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On Fixed Point Theorems. (**)

The well known conjecture that if f and g are two continuous functions which map a closed interval of real line into itself and if they commute then they have a common fixed point, has been given by ELDON DYER in 1954, by ALLEN L. SHIELDS in 1955 and by LESTER DUBINS in 1956 independently. The partial proofs of the conjecture have been given by H. COHEN [1], G. JUNGCK [7], R. DEMARR ([2], [3], [4]) and others. Similar results for analytic functions have been given by ALLEN L. SHIELDS [8] and S. P. SINGH [9].

Definition. A mapping T of a metric space X into itself is said to satisfy a LIPSCHITZ conditions with LIPSCHITZ constant α if

$$d(Tx,\ Ty) \leqslant \alpha\, d(x,\ y) \qquad (x,\ y \in X).$$

If this condition is satisfied with a LIPSCHITZ constant α such that $0 \leqslant \alpha < 1$, then T is called Contraction Mapping.

Definitions. Two functions f: $Z \rightarrow Z$ and g: $Z \rightarrow Z$ are said to be commutative if $\text{fog}(z) = \text{gof}(z)$ for all $z \in Z$. A point z_0 is said to be common fixed point for f and g if $f(z_0) = z_0 = g(z_0)$.

In this present paper we have given a few theorems on fixed points. The first three theorems are mainly related to commuting functions and fixed points, and the remaining two theorems are the generalizations of the well known theorems due to M. EDELSTEIN ([5], [6]). A few examples have been given in order to illustrate the results.

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Theorem 1. *Let $f(z)$ be an analytic function in a domain D of the complex Z-plane. Let $f(z)$ map a closed unit disk $|z|<1$ of Z into itself. Then the equation $f(z)=z$ has a unique solution provided $|f'(z)|<1$.*

Proof. The function $f(z)$ maps a closed unit disk $|z|\leqslant 1$, of a complex z-plane into itself. The closed subset of a complete metric space is complete and therefore $|z|\leqslant 1$ is a complete metric space. Now $|f'(z)|\leqslant\alpha<1$ implies that f is a contraction mapping. In fact due to LAGRANGE's formula for any z_1, z_2,

$$|f(z_1)-f(z_2)|=|f'(\xi)(z_1-z_2)|\leqslant\alpha|z_1-z_2|,$$

where $z_1\leqslant\xi\leqslant z_2$. By using a well known BANACH contraction principle we get that f has a unique fixed point, i.e. $f(z)=z$ has a unique solution.

Example. Let $f(z)=\frac{1}{2}-\frac{z}{2}$, then $|f'(z)|=\frac{1}{2}<1$. Hence $f(z)$ is a contraction mapping, and $f(z)$ has $z=1/3$ as a fixed point.

Theorem 2. *Let $f(z)$ and $g(z)$ map the closed unit disk $|z|\leqslant 1$ in the complex plane into itself in a continuous manner, if they are analytic in the open disk and if they commute $f(g(z))=g(f(z))$ for all z, then they have a common fixed point, provided $|f'(z)|<1$.* (A similar theorem has been given by ALLEN L. SHIELDS [8] in an entirely different way.)

Proof. We have proved in Theorem 1 that, under the conditions given in the theorem, f has a unique fixed point, say z_0, i.e. $f(z_0)=z_0$. Since $f(z)$ and $g(z)$ commute, therefore

$$fg(z)=gf(z) \qquad \text{for all } z \text{ in } Z\text{-plane.}$$

Now

$$fg(z_0)=gf(z_0)=g(z_0) \qquad \text{or} \qquad f[g(z_0)]=g(z_0).$$

Thus $g(z_0)$ is a fixed point for $f(z)$, but $f(z)$ has a unique fixed point z_0. Therefore $g(z_0)=z_0$, and thus z_0 is a fixed point for g. Hence the theorem.

Example. Let $f(z)$ be a function from a closed unit disk into itself defined by $f(z)=\frac{1}{2}-\frac{z}{2}$, and $g(z)$ be another function defined by $g(z)=z$. Here $|f'(z)|=\frac{1}{2}<1$. Hence $f(z)$ is a contraction mapping with a unique fixed point $z=1/3$. Now f and g both commute, i.e.

$$fg(z)=gf(z) \qquad \text{for all } z \text{ in } Z\text{-plane.}$$

Thus $fg(z) = gf(z) = (1 - z)/2$. The point $z = 1/3$ is a fixed point for $g(z) = z$. Thus they have a common fixed point say $z = 1/3$.

Theorem 3. *Let* $f(z) = \dfrac{az + b}{cz + d}$, $ad - bc \neq 0$ *and* $g(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma = 0$ *be two linear fractions. Then f and g have a common fixed point, provided they commute and* $(a - d)^2 + 4bc = 0$.

Proof. The linear fraction $f(z) = \dfrac{az + b}{cz + d}$, $ad - bc \neq 0$, has a unique fixed point say $z_0 = \dfrac{a - d}{2c}$ under the condition $(a - d)^2 + 4bc = 0$, i.e.

$$f(z_0) = z_0 = (a - 2d)/c .$$

Now since $f(z)$ and $g(z)$ commute, therefore

$$fg(z) = gf(z) \qquad \text{for all } z \text{ in } Z\text{-plane.}$$

Now

$$fg(z_0) = gf(z_0) = g(z_0) \qquad \text{or} \qquad f[g(z_0)] = g(z_0) .$$

Thus $g(z_0)$ is a fixed point for $f(z)$; but $f(z)$ has a unique fixed point z_0. Therefore $g(z_0) = z_0$, and thus z_0 is a fixed point for g. Hence the theorem. In order to illustrate the theorem we take the following

Example. Let

$$f(z) = \frac{6z + 4}{-z + 2} \qquad \text{and} \qquad g(z) = \frac{-2z + 4}{-z - 6}$$

be two linear fractions. Then $f(z)$ and $g(z)$ commute and have a common fixed point $z_0 = -2$.

Definition. A mapping T of a metric space X into itself is said to be contractive if

$$d(Tx, Ty) < d(x, y) ,$$

for all $x, y \in X$ $(x \neq y)$.

A contractive mapping of a complete metric space X into itself need not have a fixed point. For example, let $X = \{x \mid x \geqslant 1\}$ with the usual metric $d(x, y) = |x - y|$, and let $T: X \to X$ be given by

$$Tx = x + 1/x .$$

However, if T is a contractive mapping of a metric space X into a compact subset of X, then T has a unique fixed point $x \in X$. We will use this result in the proof of the following

Theorem 4. *If X is a compact metric space and T^p, where p is a positive integer, is a contractive mapping of X into itself, then T has a unique fixed point.*

Proof. Let $T^p = g$. Then we apply the above theorem to g. If x is the fixed point of g, the relation $g(x) = x$ gives $Tg(x) = T(x)$, but $T^{p+1} = T(g) = gT$: therefore $g(T(x)) = T(x)$. Hence $T(x)$ is a fixed point of g; the uniqueness of this point shows that $T(x) = x$. In other words, x is also a fixed point of T.

Definition. A metric space X is said to be well-linked if for every pair a, b of points of X and for every $\varepsilon > 0$, there exists a finite sequence $x_1, x_2, \ldots, x_n$, of points of X, with $x_1 = a$ and $x_n = b$, such that $d(x_i, x_{i+1}) \leqslant \varepsilon$ for every $i < n$. In other words, a and b can be joined by a chain of steps at most equal to ε.

Definition. A mapping T of a metric space X into itself is said to be ε-contractive if $0 < d(x, y) < \varepsilon$, then $d(Tx, Ty) < d(x, y)$.

M. Edelstein ([5], [6]) has given the following theorem: «Let T be a mapping of a complete well-linked metric space X into itself, and suppose that there is a real number k with $0 \leqslant k < 1$ such that $d(x, y) < \varepsilon$ implies that

$$d(Tx, Ty) \leqslant k\, d(x, y)\,. \tag{1}$$

Then T has a unique fixed point x in X. In case T does not satisfy condition (1) of the above theorem, but a suitable power T^p of T satisfies». Then we have the following

Theorem 5. *Let T be a mapping of a complete well-linked metric space X into itself and suppose that there is real number k with $0 \leqslant k < 1$ such that $d(x, y) < \varepsilon$ implies*

$$d(T^p x, T^p y) \leqslant k\, d(x, y)\,.$$

Then T has a unique fixed point x in X.

Proof. The proof of this theorem follows on the same lines as the proof of Theorem 4.

At the end we want to add the following

Corollary. *If X is a well-linked compact metric space and T^p is an ε-contractive mapping of X into itself, then T has a unique fixed point.*

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