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Integral Expressions for Appell's Functions F_1 and F_2. (**)

Introduction.

The integral representations ([2], pp. 230-31)

$$F_1(a;\ b,\ c;\ d;\ x,\ y) = \frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-a)} \cdot \tag{1}$$

$$\cdot \int_0^1 u^{a-1}\,(1-u)^{d-a-1}\,(1-ux)^{-b}\,(1-uy)^{-c}\,\mathrm{d}u \qquad (\mathrm{Re}\ d > \mathrm{Re}\ a > 0)$$

and

$$F_2(a;\ b,\ c;\ d,\ e;\ x,\ y) = \frac{\Gamma(d)\,\Gamma(e)}{\Gamma(b)\,\Gamma(c)\,\Gamma(d-b)\,\Gamma(e-c)} \cdot \tag{2}$$

$$\cdot \int_0^1 \int_0^1 u^{b-1}\,v^{c-1}\,(1-u)^{d-b-1}\,(1-v)^{e-c-1}\,(1-ux-vy)^{-a}\,\mathrm{d}u\,\mathrm{d}v$$

$$(\mathrm{Re}\ d > \mathrm{Re}\ b > 0, \qquad \mathrm{Re}\ e > \mathrm{Re}\ c > 0)\,,$$

where the APPELL's functions F_1 and F_2 are defined, respectively, as ([2], p. 224).

$$F_1(a;\ b,\ c;\ d;\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(c)_n}{m!\,n!\,(d)_{m+n}}\,x^m\,y^n\,, \tag{3}$$

$$F_2(a;\ b,\ c;\ d,\ e;\ x,\ y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m\,(c)_n}{m!\,n!\,(d)_m\,(c)_n}\,x^m\,y^n\,, \tag{4}$$

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have been obtained by using EULER's integral of the first kind for the beta function.

In this article, besides deriving (1) and (2) directly from the definitions (3) and (4) by using the conception of fractional derivatives, we propose to obtain new integral expressions in the shape of functional equations for F_1 and F_2 by transforming (1) and (2) by means of fractional integration by parts.

2. – For the sake of convenience the rules for fractional integration and differentiation are given below.

Following ERDÉLYI [1], we write the rule for fractional integration by parts in the form

$$\int_a^b u \frac{\mathrm{d}^\lambda v}{\mathrm{d}(b-x)^\lambda}\,\mathrm{d}x = \int_a^b v \frac{\mathrm{d}^\lambda u}{\mathrm{d}(x-a)^\lambda}\,\mathrm{d}x . \tag{5}$$

The fractional derivatives occuring in this rule can be defined by integrals, if the real part of λ is negative. Thus

$$\begin{cases} \dfrac{\mathrm{d}^\lambda u}{\mathrm{d}(x-a)^\lambda} = \dfrac{1}{\Gamma(-\lambda)} \displaystyle\int_a^x (x-y)^{-\lambda-1}\, u(y)\, \mathrm{d}y \\ \dfrac{\mathrm{d}^\lambda v}{\mathrm{d}(b-x)^\lambda} = \dfrac{1}{\Gamma(-\lambda)} \displaystyle\int_x^b (y-x)^{-\lambda-1}\, v(y)\, \mathrm{d}y \end{cases} \qquad (\mathrm{Re}\,\lambda < 0) . \tag{6}$$

If u and v are expressible by means of series of the types

$$u = \sum A_r\,(x-a)^{\beta+r-1}, \qquad v = \sum \beta_s\,(b-x)^{\alpha+s-1}, \tag{7}$$

then the fractional derivatives are obtainable by differentiating these series term by term and using the definition

$$\frac{\mathrm{d}^\lambda w^{\mu-1}}{\mathrm{d}w^\lambda} = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\, w^{\mu-\lambda-1} \tag{8}$$

for fractional derivatives, which holds for all values of λ except $\lambda = \mu$. Obviously (2.2) and (2.4) are in accordance in those cases in which both definitions have a meaning.

In the event of partial differentiation with respect to w, which arises if, besides w, there are other variables involved, $\frac{\mathrm{d}}{\mathrm{d}w}$ is to be replaced by the usual notation $\frac{\partial}{\partial w}$.

3. – Now we can derive (1) and (2) by means of fractional derivatives. Using the definition (8), we obtain

$$F_1(a;\, b,\, c;\, d;\, xt,\, xy) = \frac{\Gamma(d)}{\Gamma(a)}\, t^{1-d}\, \frac{\partial^{a-d}}{\partial t^{a-d}} \left\{ t^{a-1}\, (1-xt)^{-b}\, (1-yt)^{-c} \right\}. \tag{9}$$

Applying definition (6) of fractional derivatives, we arrive at

$$F_1(a;\, b,\, c;\, d;\, xt,\, yt) = \frac{\Gamma(d)\, t^{1-d}}{\Gamma(a)\, \Gamma(d-a)} \int_0^t u^{a-1} (t-u)^{d-a-1}\, (1-ux)^{-b}\, (1-uy)^{-c}\, \mathrm{d}u\,, \tag{10}$$

valid where $\operatorname{Re} d > \operatorname{Re} a > 0$.

Putting $t = 1$ in (10) we get (1).

Next, by (8) we have

$$F_2(a;\, b,\, c;\, d,\, e;\, x,\, y) = \frac{\Gamma(d)\, \Gamma(e)}{\Gamma(b)\, \Gamma(c)}\, x^{1-d}\, y^{1-e}\, \frac{\partial^{b-d+c-e}}{\partial x^{b-d}\, \partial y^{c-e}} \left\{ x^{b-1}\, y^{c-1}\, (1-x-y)^{-a} \right\}. \tag{11}$$

Using again the definition (6), we obtain

$$F_2(a;\; b,\; c;\; d,\; e;\; x,\; y) = \frac{\Gamma(d)\, \Gamma(e)\, x^{1-d}\, y^{1-e}}{\Gamma(b)\, \Gamma(c)\, \Gamma(d-b)\, \Gamma(e-c)}\, \cdot$$

$$\cdot \int_0^x \int_0^y z^{b-1}\, t^{c-1}\, (x-z)^{d-b-1}\, (y-t)^{e-c-1}\, (1-z-t)^{-a}\, \mathrm{d}z\, \mathrm{d}t\,,$$

valid when $\operatorname{Re} d > \operatorname{Re} b > 0$, $\operatorname{Re} e > \operatorname{Re} c > 0$.

Introducing new variables u and v of integration by the equation $z = u\,x$ and $t = v\,y$, the last relation turns into (2).

4. – Transforming (1) by means of fractional integration by parts, we shall prove the formula

$$F_1(a;\, b,\, c;\, d;\, x,\, y) = \frac{\Gamma(d)}{\Gamma(\lambda)\, \Gamma(d-\lambda)} \int_0^1 u^{\lambda-1}\, (1-u)^{d-\lambda-1}\, (1-ux)^{-b'}\, (1-uy)^{-c'}\, \cdot \tag{12}$$

$$\cdot F_1(a;\; b-b',\; c-c';\; \lambda;\; ux,\; uy)\, F_1\!\left(a-\lambda;\; b',\; c';\; d-\lambda;\; \frac{1-u}{1-ux}\, x,\; \frac{1-u}{1-uy}\, y\right) \mathrm{d}u,$$

which is valid provided that $\operatorname{Re} d > \operatorname{Re} \lambda > 0$.

By the definition (8) we have

$$(1-u)^{d-a-1}\,(1-ux)^{-b'}\,(1-uy)^{-c'}=$$

$$=(1-x)^{-b'}\,(1-y)^{-c'}\,(1-u)^{d-a-1}\left(1+\frac{1-u}{1-x}\,x\right)^{-b'}\left(1+\frac{1-u}{1-y}\,y\right)^{-c'}=$$

$$=(1-x)^{-b'}(1-y)^{-c'}\sum_{m,n=0}^{\infty}\frac{(b')_m\,(c')_n}{m!\;n!}\left(\frac{x}{x-1}\right)^m\left(\frac{y}{y-1}\right)^n(1-u)^{d-a+m+n-1}=$$

$$=(1-x)^{-b'}(1-y)^{-c'}\,\frac{\Gamma(d-a)}{\Gamma(d-\lambda)}\,\frac{\partial^{a-\lambda}}{\partial(1-u)^{a-\lambda}}\left\{(1-u)^{d-\lambda-1}\cdot\right.$$

$$\left.\cdot F_1\!\left(d-a;\;b',\;c';\;d-\lambda;\;\frac{1-u}{x-1}\,x,\;\frac{1-u}{y-1}\,y\right)\right\}.$$

It is known that ([2], p. 239)

$$F_1(\alpha;\;\beta,\;\beta';\;\gamma;\;x,\;y)=(1-x)^{-\beta}(1-y)^{-\beta'}F_1\!\left(\gamma-\alpha;\;\beta,\;\beta';\;\gamma;\;\frac{x}{x-1},\;\frac{y}{y-1}\right),$$

so that

$$(1-u)^{d-a-1}\,(1-ux)^{-b'}(1-uy)^{-c'}=\frac{\Gamma(d-a)}{\Gamma(d-\lambda)}\,\frac{\partial^{a-\lambda}}{\partial(1-u)^{a-\lambda}}\left\{(1-u)^{d-\lambda-1}\;\cdot\right.$$

$$\left.\cdot\;(1-ux)^{-b'}(1-uy)^{-c'}\cdot F_1\!\left(a-\lambda;\,b',\,c';\,d-\lambda;\,\frac{1-u}{1-ux}\,x,\;\frac{1-u}{1-uy}\,y\right)\right\}.$$

Using this result it follows from (1)

$$F_1(a;\;b,\;c;\;d;\;x,\;y)=\frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-a}\;\cdot$$

$$\cdot\int_0^1 u^{a-1}(1-ux)^{b'-b}\,(1-uy)^{c'-c}\left\{(1-u)^{d-a-1}(1-ux)^{-b'}\,(1-uy)^{-c'}\right\}du=$$

$$=\frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-\lambda)}\int_0^1 u^{a-1}(1-ux)^{b'-b}\,(1-uy)^{c'-c}\,\frac{\partial^{a-\lambda}}{\partial(1-u)^{a-\lambda}}\left\{(1-u)^{d-\lambda-1}\;\cdot\right.$$

$$\left.\cdot\;(1-ux)^{-b'}(1-uy)^{-c'}\,F_1\!\left(a-\lambda;\,b',\,c';\;d-\lambda;\;\frac{1-u}{1-ux}\,x,\;\frac{1-u}{1-uy}\,y\right)\right\}du.$$

Integrating fractionally by parts this becomes, by the rule (5),

$$F_1(a;\ b,\ c;\ d;\ x,\ y) = \frac{\Gamma(d)}{\Gamma(a)\,\Gamma(d-\lambda)} \int_0^1 (1-u)^{d-\lambda-1}(1-ux)^{-b'}(1-uy)^{-c'} \cdot$$

$$\cdot F_1\left(a-\lambda;\ b',\ c';\ d-\lambda;\ \frac{1-u}{1-ux}\,x,\ \frac{1-u}{1-uy}\,y\right) \frac{\partial^{a-\lambda}}{\partial u^{a-\lambda}} \{u^{a-1}(1-ux)^{b'-b}(1-uy)^{c'-c}\} du.$$

Now

$$\frac{\partial^{a-\lambda}}{\partial u^{a-\lambda}} \{u^{a-1}\,(1-ux)^{b'-b}(1-uy)^{c'-c}\} =$$

$$= \frac{\partial^{a-\lambda}}{\partial u^{a-\lambda}} \left\{ \sum_{m,n=0}^{\infty} \frac{(b-b')_m\ (c-c')_n}{m!\ n!}\ u^{a+m+n-1}\, x^m\, y^n \right\} =$$

$$= \frac{\Gamma(a)}{\Gamma(\lambda)}\, u^{\lambda-1}\, F_1(a;\ b-b',\ c-c';\ \lambda;\ ux,\ uy)\,,$$

and therefore

$$F_1(a;\ b,\ c;\ d;\ x,\ y) = \frac{\Gamma(d)}{\Gamma(\lambda)\,\Gamma(d-\lambda)} \int_0^1 u^{\lambda-1}(1-u)^{d-\lambda-1}(1-ux)^{-b'}(1-uy)^{-c'} \cdot$$

$$\cdot\ F_1(a;\ b-b',\ c-c';\ \lambda;\ ux,\ uy)\ F_1\left(a-\lambda;\ b',\ c';\ d-\lambda;\ \frac{1-u}{1-ux}\,x,\ \frac{1-u}{1-uy}\,y\right) du\,.$$

Thus far the proof given in this section holds only provided that

$$\operatorname{Re} d > \operatorname{Re} a > 0, \qquad \operatorname{Re} d > \operatorname{Re} \lambda > 0; \qquad |x| < 1, \qquad |y| < 1.$$

5. – Lastly, by transforming (2), we prove

$$(13)\ \begin{cases} F_2(a;\ b,\ c;\ d,\ e;\ x,\ y) = \dfrac{\Gamma(d)\,\Gamma(e)}{\Gamma(\lambda)\,\Gamma(\mu)\,\Gamma(d-\lambda)\,\Gamma(e-\mu)} \cdot \\ \cdot \displaystyle\int_0^1\!\!\int_0^1 u^{\lambda-1}v^{\mu-1}(1-u)^{d-\lambda-1}(1-v)^{e-\mu-1}(1-ux-vy)^{-a'}F_2(a-a';b,c;\lambda,\mu;ux,vy) \cdot \\ \cdot F_2\left(a';\ b-\lambda,\ c-\mu;\ d-\lambda,\ e-\mu;\ \dfrac{1-u}{1-ux-vy}\,x,\ \dfrac{1-v}{1-ux-vy}\,y\right) du\, dv\,, \end{cases}$$

valid when $\operatorname{Re} d > \operatorname{Re} \lambda > 0$, $\operatorname{Re} e > \operatorname{Re} \mu > 0$.

Using the definition (8) we see that

$$(1-u)^{d-b-1}(1-v)^{e-c-1}(1-ux-uy)^{-a'}=$$

$$=(1-u)^{d-b-1}(1-v)^{e-c-1}(1-x-y)^{-a'}\left(1+\frac{1-u}{1-x-y}\,x+\frac{1-v}{1-x-y}\,y\right)^{-a'}=$$

$$=(1-x-y)^{-a'}\sum_{m,n=0}^{\infty}\frac{(a')_{m+n}}{m!\,n!}\left(\frac{-x}{1-x-y}\right)^{m}\left(\frac{-y}{1-x-y}\right)^{n}(1-u)^{d-b+m-1}(1-v)^{e-c+n-1}=$$

$$=\frac{\Gamma(d-b)\,\Gamma(e-c)}{\Gamma(d-\lambda)\,\Gamma(e-\mu)}(1-x-y)^{-a'}\frac{\partial^{b-\lambda+c-\mu}}{\partial(1-u)^{b-\lambda}\,\partial(1-v)^{c-\mu}}\Big\{(1-u)^{d-\lambda-1}(1-v)^{e-\mu-1}\cdot$$

$$\cdot F_2\left(a';\ d-b,\ e-c;\ d-\lambda,\ e-\mu;\ -\frac{1-u}{1-x-y}\,x,\ -\frac{1-v}{1-x-y}\,y\right)\Big\}.$$

Now ([**2**], p. 240)

$$F_2(\alpha;\beta,\beta';\gamma,\gamma';x,y)=(1-x-y)^{-\alpha}F_2\left(\alpha;\gamma-\beta,\gamma'-\beta';\gamma,\gamma';\frac{-x}{1-x-y},\frac{-y}{1-x-y}\right)$$

and therefore

$$(1-u)^{d-b-1}(1-v)^{e-c-1}(1-ux-vy)^{-a'}=$$

$$=\frac{\Gamma(d-b)\,\Gamma(e-c)}{\Gamma(d-\lambda)\,\Gamma(e-\mu)}\frac{\partial^{b-\lambda+c-\mu}}{\partial(1-u)^{b-\lambda}\,\partial(1-v)^{c-\mu}}\Big\{(1-u)^{d-\lambda-1}(1-v)^{e-\mu-1}(1-ux-vy)^{-a'}\cdot$$

$$\cdot F_2\left(a';\ b-\lambda,\ c-\mu;\ d-\lambda,\ e-\mu;\ \frac{1-u}{1-ux-vy}\,x,\ \frac{1-v}{1-ux-vy}\,y\right)\Big\}.$$

Using this it follows from (2)

$$F_2(a;\ b,\ c;\ d,\ e;\ x,\ y)=\frac{\Gamma(d)\,\Gamma(e)}{\Gamma(b)\,\Gamma(c)\,\Gamma(d-b)\,\Gamma(e-c)}\,\cdot$$

$$\cdot\int_0^1\!\!\int_0^1 u^{b-1}v^{c-1}(1-ux-vy)^{a'-a}\left\{(1-u)^{d-b-1}(1-v)^{e-c-1}(1-ux-vy)^{-a'}\right\}du\,dv=$$

$$=\frac{\Gamma(d)\,\Gamma(e)}{\Gamma(b)\,\Gamma(c)\,\Gamma(d-\lambda)\,\Gamma(e-\mu)}\int_0^1\!\!\int_0^1 u^{b-1}\,v^{c-1}(1-ux-vy)^{a'-a}\,\cdot$$

$$\cdot\frac{\partial^{b-\lambda+c-\mu}}{\partial(1-u)^{b-\lambda}\,\partial(1-v)^{c-\mu}}\Big\{(1-u)^{d-\lambda-1}(1-v)^{e-\mu-1}(1-ux-vy)^{-a'}\cdot$$

$$\cdot F_2\left(a';\ b-\lambda,\ c-\mu;\ d-\lambda,\ e-\mu;\ \frac{1-u}{1-ux-vy}\,x,\ \frac{1-v}{1-ux-vy}\,y\right)\Big\}\,du\,dv\,.$$

Integrating fractionally by parts, we obtain

$$F_2(a;\ b,\ c;\ d,\ e;\ x,\ y) =$$

$$= \frac{\Gamma(d)\,\Gamma(e)}{\Gamma(b)\,\Gamma(c)\,\Gamma(d-\lambda)\,\Gamma(e-\mu)} \int_0^1\!\!\int_0^1 (1-u)^{d-\lambda-1}\,(1-v)^{e-\mu-1}\,(1-ux-vy)^{-a'}\,\cdot$$

$$\cdot\, F_2\!\left(a';\ b-\lambda,\ c-\mu;\ d-\lambda,\ e-\mu;\ \frac{1-u}{1-ux-vy}\,x,\ \frac{1-v}{1-ux-vy}\,y\right)\cdot$$

$$\cdot\, \frac{\partial^{b-\lambda+c-\mu}}{\partial u^{b-\lambda}\,\partial v^{c-\mu}}\left\{u^{b-1}\,v^{c-1}\,(1-ux-vy)^{a'-a}\right\} \mathrm{d}u\,\mathrm{d}v\,.$$

But since

$$\frac{\partial^{b-\lambda+c-\mu}}{\partial u^{b-\lambda}\,\partial v^{c-\mu}}\left\{u^{b-1}\,v^{c-1}\,(1-ux-vy)^{a'-a}\right\} =$$

$$= \frac{\partial^{b-\lambda+c-\mu}}{\partial u^{b-\lambda}\,\partial v^{c-\mu}}\left\{\sum_{m,n=0}^{\infty} \frac{(a-a')_{m+n}}{m!\ n!}\,u^{b+m-1}\,v^{c+n-1}\,x^m\,y^n\right\} =$$

$$= \frac{\Gamma(b)\,\Gamma(c)}{\Gamma(\lambda)\,\Gamma(\mu)}\,u^{\lambda-1}\,v^{\mu-1}\,F_2(a-a';\ b,\ c;\ \lambda,\ \mu;\ ux,\ vy)\,,$$

it follows that

$$F_2(a;\ b,\ c;\ d,\ e;\ x,\ y) = \frac{\Gamma(d)\,\Gamma(e)}{\Gamma(\lambda)\,\Gamma(\mu)\,\Gamma(d-\lambda)\,\Gamma(e-\mu)}\,\cdot$$

$$\cdot \int_0^1\!\!\int_0^1 u^{\lambda-1}\,v^{\mu-1}\,(1-u)^{d-\lambda-1}(1-v)^{e-\mu-1}(1-ux-vy)^{-a'}F_2(a-a';\ b, c;\ \lambda, \mu;\ ux,\ vy)\,\cdot$$

$$\cdot\, F_2\!\left(a';\ b-\lambda,\ c-\mu;\ d-\lambda,\ e-\mu;\ \frac{1-u}{1-ux-vy}\,x,\ \frac{1-v}{1-ux-vy}\,y\right)\mathrm{d}u\,\mathrm{d}v\,.$$

This completes the proof of (13).

Thus far this proof is valid provided that

$$\operatorname{Re} d > \operatorname{Re} b > 0, \quad \operatorname{Re} e > \operatorname{Re} c > 0, \quad \operatorname{Re} d > \operatorname{Re} \lambda > 0, \quad \operatorname{Re} e > \operatorname{Re} \mu > 0;$$
$$|x| + |y| < 1 .$$

References.

[1] A. Erdélyi, *On fractional integration and its application to the theory of Hankel transforms*, Quart. J. Math. Oxford. **11** (1940), 293-303.

[2] A. Erdélyi, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York 1953.

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