Riv. Mat. Univ. Parma (2) 9 (1968), 91-96

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Some Results Involving Generalized Hypergeometric Functions. (**)

1. - Introduction.

The well-known GEGENBAUER's addition theorem ([11], p. 363) has been modified as follows:

$$(1.1) \qquad t^{-\nu} J_\nu(c_1 t) = \left(\frac{2c_1}{ab}\right)^\nu \Gamma(\nu) \sum_{n=0}^{\infty} (\nu+n)\, C_n^\nu(\cos\theta)\, t^{-2\nu} J_{\nu+n}(at)\, J_{\nu+n}(bt),$$

valid for $\nu \neq 0, -1, -2, \ldots$ and θ is given by $c_1^2 = a^2 + b^2 - 2ab\cos\theta$.

The object of this Paper is to use the result (1.1) to sum up:

(a) an infinite series involving products of GEGENBAUER's polynomial $C_n^\nu(\cos\theta)$ and LAURICELLA's function F_c ([1], p. 114) in terms of F_c;

(b) an infinite series involving products of GEGENBAUER's polynomial and generalized confluent hypergeometric series ψ_2 in terms of ψ_2; and

(c) an infinite series involving GEGENBAUER's polynomial and the H-function of FOX [6].

A few interesting particular cases of these results have also been derived.

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(**) Ricevuto: 29-II-1968.

2. - Main Results.

The main results to be proved are:

$$(2.1)\quad \begin{cases} F_c\left[\beta-\varrho,\ \beta+\varrho;\ 1+\nu,\ 1+\nu_1,\ \dots,\ 1+\nu_r;\ -\dfrac{c_1^2}{\alpha^2},\ -\dfrac{\alpha_1^2}{\alpha^2},\ \dots,-\dfrac{\alpha_r^2}{\alpha^2}\right]= \\ \qquad = \sum_{n=0}^{\infty}\dfrac{(ab/\alpha^2)^n(\beta-\varrho)_n(\beta+\varrho)_n}{(\nu)_n\,(1+\nu)_n}\,C_n^\nu(\cos\theta)\cdot \\ \cdot F_c\left[\beta-\varrho+n,\beta+\varrho+n;1+\nu+n,1+\nu+n,1+\nu_1,\dots,1+\nu_r;\right. \\ \qquad\left. -\dfrac{a^2}{\alpha^2},-\dfrac{b^2}{\alpha^2},-\dfrac{\alpha_1^2}{\alpha^2},\ \dots,-\dfrac{\alpha_r^2}{\alpha^2}\right], \end{cases}$$

valid for $\operatorname{Re}\alpha>\sum_{1}^{r}|\operatorname{Im}\alpha_i|$, $\operatorname{Re}(\beta\pm\varrho)>0$, where $\cos\theta$ is given by $c_1^2=a^2+b^2-2ab\cos\theta$.

$$(2.2)\quad \begin{cases} \psi_2\left[\beta;\ 1+\nu,\ 1+\nu_1,\ \dots,\ \nu+1_r;\ -\dfrac{c_1^2}{4\alpha},\ -\dfrac{\alpha_1^2}{4\alpha},\ \dots,\ -\dfrac{\alpha_r^2}{4\alpha}\right]= \\ \qquad = \sum_{n=0}^{\infty}\dfrac{\{ab/(4\alpha)\}^n(\beta)_n}{(\nu)_n\,(1+\nu)_n}\,C_n^\nu(\cos\theta)\cdot \\ \cdot\psi_2\left[\beta+n;\ 1+\nu+n,\ 1+\nu+n,\ 1+\nu_1,\ \dots,\ 1+\nu_r;\right. \\ \qquad\left. -\dfrac{a^2}{4\alpha},-\dfrac{b^2}{4\alpha},-\dfrac{\alpha_1^2}{4\alpha},\ \dots,-\dfrac{\alpha_r^2}{4\alpha}\right], \end{cases}$$

valid for $\operatorname{Re}\alpha>0$, $\operatorname{Re}\beta>0$, where $\cos\theta$ is given by $c_1^2=a^2+b^2-2ab\cos\theta$.

$$(2.3)\quad \begin{cases} H_{p+2,q}^{m,l+1}\left[z\,a^{-2\sigma}\sin^{-2\sigma}(\theta/2)\,\middle|\,\begin{matrix}(1-\varrho/2,\sigma),(a_1,e_1),\dots,(a_p,e_p),(1-\varrho/2+\nu,\sigma)\\(b_1,f_1),\dots,(b_q,f_q)\end{matrix}\right]= \\ \qquad = 2^{2\nu}\pi^{-1/2}\sin^{\varrho}(\theta/2)\,\Gamma(\nu)\sum_{n=0}^{\infty}(\nu+n)\,C_n^\nu(\cos\theta)\cdot \\ \cdot H_{p+3,q+1}^{m+1,l+1}\left[z\,a^{-2\sigma}\,\middle|\,\begin{matrix}(a_1,e_1),\dots,(a_l,e_l),(1-n-\varrho/2,\sigma),(1-\varrho/2+\nu,\sigma),\\(1+2\nu+n-\varrho/2,\sigma),(a_{l+1},e_{l+1}),\dots,(a_p,e_p)\\(b_1,f_1),\dots,(b_m,f_m),(1/2-\varrho/2+\nu,\sigma),(b_{m+1},f_{m+1}),\\\dots,(b_q,f_q)\end{matrix}\right], \end{cases}$$

valid for $\nu \neq 0, -1, -2, \ldots, \sigma > 0, a \sin(\theta/2) > 0$, $\mathrm{Re}\{\varrho + 2\sigma \min(b_h/f_h)\} > 0$,

$$h = 1, \ldots, m; \qquad \mathrm{Re}\left(\varrho - \nu - 3/2 + 2\sigma \max \frac{a_j - 1}{e_j}\right) < 0; \qquad j = 1, \ldots, l;$$

$$\lambda \equiv \sum_1^l e_j - \sum_{l+1}^p e_j + \sum_1^m f_j - \sum_{m+1}^q f_j, \qquad \lambda > 0, \qquad |\arg z| < \lambda\pi/2.$$

Here

$$H_{p,q}^{m,l}\left[z \,\middle|\, \begin{matrix}(a_1, e_1), \ldots, (a_p, e_p)\\ (b_1, f_1), \ldots, (b_q, f_q)\end{matrix}\right]$$

is the Fox's H-function [6].

Proofs of results (2.1), (2.2) and (2.3): Multiplying (1.1) by $t^\sigma f(t)$, integrating with respect to t from 0 to ∞, and interchanging the order of integration and summation, we have

$$(2.4) \quad \int_0^\infty t^{\sigma-\nu} J_\nu(c_1 t) f(t) \mathrm{d}t = \left(\frac{2c_1}{ab}\right)^\nu \Gamma(\nu) \sum_{n=0}^\infty (\nu+n) C_n^\nu(\cos\theta) \int_0^\infty t^{\sigma-2\nu} J_{\nu+n}(at)\, J_{\nu+n}(bt) f(t)\, \mathrm{d}t.$$

(a) Taking

$$f(t) = K_{2\varrho}(\alpha t) \prod_{i=1}^r J_{\nu_i}(\alpha_i t)$$

in (2.4) and using ([9], p. 162), we arrive at (2.1) on replacing $\frac{1}{2}(\sigma + 1 + \sum_1^r \nu_i)$ by β.

(b) Again taking

$$f(t) = e^{-\alpha t^2} \prod_{i=0}^r J_{\nu_i}(\alpha_i t)$$

in (2.4) and using ([5], p. 187, (43)), we arrive at (2.2) on replacing $\frac{1}{2}(\sigma + 1 + \sum_1^r \nu_i)$ by β.

(c) Once again, taking

$$f(t) = t^{\varrho-\sigma-1} H_{p,q}^{m,l}\left[z t^{2\sigma} \,\middle|\, \begin{matrix}(a_1, e_1), \ldots, (a_p, e_p)\\ (b_1, f_1), \ldots, (b_q, f_q)\end{matrix}\right]$$

in (2.4) when $a = b$ and using [7] and [8] we arrive at the result (2.3).

The change of the order of integration and summation is justified for the condition given with the results (2.1), (2.2) and (2.3) respectively by ([2], p. 500).

3. - Special Cases of (2.2), (2.1) and (2.3).

In this section, we shall derive a few interesting special cases of the results (2.1), (2.2) and (2.3) respectively.

(i) For $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$, (2.1) reduces to

$$(3.1)\quad \begin{cases} {}_2F_1[\beta-\varrho,\ \beta+\varrho;\ 1+\nu;\ -c_1^2/\alpha^2] = \sum_{n=0}^{\infty} \dfrac{(ab/\alpha^2)^n(\beta-\varrho)_n(\beta+\varrho)_n}{(\nu)_n\,(1+\nu)_n}\ \cdot \\ \cdot\ C_n^{\nu}(\cos\theta)\, F_4\!\left[\beta-\varrho+n,\ \beta+\varrho+n;\ 1+\nu+n,\ 1+\nu+n;\ -\dfrac{a^2}{\alpha^2},\ -\dfrac{b^2}{\alpha^2}\right], \end{cases}$$

for $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,(\beta \pm \varrho) > 0$ and $c_1^2 = a^2 + b^2 - 2ab\cos\theta$. Using the result ([3], p. 101, (37))

$$(3.2)\qquad F_4[\alpha,\ \beta,\ \gamma,\ \delta;\ x,\ x] = {}_4F_3\left[\begin{matrix} \alpha,\ \beta,\ (\gamma+\delta-1)/2,\ (\gamma+\delta)/2;\ 4x \\ \gamma,\ \delta,\ \gamma+\delta-1 \end{matrix}\right]$$

in (3.1) when $a = b$, we have

$$(3.3)\quad \begin{cases} {}_2F_1\!\left(\beta-\varrho,\ \beta+\varrho;\ 1+\nu;\ -\dfrac{4a^2\sin^2(\theta/2)}{\alpha^2}\right) = \sum_{n=0}^{\infty} \dfrac{(a/\alpha)^{2n}\,(\beta-\varrho)_n\,(\beta+\varrho)_n}{(\nu)_n\,(1+\nu)_n}\ \cdot \\ \cdot\ C_n^{\nu}(\cos\theta)\ {}_3F_2\left[\begin{matrix} \beta-\varrho+n,\ \beta+\varrho+n,\ \tfrac{1}{2}+\nu+n;\ -4a^2/\alpha^2 \\ 1+\nu+n,\ 1+2\nu+2n \end{matrix}\right], \end{cases}$$

for $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,(\beta \pm \varrho) > 0$.

(ii) When $\alpha_1 = \ldots = \alpha_r = 0$, (2.2) reduces to

$$(3.4)\quad \begin{cases} {}_1F_1(\beta;\ 1+\nu;\ -c_1^2/(4\alpha)) = \sum_{n=0}^{\infty} \dfrac{\{ab/(4\alpha)\}^n\,(\beta)_n}{(\nu)_n\,(1+\nu)_n}\ \cdot \\ \cdot\ C_n^{\nu}(\cos\theta)\ \psi_2\!\left[\beta+n;\ 1+\nu+n,\ 1+\nu+n;\ -\dfrac{a^2}{4\alpha},\ -\dfrac{b^2}{4\alpha}\right], \end{cases}$$

valid for $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$ and $c_1^2 = a^2 + b^2 - 2ab\cos\theta$. Again, taking $a = b$ in (3.4) and using the result ([4], p. 124)

$$(3.5)\qquad \psi_2(a;\ b,\ c;\ x,\ x) = {}_3F_3\left[\begin{matrix} a,\ (b+c)/2,\ (b+c-1)/2;\ 4x \\ b,\ c,\ b+c-1 \end{matrix}\right],$$

we get

$$(3.6)\quad \begin{cases} {}_1F_1\left[\beta;\ \nu+1;\ -\dfrac{a^2\sin^2(\theta/2)}{\alpha}\right] = \\ = \displaystyle\sum_{n=0}^{\infty} \frac{\{a^2/(4\alpha)\}^n\,(\beta)_n}{(\nu)_n\,(1+\nu)_n}\, C_n^\nu(\cos\theta)\ {}_2F_2\left[\begin{matrix}\beta+n,\ \frac{1}{2}+\nu+n;\ -a^2/\alpha\\ 1+\nu+n,\ 1+2\nu+2n\end{matrix}\right], \end{cases}$$

valid for $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$. Taking $\beta = 1+\nu$ in (3.6),and using ([**10**], p. 12, (1.8.3))

$$(3.7)\qquad {}_1F_1(a;\ 2\,a;\ x) = (x/4)^{\frac{1}{2}-a}\, e^{x/2}\,\Gamma(a+1/2)\, I_{a-1/2}(x/2)\,,$$

$$(2a \neq 0, -1, -2, \ldots),$$

we arrive at

$$(3.8)\qquad \left(\frac{a^2}{4\alpha}\right)^{\nu} e^{a^2\cos(\theta/(2\alpha))} = \sum_{n=0}^{\infty} (\nu+n)\,\Gamma(\nu)\, C_n^\nu(\cos\theta)\, I_{\nu+n}\!\left(\frac{a^2}{2\alpha}\right),$$

valid for $\operatorname{Re}\alpha > 0$, $\operatorname{Re}(\nu+1) > 0$.

(iii) For $\sigma = 1$, $e_1 = \ldots = e_p = f_1 = \ldots = f_q = 1$, (2.3) reduces to the following result under a set of conditions easily deduciable from that given with (2.3):

$$(3.9)\quad \begin{cases} G_{p+2,q}^{m,l+1}\left| z\,a^{-2}\sin^{-2}(\theta/2)\left[\begin{matrix}1-\varrho/2,\ a_1,\ \ldots,\ a_p,\ 1-\varrho/2+\nu\\ b_1,\ \ldots,\ b_q\end{matrix}\right]\right. = \\ = \pi^{-1/2}\, 2^{2\nu}\sin^{\varrho}(\theta/2)\,\Gamma(\nu)\displaystyle\sum_{n=0}^{\infty}(\nu+n)\, C_n^\nu(\cos\theta)\cdot \\ \cdot\, G_{p+3,q+1}^{m+1,l+1}\left[z\,a^{-2}\left|\begin{matrix}a_1,\ldots,a_l,\ 1-n-\varrho/2,1-\varrho/2+\nu,\ 1+2\nu+n-\varrho/2,\ a_{l+1},\ldots,a_p\\ b_1,\ \ldots,\ b_m,\ 1/2-\varrho/2+\nu,\ b_{m+1},\ \ldots,\ b_q\end{matrix}\right.\right], \end{cases}$$

where $G_{p\,q}^{m,l}$ is the well-known MEIJER's G-function.

References.

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Summary.

The well-known Gegenbauer's addition theorem has been used to express: (i) *Lauricella's generalised hypergeometric function F_c as an infinite series involving products of $C_n^\nu(\cos\theta)$, the Gegenbauer polynomial and F_c;* (ii) *the generalised confluent hypergeometric function ψ_2 as an infinite series involving products of $C_n^\nu(\cos\theta)$ and ψ_2; and* (iii) *the H-function of Fox as an infinite series involving products of $C_n^\nu(\cos\theta)$ and H-function, respectively. Some very interesting special cases involving Meijer's G-function, Appell's function F_4, hypergeometric functions ${}_3F_3$, ${}_2F_1$, ${}_1F_1$, etc. have been derived.*

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