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On the Absolute Cesàro Summability of Factored Fourier Series. (**)

1. – Let $f(t)$ be a Lebesgue integrable function in the interval $(-\pi, \pi)$ and periodic with period 2π. Let its Fourier series be given by

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t).$$

We write

$$\varphi(t) = \tfrac{1}{2}[f(x+t) + f(x-t) - 2\, f(x)], \qquad S_n(x) = \sum_{\nu=0}^{n} A_\nu(x).$$

2. – Cheng [1] in 1947 obtained the following theorem concerning the summability $|C, \alpha|$ of the factored Fourier series:

Theorem A. If

(2.1) $$\int_0^t |\varphi(u)|\, du = O(t\,(\log t^{-1})^{\beta}) \qquad (\beta \geqslant 0,\ t \to 0),$$

then $\sum \frac{A_n(t)}{(\log n)^{1+\beta+\varepsilon}}$ $(\varepsilon > 0)$ is summable $|C, \alpha|$ $(\alpha > 1)$ at the point $t = x$.

This result was recently extended to a general class of sequences by Dikshit [2] in the following way:

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Theorem B. If $\{\lambda_n\}$ is a convex sequence such that the series $\sum_1^\infty \frac{\lambda_n}{n}$ is convergent, then the series $\sum_1^\infty \frac{\lambda_n A_n(t)}{(\log \overline{n+1})^\beta}$, at $t = x$, is summable $|C, \alpha|$ for every $\alpha > 1$, provided that (2.1) holds.

The object of this paper is to find suitable conditions so that the above factored FOURIER series may be summable $|C, 1|$. It may be remarked that our result generalizes the following theorem of PATI [4] in the sense that our condition for $\beta = 0$ is lighter than the condition (2.2) (see below) of PATI:

Theorem C. If

(2.2) $$\int_0^t |\varphi(u)|\, du = o(t) \qquad \text{as} \quad t \to 0,$$

and $\{\lambda_n\}$ be a convex sequence such that $\sum \lambda_n (\log n)^{1/2}/n < \infty$, then, at the point $t = x$, $\sum \lambda_n A_n(t)$ is summable $|C, 1|$.

3. – In what follows we prove the following

Theorem. *If $\{\lambda_n\}$ is a convex sequence such that $\sum \lambda_n (\log n)^{1/2}/n < \infty$, then the factored Fourier series $\sum \frac{\lambda_n A_n(t)}{(\log \overline{n+1})^\beta}$ is summable $|C, 1|$ at the point $t = x$, provided (2.1) holds.*

4. – For the proof of the theorem we require the following lemmas:

Lemma 1 [1]. If (2.1) holds, then

$$\sum_{\nu=0}^n |S_\nu(x) - f(x)| = O\left(n(\log n)^{1/2+\beta}\right).$$

Lemma 2 [1].

$$\sum_{\nu=0}^n \nu \cos(\nu t) = \begin{cases} O(n^2) \\ O(n/t). \end{cases}$$

Lemma 3 [3]. If $\{\lambda_n\}$ is a convex sequence such that $\sum \lambda_n/n < \infty$, then $\sum (\log n)\, \Delta\lambda_n = O(1)$.

Lemma 4. *Under the condition* (2.1) *we have*

$$\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t}\, dt = O\big((\log n)^{\beta+1}\big)\,.$$

Proof. Integrating by parts we have

$$\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t}\, dt = \left[\frac{\Phi(t)}{t}\right]_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi(t)}{t^2}\, dt \quad (*)$$

$$= O\big((\log n)^{\beta}\big) + O\Big(\int_{1/n}^{\pi} \frac{(\log t^{-1})^{\beta}}{t}\, dt\Big)$$

$$= O\big((\log n)^{\beta}\big) + O\big((\log n)^{\beta+1}\big) = O\big((\log n)^{\beta+1}\big)\,.$$

This completes the proof of the lemma.

5. – Proof of the Theorem. We observe that the summability $|C, 1|$ of $\sum \frac{\lambda_n A_n(x)}{(\log \overline{n+1})^{\beta}}$ is the same as the convergence of $\sum |\zeta_n(x)|/n$, where $\zeta_n(x)$ is the n-th CESÀRO mean of order 1 of the sequence $\left\{\frac{n\,\lambda_n A_n(x)}{(\log \overline{n+1})^{\beta}}\right\}$. Hence we have only to prove that $\sum |\zeta_n(x)|/n < \infty$.

Now

$$\zeta_n(n) = \frac{2}{\pi}\int_0^{\pi} \varphi(t)\, \frac{1}{n+1} \sum_{\nu=0}^{n} \frac{(\nu+2)\,\lambda_{\nu+2}\cos(\nu+2)t}{(\log \overline{\nu+3})^{\beta}}\, dt\,.$$

ABEL's transformation gives

$$\zeta_n(x) = \frac{2}{\pi}\int_0^{\pi} \varphi(t)\, \frac{1}{n+1}\left\{\sum_{\nu=0}^{n-1} T_\nu(t)\, \Delta \frac{\lambda_{\nu+2}}{(\log \overline{\nu+3})^{\beta}}\right\} dt +$$

$$+ \frac{2}{\pi}\int_0^{\pi} \frac{\varphi(t)\, T_n(t)\, \lambda_{n+2}}{(n+1)\,(\log \overline{n+3})^{\beta}}\, dt = I + J \qquad \text{(say)},$$

where

$$T_n(t) = \sum_{\nu=0}^{n} (\nu+2)\cos(\nu+2)t\,.$$

We have

$$I = \frac{2}{\pi}\int_0^{\pi} \varphi(t)\, \frac{1}{n+1}\left\{\sum_{\nu=0}^{n-1} T_\nu(t)\, \Delta \frac{\lambda_{\nu+2}}{(\log \overline{\nu+3})^{\beta}}\right\} dt =$$

(*) Where $\Phi(t) = \int_0^{t} |\varphi(u)|\, du$.

$$= \frac{2}{\pi}\left(\int_0^{1/n} \dots \, dt + \int_{1/n}^{\pi} \dots \, dt\right) = I_1 + I_2, \qquad \text{say.}$$

Substituting the order of $T_\nu(t)$ from Lemma 2, we get

$$I_1 = O\left(\int_0^{1/n} |\varphi(t)| \, dt \cdot \frac{1}{n+1}\left\{\sum_{\nu=0}^{n-1} \nu^2 \left| \Delta \frac{\lambda_{\nu+2}}{(\log \overline{\nu+3})^\beta} \right| \right\}\right)$$

$$= O\left(\int_0^{1/n} |\varphi(t)| \, dt \cdot \frac{1}{n+1}\left\{\sum_{\nu=0}^{n-1} \frac{\nu \, \lambda_{\nu+2}}{(\log \overline{\nu+3})^{\beta+1}} + \sum_{\nu=0}^{n-1} \frac{\nu^2 \, \Delta\lambda_{\nu+2}}{(\log \overline{\nu+3})^\beta}\right\}\right)$$

$$= O\left(\frac{(\log n)^\beta}{n} \frac{1}{n+1}\left\{\sum_{\nu=0}^{n-1} \frac{\nu \, \lambda_{\nu+2}}{(\log \overline{\nu+3})^{\beta+1}} + \sum_{\nu=0}^{n-1} \frac{\nu^2 \, \Delta\lambda_{\nu+2}}{(\log \overline{\nu+3})^\beta}\right\}\right)$$

$$= O\left(\frac{(\log n)^\beta}{n^2} \sum_{\nu=0}^{n-1} \frac{\nu \, \lambda_{\nu+2}}{(\log \overline{\nu+2})^{\beta+1}}\right) + O\left(\frac{(\log n)^\beta}{n^2} \sum_{\nu=0}^{n-1} \frac{\nu^2 \, \Delta\lambda_{\nu+2}}{(\log \overline{\nu+2})^\beta}\right)$$

$$= O\left(\frac{1}{n \log(n+1)} \sum_{\nu=0}^{n} \lambda_{\nu+2}\right) + O\left(\frac{1}{n} \sum_{\nu=0}^{n} \nu \, \Delta\lambda_{\nu+2}\right).$$

Again substituting the second order of $T_\nu(t)$ from Lemma 2 in I_2 and applying Lemma 4, we get

$$I_2 = O\left(\int_{1/n}^{\pi} |\varphi(t)| \frac{1}{n+1}\left\{\sum_{\nu=0}^{n-1} \frac{\nu}{t} \, \Delta \frac{\lambda_{\nu+2}}{(\log \overline{\nu+3})^\beta}\right\} dt\right)$$

$$= O\left(\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} \frac{1}{n+1}\left\{\sum_{\nu=0}^{n-1} \nu \, \Delta \frac{\lambda_{\nu+2}}{(\log \overline{\nu+3})^\beta}\right\} dt\right)$$

$$= O\left(\frac{1}{n+1}\left\{\sum_{\nu=0}^{n} \nu \, \Delta \frac{\lambda_{\nu+2}}{(\log \overline{\nu+2})^\beta}\right\} \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} \, dt\right)$$

$$= O\left(\frac{1}{n+1}\left\{\sum_{\nu=0}^{n} \frac{\lambda_{\nu+2}}{(\log \overline{\nu+2})^{\beta+1}}\right\} \int_{1/n}^{\pi} \frac{|\varphi(t)|}{t} \, dt\right) +$$

$$+ O\left(\frac{1}{n+1}\left\{\sum_{\nu=0}^{n} \frac{\nu\ \Delta\lambda_{\nu+2}}{(\log \overline{\nu+2})^{\beta}}\right\}\int_{1/n}^{\pi} \frac{|\varphi(t)|}{t}\, dt\right)$$

$$= O\left(\frac{(\log n)^{\beta+1}}{n+1}\sum_{\nu=0}^{n} \frac{\lambda_{\nu+2}}{(\log \overline{\nu+2})^{\beta+1}}\right) + O\left(\frac{(\log n)^{\beta+1}}{n+1}\sum_{\nu=0}^{n} \frac{\nu\ \Delta\lambda_{\nu+2}}{(\log \overline{\nu+2})^{\beta}}\right).$$

Therefore, we have

$$\sum |I|/n \leqslant \sum |I_1|/n + \sum |I_2|/n =$$

$$= O\left(\sum_{n=1}^{\infty} \frac{1}{n^2 \log(n+1)} \sum_{\nu=1}^{n-1} \lambda_{\nu+1}\right) + O\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\sum_{\nu=1}^{n-1} \nu\ \Delta\lambda_{\nu+1}\right) +$$

$$+ O\left(\sum_{n=1}^{\infty} \frac{(\log n)^{\beta+1}}{n^2} \sum_{\nu=1}^{n-1} \frac{\lambda_{\nu+1}}{(\log \overline{\nu+1})^{\beta+1}}\right) + O\left(\sum_{n=1}^{\infty} \frac{(\log n)^{\beta+1}}{n^2} \sum_{\nu=1}^{n-1} \frac{\nu\ \Delta\lambda_{\nu+1}}{(\log \overline{\nu+1})^{\beta}}\right)$$

$$= O\left(\sum_{\nu=1}^{\infty} \lambda_{\nu+1} \sum_{n=\nu+1}^{\infty} \frac{1}{n^2 \log(n+1)}\right) + O\left(\sum_{\nu=1}^{\infty} \nu\ \Delta\lambda_{\nu+1} \sum_{n=\nu+1}^{\infty} \frac{1}{n^2}\right) +$$

$$+ O\left(\sum_{\nu=1}^{\infty} \frac{\lambda_{\nu+1}}{(\log \overline{\nu+2})^{\beta+1}} \sum_{n=\nu+1}^{\infty} \frac{(\log n)^{\beta+1}}{n^2}\right) + O\left(\sum_{\nu=1}^{\infty} \frac{\nu\ \Delta\lambda_{\nu+1}}{(\log \overline{\nu+2})^{\beta}} \sum_{n=\nu+1}^{\infty} \frac{(\log n)^{\beta+1}}{n^2}\right)$$

$$= O\left(\sum_{\nu=1}^{\infty} \frac{\lambda_{\nu+1}}{\nu+1}\right) + O\left(\sum_{\nu=1}^{\infty} \Delta\lambda_{\nu+1}\right) + O\left(\sum_{\nu=1}^{\infty} \frac{\lambda_{\nu+1}}{\nu+1}\right) + O\left(\sum_{\nu=1}^{\infty} \log(\nu+1)\ \Delta\lambda_{\nu+1}\right) = O(1),$$

by virtue of Lemma 3 and the fact that

$$\sum_{n=\nu+1}^{\infty} \frac{(\log n)^{\beta+1}}{n^2} = O\left(\frac{(\log \overline{\nu+1})^{\beta+1}}{\nu+1}\right).$$

Now we have only to prove that

$$\sum_{n} |J|/n = O(1).$$

Therefore

$$J = \frac{2}{\pi}\int_{0}^{\pi} \frac{\varphi(t)\ T_n(t)\ \lambda_{n+2}}{(n+1)\ (\log \overline{n+3})^{\beta}}\, dt =$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \varphi(t) \sum_{\nu=0}^{n} (\nu+2)\cos(\nu+2)t\ \frac{\lambda_{n+2}}{(n+1)\ (\log \overline{n+3})^{\beta}}\, dt$$

$$= \frac{2}{\pi} \frac{\lambda_{n+2}}{(n+1)\,(\log \overline{n+3})^{\beta}} \int_0^{\pi} \varphi(t) \sum_{\nu=0}^{n} (\nu+2) \cos(\nu+2)t \, dt$$

$$= \frac{2}{\pi} \frac{\lambda_{n+2}}{(n+1)\,(\log \overline{n+3})^{\beta}} \sum_{\nu=0}^{n} (\nu+2) \int_0^{\pi} \varphi(t) \cos(\nu+2)t \, dt$$

$$= \frac{\lambda_{n+2}}{\log(\overline{n+3})^{\beta}} \left\{ \frac{1}{n+1} \sum_{\nu=0}^{n} (\nu+2)\, A_{\nu+2}(x) \right\}.$$

Put

$$H_{n+2}(x) = \frac{1}{n+3} \sum_{\nu=2}^{n+2} \nu\, A_{\nu}(x), \qquad S_n(x) = \sum_{\nu=0}^{n} A_{\nu}(x), \qquad \sigma_n(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} S_{\nu}(x),$$

so that

$$H_{n+2}(x) = S_{n+2}(x) - \sigma_{n+2}(x) - \frac{1}{n+3} A_1(x),$$

and therefore

$$| H_{n+2}(x) | \leqslant | S_{n+2}(x) - f(x) | + | \sigma_{n+2}(x) - f(x) | .$$

From Lemma 1 we have

$$| \sigma_{n+2}(x) - f(x) | = O((\log n)^{1/2+\beta}),$$

and hence

$$J = O\left((\log n)^{1/2+\beta} \frac{\lambda_{n+2}}{(\log \overline{n+2})^{\beta}} \right) + O(1) + O\left(\frac{\lambda_{n+2}}{(\log \overline{n+2})^{\beta}} | S_{n+2}(x) - f(x) | \right),$$

$$\sum_{n=1}^{\infty} \frac{|J|}{n} = O\left(\sum_{n=1}^{\infty} \frac{(\log n)^{1/2+\beta} \lambda_{n+2}}{n\,(\log \overline{n+2})^{\beta}} \right) + O(1) + O\left(\sum_{n=1}^{\infty} \frac{\lambda_{n+2}}{n\,(\log \overline{n+2})^{\beta}} |S_{n+2}(x) - f(x)| \right)$$

$$= O\left(\sum_{n=1}^{\infty} \frac{\lambda_n (\log n)^{1/2}}{n} \right) + O(1) + \sum_{n=1}^{\infty} \frac{\lambda_{n+2}}{n\,(\log \overline{n+2})^{\beta}} | S_{n+2}(x) - f(x) |$$

$$= O(1) + \sum_{n=1}^{\infty} \frac{\lambda_{n+2}}{n\,(\log \overline{n+2})^{\beta}} | S_{n+2}(x) - f(x) |,$$

by hypothesis.

Now we have only to show that

$$\sum_{n=1}^{\infty} \frac{\lambda_{n+2}}{n\,(\log \overline{n+2})^{\beta}} \,|\, S_{n+2}(x) - f(x) \,|\, = O(1)\,.$$

Since by ABEL's transformation we have

$$\sum_{n=1}^{n} \frac{\lambda_{n+2}}{n\,(\log \overline{n+2})^{\beta}} \,|\, S_{n+2}(x) - f(x) \,|\, =$$

$$= \sum_{n=1}^{m-1} \sum_{\nu=0}^{n} |\, S_{\nu+2}(x) - f(x) \,|\, \Delta \frac{\lambda_{n+2}}{n\,(\log \overline{n+2})^{\beta}} +$$

$$+ \sum_{n=0}^{m} |\, S_{n+2}(x) - f(x) \,|\, \frac{\lambda_{m+2}}{m\,(\log \overline{m+2})^{\beta}} + O(1)\,.$$

Applying Lemma 1 we get

$$\sum_{n=1}^{m} \frac{\lambda_{n+2}}{n\,(\log \overline{n+2})^{\beta}} \,|\, S_{n+2}(x) - f(x) \,|\, =$$

$$= O\left(\sum_{n=1}^{m-1} n(\log n)^{\frac{1}{2}+\beta} \;\Delta \frac{\lambda_{n+2}}{n\,(\log \overline{n+2})^{\beta}}\right) + O\left(m(\log m)^{\frac{1}{2}+\beta} \frac{\lambda_{m+2}}{m\,(\log \overline{m+2})^{\beta}}\right. + O(1)$$

$$= O\left(\sum_{n=1}^{\infty} n\,(\log n)^{\frac{1}{2}+\beta} \frac{\Delta\, \lambda_{n+2}}{n\,(\log \overline{n+2})^{\beta}}\right. + O\left(\sum_{n=1}^{\infty} n\,(\log n)^{\frac{1}{2}+\beta} \frac{\lambda_{n+2}}{n^{2}\,(\log \overline{n+2})^{\beta}}\right) +$$

$$+ O\big((\log m)^{\frac{1}{2}}\, \lambda_{m+2}\big) + O(1) =$$

$$= O\left(\sum_{n=1}^{\infty} (\log n)^{\frac{1}{2}}\, \lambda_{n+2}\right) + O\left(\sum_{n=1}^{\infty} \frac{\lambda_{n+2}\,(\log n)^{\frac{1}{2}}}{n}\right) + O\big((\log m)^{\frac{1}{2}}\, \lambda_{m}\big) + O(1) =$$

$$= O\left(\sum_{n=1}^{\infty} \frac{\lambda_{n}\,(\log n)^{\frac{1}{2}}}{n}\right) + O(1) = O(1)\,,$$

by virtue of Lemma 3, the hypothesis of the theorem and the fact that $(\log n)\ \lambda_n = o(1)$.

Thus finally

$$\sum |\zeta_n(x)| /n < \infty .$$

This completes the proof of the theorem.

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