \| .$$

Then if we let

$$b_\alpha = a_{1\alpha} + a_{2\alpha} + \ldots + a_{k\alpha} \qquad (\alpha \in T),$$

we obtain for any finite subset J of the set T the estimation

$$\sum_{\alpha\in J} b_\alpha \leqslant 2M \sum_{i=1}^{k} \| y_i' \| = \overline{M}.$$

This means that the family of non-negative numbers b_α is summable to a number b and therefore it si true that

$$\Big| \sum_{\alpha\in J} b_\alpha - b \Big| < \varepsilon \qquad \text{if} \qquad J \supset J_0$$

for any finite subset J of the set T and for some finite subset J_0 of T. But this can be written in the form:

$$\sum_{\alpha/J} b_\alpha < \varepsilon \qquad \text{if} \qquad J \supset J_0 .$$

Now let us consider any family of points $x_\alpha \in A_\alpha$. Then

$$\Big| y_i'\big(\sum_{T/J} x_\alpha\big) \Big| = \Big| \sum_{T/J} y_i'(x_\alpha) \Big| \leqslant \sum_{T/J} | y_i'(x_\alpha) | \leqslant \sum_{T/J} b_\alpha < \varepsilon \qquad \text{if} \qquad J \supset J_0 .$$

Then this implies that

$$\sum_{T/J} x_\alpha \in V \qquad \text{if} \qquad J \supset J_0 , \qquad x_\alpha \in A_\alpha \quad (\alpha \in T) .$$

To show now that the family of sets A_α is summable in the weak topology of the space $C(Y)$ to the set A. Let us therefore consider any neighborhood of zero in the weak topology of the space Y, where

$$V = \{ y \in Y : \quad | y_i'(y) | < \varepsilon \qquad (i = 1, 2, \ldots, k) \},$$

and let

$$U = \{ y \in Y : \quad | y_i'(y) | \leqslant \tfrac{1}{2}\varepsilon \qquad (i = 1, 2, \ldots, k) \},$$

$$V_0 = \{ y \in Y : \quad | y_i'(y) | < \tfrac{1}{4} \qquad (i = 1, 2, \ldots, k) \}.$$

We have

$$V_0 \subset U \subset V .$$

Now using Lemma 3 let us find a set J_0 with the property that the following condition holds

$$x = \sum_{T/J} X_\alpha \in V_0 \qquad \text{if} \qquad J \supset J_0 , \qquad x \in A .$$

We will prove that

$$A \subset \sum_J A_\alpha + V \qquad \text{if} \qquad J \supset J_0$$

for any finite subset J of the set T. For this purpose, let us select any point $x \in B$. Then

$$x = \sum_J x_\alpha + \sum_{T/J} x_\alpha \in \sum_J A_\alpha + V_0 .$$

This means that

$$B \subset \sum_J A_\alpha + U \qquad \text{if} \qquad J \supset J_0 .$$

Since the set $\sum_J A_\alpha$ is compact in the weak topology and the set U is closed in that topology, we conclude that the algebraic sum of the two sets is a closed set in the weak topology. Therefore, the weak closure of the set B must be contained in their sum, that is

$$\text{(a)} \qquad A = \overline{B} \subset \sum_J A_\alpha + U \subset \sum_J A_\alpha + V \qquad \text{if} \qquad J \supset J_0 .$$

Now let us take any point x and y of the form

$$x \in \sum_J A_\alpha , \qquad y = \sum_{T/J} x_\alpha , \qquad x_\alpha \in A_\alpha .$$

Then

$$x + y = z \in B \subset A ,$$

and

$$y = \sum_{T/J} x_\alpha \in V \qquad \text{if} \qquad J \supset J_0 .$$

Since the set V is symmetric, then $-y \in V$ and this implies that

$$x = z - y \in A + V \qquad \text{if} \qquad J \supset J_0 ,$$

or which is equivalent

$$\text{(b)} \qquad \sum_J A_\alpha \subset A + V \qquad \text{if} \qquad J \supset J_0 .$$

Now conditions (a) and (b) together are equivalent to the condition

$$\sum_J A_\alpha \in N_V(A) \qquad \text{if} \qquad J \subset J_0$$

and this by the definition of summation of a family of sets to a set means that $\sum_T A_\alpha = A$, where the summation is understood in the weak topology of the space $C(Y)$.

Proof of Necessary Condition. The proof of the necessary condition of Theorem 2 follows from the following

Theorem 3. *If a family $A_\alpha\,(\alpha \in T)$ is summable in the strong or weak topology of the space $C(Y)$, then there exists a constant M such that*

$$\| \sum_{\alpha \in J} A_\alpha \| \leqslant M ,$$

for all finite subsets J of the set T.

Proof. Since every strong neighborhood of zero of the space Y is contained in a weak neighborhood of zero of the space Y, it is sufficient to prove the theorem for the weak topology only.

Choose a family A_α summable to an element A and let V be any neighborhood of zero in the space Y which is generated by only one linear functional, that is

$$V = \{z \in Y : \ |y'(z)| < 1\}.$$

Then $\sum_{\alpha \in J} A_\alpha \subset A + V$ if the set of indexes J is greater than a fixed finite set of indexes J_0. Take an element $x \in \sum_{\alpha \in J} A_\alpha$. Then $x = y + z$, $y \in A$, $z \in V$, and therefore

$$\| y'(x) | \leqslant | y'(y) | + | y'(z) | \leqslant \| y' \| \| A \| + 1.$$

Similarly one obtains $\| \sum_{\alpha \in J} A_\alpha \| \leqslant \overline{M}$ if $J_0 \subset J$. Let $M = \sum_{\alpha \in J_0} \| A_\alpha \| + \overline{M}$. Then $\| \sum_{\alpha \in J_0} A_\alpha \| \leqslant \sum_{J \cap J} A_\alpha \| + \| \sum_{J/J_0} A_\alpha \| \leqslant M$ for any finite $J \subset T$, which proves Theorem 3.

3. - Structure theorems for set-valued additive functions.

The above theorems are essential to find structure theorems of additive set-valued functions defined on base-cones in BANACH spaces. See [3], [4], [5] where the following theorems have been proven:

Theorem 4. *Let $A(t)$ be an additive function defined on the set S of positive reals and have values in the space $C(Y)$. Then the following conditions are equivalent:*

(a) *$A(t)$ is bounded in an open interval (c, d);*

(b) *$A(t)$ is continuous at a point $t \in S$ in the weak topology of $C(Y)$;*

(c) *$A(t)$ is continuous at a point $t \in S$ in the strong topology of $C(Y)$;*

(d) *$A(t)$ is continuous at every point $t \in S$ in the strong topology of $C(Y)$;*

(e) *$A(t)$ is of the form $A(t) = tK$ for all $t \in S$, where $K = A(1) \in C(Y)$.*

Theorem 5. *Let X be a Banach space whit a biorthogonal base system e_n, f_n, and let S denote the base cone, that is $S = \{x \in X: \ f_n(x) \geqslant 0$ for $n = 1, 2, \ldots\}$. Let $C(Y)$ be defined as previously. Then for avery continuous function A defined on S and with values in $C(Y)$, there exists sets $K_n \in C(Y)$ such that $A(x) = \sum_{n=1} f_n(x) K_n$ for all $x \in S$, the last series being convergent in the strong topology of $C(Y)$.*

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